Korovkin-type theorems and their statistical versions in grand Lebesgue spaces

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Abstract: The analogs of Korovkin theorems in grand-Lebesgue spaces are proved. The subspace $G^{p)}(-\pi;\pi)$ of grand Lebesgue space is defined using shift operator. It is shown that the space of infinitely differentiable finite functions is dense in $G^{p)}(-\pi;\pi)$. The analogs of Korovkin theorems are proved in $G^{p)}(-\pi;\pi)$. These results are established in $G^{p)}(-\pi;\pi)$ in the sense of statistical convergence. The obtained results are applied to a sequence of operators generated by the Kantorovich polynomials, to Fejer and Abel-Poisson convolution operators.

Key words: Grand Lebesgue space, Korovkin theorems, shift operator, statistical convergence, positive linear operator, approximation process

1. Introduction

The concept of statistical convergence was first introduced by Fast ([20]) and Steinhaus in 1951 ([46]). For scalar sequences, this concept was treated as an almost everywhere convergence by Zygmund in the monograph [50], where it was introduced in the context of pointwise convergence of the Fourier series of summable function. This theory was further developed by Schoenberg [44], Peterson [39], Brown and Freedman [13], Connor [16], Erdös and Tenenbaum [19], Freedman and Sember [25], Fridy [26], Fridy and [27], Kuchukaslan et al. [33], Maddox [35], Maharam [36], etc. (see also [41]) Statistical convergence has important applications in different areas of mathematics, such as summation theory, number theory, probability theory, and approximation theory. Statistical convergence is related to the concept of statistical fundamentals, considered first by Fridy [26], who established the equivalence of these concepts for numerical sequences. Maio and Kocinac [37] proved that, in uniform spaces, the statistical convergence implies the statistical fundamentals. Statistical convergence in arbitrary metric and uniform topological spaces was studied in [5–8, 33–37]. The equivalence of the concepts of statistical convergence and statistical fundamentals in metric spaces was established in [5]. Continuous analog of statistical convergence was considered in [8].

Lately, there have been great interests in various nonstandard spaces in the context of applications in different areas of mathematics. Among those spaces, we can mention Lebesgue spaces with variable summability index, Morrey spaces, grand-Lebesgue spaces, etc. These spaces have been considered by many authors such as Xianling and Dun [47], Sharapudinov [45], Zorko [49], Morrey [38], Cruz-Uribe and Fiorenza [17], Adams [1], Samko [42], Kokilashvili et al. [30], Bilalov and Guseynov [3, 4], Fiorenza and Krbec [23], Castilo and

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Rafeiro [15], Fiorenza and Karadzhov [22], Capone and Fiorenza [14], Samko and Umarkhadzhiev [43], Zeren [48], etc. Note that Iwaniec and Sbordone [29] were first to introduce the concept of grand-Lebesgue space in 1992. A lot of research articles, reviews, and monographs have been dedicated to these spaces since then ([18, 21, 24, 28, 31, 32, 40]).

Korovkin theorems are important tools in approximation theory. They help to establish the convergence of the sequence of linear positive operators to the identity operator in the space $C([0;1]]$. When solving the problems of the theory of differential equations in grand-Lebesgue space, you have to consider some subspaces of it in which the space of continuous functions is dense. It is of interest to study the analogs of Korovkin theorems and their statistical versions in such subspaces.

This work deals with Korovkin type Korovkin theorems in grand-Lebesgue spaces and their statistical versions. The paper is organized as follows. In introduction, the grand-Lebesgue space $L^{p)}(-\pi;\pi)$ is defined. This space is nonseparable. Using shift operator, the subspace $G^{p)}(-\pi;\pi) \subset L^{p)}(-\pi;\pi)$ is constructed, where the set of continuous functions is dense. Classical Korovkin theorems on the approximation by positive operators are stated. Some concepts and facts concerning statistical convergence are also stated in Section 3, the density of infinitely differentiable finite on $(-\pi;\pi)$ functions in $G^{p)}(-\pi;\pi)$ is proved. The analogs of Korovkin theorems in the spaces $G^{p)}(-\pi;\pi)$ are also established in this section. In Section 3, statistical versions of the theorems obtained in Section 2 are proved.

2. Preliminaries and some related results

Let us state some concepts from the theories of grand-Lebesgue spaces and statistical convergence in metric spaces. Let $L^{p)}(-\pi;\pi)$, $1 < p < +\infty$ be a grand-Lebesgue space of measurable functions f on $[-\pi;\pi]$, which satisfy the condition

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

The space $L^{p)}(-\pi;\pi)$ is a nonseparable complete normed space with the norm $\|f\|_{p)}$. In fact, consider a family of functions

$$f_\alpha(t) = \left\{ \begin{array}{ll} 0, & 0 \leq t \leq \alpha, \\ (t-\alpha)^{-\frac{1}{p}}, & \alpha < t \leq 1, \end{array} \right. .$$

$\alpha \in [0;1)$. We have $\{f_\alpha\} \subset L_{p)}(0;1)$. In fact,

$$\|f_\alpha\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_\alpha^1 (t-\alpha)^{-1+\frac{\varepsilon}{p}} dt \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon<p-1} \left(p(t-\alpha)^{\frac{\varepsilon}{p}} |_\alpha^1 \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} \left(p(1-\alpha)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

For any different $\alpha, \beta \in [0;1)$ with $\alpha < \beta$ we have

$$\|f_\alpha - f_\beta\|_{p)} \geq \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_\alpha^\beta (t-\alpha)^{-1+\frac{\varepsilon}{p}} dt \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} \left(p(t-\alpha)^{\frac{\varepsilon}{p}} |_\alpha^\beta \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0 < \varepsilon < p-1} \left(p(\beta - \alpha)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{p-\varepsilon}} \geq \lim_{\varepsilon \to +0} \left(p(\beta - \alpha)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{p-\varepsilon}} = p^{\frac{1}{p}}.$$

This directly implies the nonseparability of the space $L_{p)}(0;1)$.

Similar to Morrey space (see [9–12, 49]) we construct the needed separable subcpace of $L^{p)}(-\pi, \pi)$ as follows. For $\forall f \in L^{p)}(-\pi; \pi)$ and $\forall \delta > 0$ we set

$$T_\delta f(x) = \begin{cases} f(x + \delta), x + \delta \in [-\pi; \pi] \\ 0, x + \delta \in R \backslash [-\pi; \pi] \end{cases}.$$

By $\tilde{G}^{p)}(-\pi; \pi)$ we denote a linear manifold of the functions $f \in L^{p)}(-\pi; \pi)$, which satisfy the condition

$$\|T_\delta f - f\|_{p)} \to 0, \delta \to 0.$$

Let $G^{p)}(-\pi; \pi)$ be a closure of $\tilde{G}^{p)}(-\pi; \pi)$ in $L^{p)}(-\pi; \pi)$.

Let us state some concepts from the theory of positive operators [see 45]. Let (X, ρ) and (Y, d) be metric spaces, $F(X)$ be a linear space of functions $f : X \to R$, $F(Y)$ be a linear space of functions $f : Y \to R$ and E be a subspace of $F(X)$. Linear functional $\mu : E \to R$ is said to be positive if for $\forall f \in E$ from $f \geq 0$ it follows $\mu(f) \geq 0$. Linear operator $T : E \to F(Y)$ is said to be positive if for $\forall f \in E$ from $f \geq 0$ it follows $T(f) \geq 0$.

Now let us state well-known Korovkin theorems which have important applications in the study of approximation problems in the spaces of continuous functions as well as in Lebesgue spaces ([2]).

Theorem 2.1 *(Korovkin's first theorem) Let $\{L_n\}_{n \in N}$ be a sequence of positive operators from $C([0;1])$ into $F([0;1])$, satisfying the condition*

$$\lim_{n \to \infty} \|L_n g - g\|_\infty = 0, \forall g \in \left\{ 1, t, t^2 \right\}.$$

Then

$$\lim_{n \to \infty} \|L_n f - f\|_\infty = 0, \forall f \in C([0;1]),$$

where $\|f\|_\infty = \sup_{[0;1]} |f(x)|$.

Using this theorem for the sequence of operators generated by the Kantorovich polynomials of the form

$$K_n(f)(x) = \sum_{k=0}^{n} \left((n+1) \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt \right) \binom{n}{k} x^k (1-x)^{n-k}, \tag{2.1}$$

$0 \leq x \leq 1, \forall f \in L_p([0;1])$, it is easy to prove its analog in the spaces $L_p([0;1])$. The following is valid:

Theorem 2.2 *Let $1 \leq p < +\infty$ and $\{K_n\}_{n \in N}$ be a sequence of operators generated by the polynomials (2.1). Then K_n is a positive operator acting from $L_p([0;1])$ into $L_p([0;1])$ and the relation*

$$\lim_{n \to \infty} \|L_n f - f\|_p = 0, \forall f \in L_p([0;1]),$$

holds, where $\|f\|_p = \left(\int_0^1 |f(t)|^p \, dt \right)^{\frac{1}{p}}$.

By $C_{2\pi}(R)$ we denote a space of 2π-periodic continuous functions on R, and by $L_{2\pi}^p(R)$, $1 \le p \le +\infty$, we denote a space of 2π-periodic functions from $L^p(R)$.

The following Korovkin theorem is true in the space $C_{2\pi}(R)$.

Theorem 2.3 *(Korovkin's second theorem) Let $\{L_n\}_{n \in N}$ be a sequence of positive operators from $C_{2\pi}(R)$ into $F(R)$, satisfying the condition*

$$\lim_{n \to \infty} \|L_n g - g\|_\infty = 0, \forall g \in \{1, \sin, \cos\}.$$

Then

$$\lim_{n \to \infty} \|L_n f - f\|_\infty = 0, \forall f \in C_{2\pi}(R).$$

A sequence $\{\varphi_n\}_{n \in N} \subset L_{2\pi}^1(R)$ is called a positive periodic kernel if $\varphi_n \ge 0$, $\forall n \in N$, almost everywhere and

$$\lim_{n \to \infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(t) dt = 1. \tag{2.2}$$

Every positive periodic kernel $\{\varphi_n\}_{n \in N} \subset L_{2\pi}^1(R)$ generates a sequence of linear operators $\{L_n\}_{n \in N}$ in $L_{2\pi}^1(R)$ by the formula

$$L_n(f)(x) = (f * \varphi_n)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x - t) \varphi_n(t) dt =$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \varphi_n(x - t) dt. \tag{2.3}$$

From Hölder's inequality it follows that if $f \in L_{2\pi}^p(R)$, $1 < p < +\infty$, then $L_n(f) \in L_{2\pi}^p(R)$. Moreover, the inequalities

$$\|L_n(f)\|_p \le \|\varphi_n\|_1 \|f\|_p, f \in C_{2\pi}(R), \tag{2.4}$$

$$\|L_n(f)\|_\infty \le \|\varphi_n\|_1 \|f\|_\infty, f \in C_{2\pi}(R) \tag{2.5}$$

hold.

A positive periodic kernel $\{\varphi_n\}_{n \in N}$ is called identically approximative if $\forall \delta \in (0; \pi)$ the equality

$$\lim_{n \to \infty} \left[\int_{-\pi}^{-\delta} \varphi_n(t) dt + \int_{\delta}^{\pi} \varphi_n(t) dt \right] = 0$$

holds.

The next theorem presents the equivalent conditions which provide the validity of Korovkin theorem for a sequence of operators defined by (2.3) in the spaces $C_{2\pi}(R)$ and $L_{2\pi}^p(R)$, $1 \le p < +\infty$.

Theorem 2.4 *Let $\{\varphi_n\}_{n \in N} \subset L_{2\pi}^1(R)$ be a positive periodic kernel and $\{L_n\}_{n \in N}$ be a sequence of operators defined by (2.3). Let*

$$\beta_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(t) \sin^2 \frac{t}{2} dt. \tag{2.6}$$

Then the following properties are equivalent:

i) for every $1 \le p < +\infty$ and $f \in L^p_{2\pi}(R)$, the relation

$$\lim_{n \to \infty} \|L_n f - f\|_p = 0$$

holds, and

$$\lim_{n \to \infty} \|L_n f - f\|_\infty = 0, \forall f \in C_{2\pi}(R);$$

ii) $\lim_{n \to \infty} \beta_n = 0;$

iii) $\{\varphi_n\}_{n \in N}$ *is identically approximative.*

Let the positive periodic kernel $\{\varphi_n\}_{n \in N}$ have the form of

$$\varphi_n = \begin{cases} \frac{\sin^2 \frac{(n+1)t}{2}}{(n+1)\sin^2 \frac{t}{2}}, t \ne 2\pi k, k \in Z \\ n+1, t = 2\pi k, k \in Z \end{cases}.$$

For this kernel we have $\beta_n = \frac{1}{2(n+1)}$. Then, by Theorem 2.4, the relations

$$\lim_{n \to \infty} \|F_n f - f\|_p = 0$$

and

$$\lim_{n \to \infty} \|F_n f - f\|_\infty = 0, \forall f \in C_{2\pi}(R),$$

hold for the Fejer convolution operator F_n given by

$$F_n(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \varphi_n(x - t) dt. \tag{2.7}$$

Similar assertions hold true for the Abel-Poisson convolution operator

$$P_r(f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P(r, x - t) dt, \tag{2.8}$$

where

$$P(r, t) = \frac{1 - r^2}{1 - 2r \cos t + r^2}, \quad t \in R, \ 0 \le r < 1,$$

is a Poission kernel for unit ball. Namely, in this case

$$\beta_r = \frac{1 - r}{2},$$

and, consequently, by Theorem 2.4 we obtain

$$\lim_{r \to 1-0} \|P_r f - f\|_p = 0,$$

and

$$\lim_{r \to 1-0} \|P_r f - f\|_\infty = 0, \ \forall f \in C_{2\pi}(R).$$

Let us recall some auxiliary concepts and facts concerning statistical convergence in metric spaces (see [18]). Let (X, ρ) be a metric space, $A \subset N$ and

$$\delta_n(A) = \frac{1}{n} \sum_{k=1}^{n} \chi_A(k).$$

If there exists a limit $\delta(A) = \lim_{n\to\infty} \delta_n(A)$, then the number $\delta(A)$ is called a statistical density of the set A. The sequence $\{x_n\}_{n\in N} \subset X$ is called statistically convergent to $x \in X$ ($st - \lim_{n\to\infty} x_n = x$), if

$$\forall \varepsilon > 0, \ \delta(A_\varepsilon) = 0, \ where \ A_\varepsilon = \{n \in N : \rho(x_n, x) \geq \varepsilon\}.$$

The sequence $\{x_n\}_{n\in N} \subset X$ is called statistically fundamental ($st - fundamental$) if

$$\forall \varepsilon > 0, \ \exists n_\varepsilon, \ \forall n \geq n_\varepsilon : \delta(\Delta_{n_\varepsilon}) = 0,$$

where $\Delta_{n_\varepsilon} = \{n \in N : \rho(x_n, x_{n_\varepsilon}) \geq \varepsilon\}$.

We set $\mathscr{K} = \{A \subset N : \delta(A) = 1\}$.

The following is valid:

Theorem 2.5 *([5]) Let (X, ρ) be a metric space and $\{x_n\}_{n\in N} \subset X$. The following conditions are equivalent:*

i) $\exists st - \lim_{n\to\infty} x_n = x$;

ii) $\{x_n\}_{n\in N}$ *is st-fundamental;*

iii) $\exists \{y_n\}_{n\in N} \subset X : \exists \lim_{n\to\infty} y_n = x, \{n \in N : x_n = y_n\} \in \mathscr{K}$.

This theorem has the following corollary.

Corollary 2.6 *Let (X, ρ) be a metric space and the sequence $\{x_n\}_{n\in N} \subset X$ be such that $\exists st - \lim_{n\to\infty} x_n = x$. Then*

$$\exists \{n_k\}_{k\in N} \in \mathscr{K} \ (n_1 < n_2 < ... < n_k < ...) : \lim_{k\to\infty} x_{n_k} = x.$$

3. Korovkin theorems in the spaces $G^{p)}(-\pi; \pi)$

Let $L^{p)}(-\pi; \pi)$, $1 < p < +\infty$, be a grand-Lebesgue space. We extend some function from $L^{p)}(-\pi; \pi)$ by zero and consider the closure in $L^{p)}(-\pi; \pi)$ of the linear manifold of functions $f \in L^{p)}(-\pi; \pi)$ such that

$$\|f(\cdot + \delta) - f(\cdot)\|_{L^{p)}} \to 0, \delta \to 0.$$

Denote this subspace by $G^{p)}(-\pi; \pi)$. We prove the following.

Lemma 3.1 *The space $C_0^\infty[-\pi; \pi]$ is dense in $G^{p)}(-\pi; \pi)$.*

Proof Consider an arbitrary number $\eta > 0$ and an arbitrary function $f \in G^{p)}(-\pi; \pi)$. Denote by $\omega_\eta(t)$ the following kernel

$$\omega_\eta(t) = \begin{cases} c_\eta \exp(-\frac{\eta^2}{\eta^2 - t^2}), & |t| \leq \eta, \\ 0, & |t| > \eta, \end{cases}$$

where the constant c_η is such that $\int_{-\infty}^{+\infty} \omega_\eta(t)dt = 1$. Let the function $f_\eta(\cdot)$ be a convolution with a kernel $\omega_\eta(\cdot)$, i.e.

$$f_\eta(t) = \int_{-\infty}^{+\infty} f(t-s)\omega_\eta(s)ds = \int_{-\infty}^{+\infty} \omega_\eta(t-s)f(s)ds.$$

Correctness of such definition follows from the inclusion $L^{p)}(-\pi;\pi) \subset L^1(-\pi;\pi)$. It is clear that $f_\eta(t)$ is an infinitely differentiable function. Using Minkowski's inequality, we obtain

$$\|f - f_\eta\|_{p)} = \left\| \int_{-\infty}^{+\infty} f(\cdot)\omega_\eta(s)ds - \int_{-\infty}^{+\infty} f(\cdot-s)\omega_\eta(s)ds \right\|_{p)} =$$

$$= \left\| \int_{-\infty}^{+\infty} [f(\cdot-s) - f(\cdot)]\omega_\eta(s)ds \right\|_{p)} =$$

$$= \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} \left| \int_{-\infty}^{+\infty} [f(t-s) - f(t)]\omega_\eta(s)ds \right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \int_{-\infty}^{+\infty} \omega_\eta(s) \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t-s) - f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} ds \leq$$

$$\leq (p-1) \sup_{|s| \leq \eta} \|f(\cdot-s) - f(\cdot)\|_{p)} \to 0, \eta \to 0.$$

Consequently, $C^\infty[-\pi;\pi]$ is dense in $G^{p)}(-\pi;\pi)$. Consider an arbitrary $\eta > 0$ and an arbitrary function $f \in G^{p)}(-\pi;\pi)$. As $C^\infty[-\pi;\pi]$ is dense in $G^{p)}(-\pi;\pi)$, there exists $g \in C^\infty[-\pi;\pi]$ such that

$$\|f - g\|_{p)} < \frac{\eta}{3}. \tag{3.1}$$

Choose the number $\delta > 0$ such that $\delta < \pi \left(\frac{\eta}{3(p-1)\|g\|_\infty} \right)^p$. Consider the intervals $E_\delta^+ = (\pi - \delta;\pi)$ and $E_\delta^- = (-\pi;-\pi+\delta)$ of length δ and define the function

$$g_\delta(t) = \begin{cases} g(t), & t \in (-\pi;\pi)\backslash(E_\delta^+ \bigcup E_\delta^-), \\ 0, & t \in E_\delta^+ \bigcup E_\delta^-. \end{cases}$$

We have

$$\|g - g_\delta\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{E_\delta^+ \bigcup E_\delta^-} |g(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \|g\|_\infty \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} 2\delta \right)^{\frac{1}{p-\varepsilon}} < \frac{\|g\|_\infty (p-1)\delta^{\frac{1}{p}}}{\pi^{\frac{1}{p}}} < \frac{\eta}{3}. \tag{3.2}$$

Let

$$g_{\delta,\tau}(t) = \int_{-\infty}^{+\infty} g_\delta(t-s)\omega_\tau(s)ds, \tau \in R.$$

Obviously, for $\tau < \frac{\delta}{2}$ we have $g_{\delta,\tau} \in C_0^\infty[-\pi;\pi]$. As $\|g_\delta - g_{\delta,\tau}\|_{p)} \to 0$ for $\tau \to 0$, there exists $\tau < \frac{\delta}{2}$ such that

$$\|g_\delta - g_{\delta,\tau}\|_{p)} < \frac{\eta}{3}. \tag{3.3}$$

Consequently, using (3.1), (3.2), and (3.3), we obtain

$$\|f - g_{\delta,\tau}\|_{p)} \le \|f - g\|_{p)} + \|g - g_\delta\|_{p)} + \|g_\delta - g_{\delta,\tau}\|_{p)} < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta,$$

i.e. $C_0^\infty[-\pi;\pi]$ is dense in $G^{p)}(-\pi;\pi)$. The lemma is proved. $\qquad\square$

The next theorem is an analog of Korovkin theorem in the spaces $G^{p)}(0;1)$.

Theorem 3.2 *Let* $\{L_n\}_{n\in N}$ *be a sequence of positive linear operators on* $G^{p)}(0;1)$, $1 < p < +\infty$, *satisfying the condition*

$$\lim_{n\to\infty} \|L_n g - g\|_\infty = 0, \;\; \forall g \in \{1, t, t^2\}.$$

Then the relation

$$\lim_{n\to\infty} \|L_n f - f\|_{p)} = 0, \;\; \forall f \in G^{p)}(0;1),$$

holds if and only if $\sup_n \|L_n\| = c < +\infty$.

Proof Necessity follows from the Banach–Steinhaus theorem. We prove the sufficiency. Let $\eta > 0$ be an arbitrary number. Consider an arbitrary function $f \in G^{p)}(0;1)$. From Lemma 3.1 it follows that there exists $g \in C[0;1]$ such that

$$\|f - g\|_{p)} < \eta. \tag{3.4}$$

By Theorem 2.1, there exists n_η such that for $\forall n > n_\eta$

$$\|L_n g - g\|_\infty < \eta. \tag{3.5}$$

For $\forall f \in C([0;1])$ we have

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |f(t)|^{p-\varepsilon}\, dt \right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_\infty = (p-1) \|f\|_\infty. \tag{3.6}$$

Applying triangle inequality, we obtain

$$\|L_n f - f\|_{p)} \le \|L_n f - L_n g\|_{p)} + \|L_n g - g\|_{p)} + \|f - g\|_{p)}.$$

Hence, using (3.4), (3.5), and (3.6), $\forall n > n_\eta$ we obtain

$$\|L_n f - f\|_{p)} < c \|f - g\|_p + (p-1) \|L_n g - g\|_\infty + \|f - g\|_{p)} < (c+p)\eta.$$

Thus, $\lim_{n\to\infty} \|L_n f - f\|_{p)} = 0$. The theorem is proved. $\qquad\square$

We apply the results obtained here to the sequence of operators generated by the Kantorovich polynomials.

Theorem 3.3 *Let $\{K_n\}_{n\in N}$ be a sequence of positive linear operators defined by* (2.1). *Then the following equality holds*

$$\lim_{n\to\infty} \|K_n f - f\|_{p)} = 0, \forall f \in G^{p)}(0;1), 1 < p < +\infty.$$

Proof It is known ([49]) that K_n is a positive linear operator acting boundedly in $L^p(0;1)$ and $\|K_n\| \le 1$, $\forall n \in N$. We have

$$\|K_n f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 |(K_n f)(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|K_n f\|_{p-\varepsilon} \le \|f\|_{p)}, \forall f \in G^{p)}(0;1),$$

i.e. $\{K_n\}_{n\in N}$ is uniformly bounded in $G^{p)}(0;1)$. As

$$\lim_{n\to\infty} \|K_n g - g\|_{\infty} = 0, \forall g \in \{1, t, t^2\},$$

from Theorem 3.2 it follows that $\lim_{n\to\infty} \|K_n f - f\|_{p)} = 0$, $\forall f \in G^{p)}(0;1)$. The theorem is proved. $\square$

Now let us state the analog of Korovkin's second theorem in the space $G^{p)}(0;1)$.

Theorem 3.4 *Let $\{L_n\}_{n\in N}$ be a sequence of positive linear operators in $G^{p)}(-\pi;\pi)$, $1 < p < +\infty$, satisfying the condition*

$$\lim_{n\to\infty} \|L_n g - g\|_{\infty} = 0, \forall g \in \{1, \sin, \cos\}.$$

Then the relation

$$\lim_{n\to\infty} \|L_n f - f\|_{p)} = 0, \forall f \in G^{p)}(-\pi;\pi)$$

holds if and only if $\sup_n \|L_n\| = c < +\infty$.

Proof The necessity follows from the Banach–Steinhaus theorem. We prove the sufficiency. Consider an arbitrary number $\eta > 0$ and an arbitrary function $f \in G^{p)}(-\pi;\pi)$. As $C_{2\pi}(R)$ is dense in $G^{p)}(-\pi;\pi)$, we can find a function $g \in C_{2\pi}(R)$ such that

$$\|f - g\|_{p)} < \eta. \tag{3.7}$$

By Theorem 2.1, there exists n_η such that for $\forall n > n_\eta$

$$\|L_n g - g\|_{\infty} < \eta. \tag{3.8}$$

We have

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \le$$

$$\le \sup_{0<\varepsilon<p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{\infty} = (p-1) \|f\|_{\infty}. \tag{3.9}$$

Then, taking into account (3.7) , (3.8), and (3.9), we obtain

$$\|L_n f - f\|_{p)} \le \|L_n f - L_n g\|_{p)} + \|L_n g - g\|_{p)} + \|f - g\|_{p)} <$$

$$< (c+1)\eta + (p-1)\eta = (c+p)\eta.$$

Thus, $\lim\limits_{n\to\infty} \|L_n f - f\|_{p)} = 0$, $\forall f \in G^{p)}(-\pi;\pi)$. The theorem is proved. $\qquad\square$

Also, the analog of Theorem 2.4 in the space $G^{p)}(-\pi;\pi)$ is true.

Theorem 3.5 *Let $\{\varphi_n\}_{n\in N} \subset L^1_{2\pi}(R)$ be a positive periodic kernel, $\{L_n\}_{n\in N}$ be a sequence of operators defined by (2.3), and the sequence of numbers $\{\beta_n\}_{n\in N}$ be given by the formula (2.6). Then the following properties are equivalent:*

i) for every $1 < p < +\infty$ and $\forall f \in G^{p)}(-\pi;\pi)$

$$\lim_{n\to\infty} \|L_n f - f\|_{p)} = 0, \tag{3.10}$$

and also

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0, \forall f \in C_{2\pi}(R); \tag{3.11}$$

ii) $\lim\limits_{n\to\infty} \beta_n = 0$;

iii) $\{\varphi_n\}_{n\in N}$ is identically approximative.

Proof Obviously, to prove this theorem it suffices to show the validity of the equivalence $i) \Leftrightarrow ii)$. Let the condition $i)$ hold. As the positive kernel $\{\varphi_n\}_{n\in N}$ is bounded, by (2.6) there exists a number $c > 0$ such that $\|L_n\| \le c$, $\forall n \in N$. Therefore, due to the density of $C_{2\pi}(R)$ in $L^p_{2\pi}(R)$ and the equality (3.11), we obtain

$$\lim_{n\to\infty} \|L_n f - f\|_p = 0, \forall f \in L^p_{2\pi}(R).$$

Consequently, by Theorem 2.4, the condition $ii)$ holds.

Conversely, let the condition $ii)$ hold. By Theorem 2.4, we have

$$\lim_{n\to\infty} \|L_n f - f\|_p = 0, \forall f \in L^p_{2\pi}(R),$$

$$\lim_{n\to\infty} \|L_n f - f\|_\infty = 0, \forall f \in C_{2\pi}(R).$$

Taking into account (2.4), for $\forall f \in G^{p)}(-\pi;\pi)$ we obtain

$$\|L_n(f)\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}} \|L_n(f)\|_{p-\varepsilon} \le$$

$$\le \|\varphi_n\|_1 \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}} \|f\|_{p-\varepsilon} = \|\varphi_n\|_1 \|f\|_{p)}.$$

Consider $\forall f \in G^{p)}(-\pi;\pi)$ and $\forall \eta > 0$. Due to the density of $C_{2\pi}(R)$ in $G^{p)}(-\pi;\pi)$, there exists $g \in C_{2\pi}(R)$ such that

$$\|f - g\|_{p)} < \eta. \tag{3.12}$$

From the condition of the theorem it follows that there exists n_η such that for $\forall n > n_\eta$:

$$\|L_n g - g\|_\infty < \eta. \tag{3.13}$$

It is clear that $\sup_n \|\varphi_n\|_1 = c < +\infty$. Then, taking into account (3.12), (3.9), and (3.10), we obtain

$$\|L_n f - f\|_{p)} \leq \|L_n f - L_n g\|_{p)} + \|L_n g - g\|_{p)} + \|f - g\|_{p)} <$$

$$< (c+1)\eta + (p-1)\eta = (c+p)\eta,$$

i.e. the relation (3.10) is true. Thus, the condition $i)$ holds. The theorem is proved. □

The above-proved theorem has the following corollaries, in particular, for Fejer and Abel-Poisson convolution operators.

Corollary 3.6 *Let F_n be a linear Fejer convolution operator in $G^{p)}(-\pi; \pi)$, $1 < p < +\infty$, defined by the formula (2.7). Then $\forall f \in G^{p)}(-\pi; \pi)$ the following relation holds:*

$$\lim_{n \to \infty} \|F_n f - f\|_{p)} = 0. \tag{3.14}$$

Proof Consider the sequence $\{\beta_n\}_{n \in N}$ defined by the formula (2.6). We have

$$\beta_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi_n(t) \sin^2 \frac{t}{2} dt = \frac{1}{2(n+1)},$$

where

$$\varphi_n = \begin{cases} \frac{\sin^2 \frac{(n+1)t}{2}}{(n+1) \sin^2 \frac{t}{2}}, t \neq 2\pi k, k \in Z \\ n+1, t = 2\pi k, k \in Z \end{cases}.$$

Consequently, by Theorem 2.4, the relation (3.14) holds. The corollary is proved. □

Corollary 3.7 *Let P_r, $0 \leq r < 1$, be a linear Abel-Poisson convolution operator in $G^{p)}(-\pi; \pi)$, $1 < p < +\infty$, defined by the formula (2.8). Then $\forall f \in G^{p)}(-\pi; \pi)$ the following relation holds:*

$$\lim_{r \to 1-0} \|P_r f - f\|_{p)} = 0. \tag{3.15}$$

Proof We have

$$\beta_r = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(r, t) \sin^2 \frac{t}{2} dt = \frac{1-r}{2},$$

where $P(r, t) = \frac{1-r^2}{1-2r\cos t + r^2}$. Hence, $\lim_{r \to 1-0} \beta_r = 0$; therefore, by Theorem 2.4, the relation (3.15) holds. The corollary is proved. □

4. Statistical versions of Korovkin theorems in the space $G^{p)}(-\pi;\pi)$

In this section, we establish statistical versions of results obtained in the previous section. We will need the following easy-to-prove lemma.

Lemma 4.1 *If $A, B \in \mathscr{K}$, then $A \bigcap B \in \mathscr{K}$.*

For the proof of this lemma we refer the readers to [5–7].

Theorem 4.2 *Let $\{L_n\}_{n \in N}$ be a sequence of positive linear operators on $G^{p)}(0;1), 1 < p < +\infty$, such that $\sup_n \|L_n\| = c < +\infty$. If in $C([0;1])$*

$$\exists st - \lim_{n \to \infty} L_n g = g, \forall g \in \left\{1, t, t^2\right\},$$

then in $G^{p)}(0;1)$

$$\exists st - \lim_{n \to \infty} L_n f = f, \forall f \in G^{p)}(0;1).$$

Proof From $\exists st - \lim_{n \to \infty} L_n g_i = g_i$, where $g_i(t) = t^i$, $i = 0; 1; 2$, by Corollary 2.6, it follows $\exists \left\{n_k^{(i)}\right\}_{k \in N} \in K$ $\left(n_1^{(i)} < n_2^{(i)} < ... < n_k^{(i)} < ...\right)$

$$\lim_{k \to \infty} \left\|L_{n_k^{(i)}} g_i - g_i\right\|_\infty = 0, i = 0; 1; 2.$$

Let $\{n_k\} = \bigcap_{i=0;1;2} \left\{n_k^i\right\}$, $(n_1 < n_2 < ... < n_k < ...)$. It follows directly from Lemma 4.1 that $\delta\left(\{n_k\}\right) = 1$. Obviously,

$$\lim_{k \to \infty} \|L_{n_k} g_i - g_i\|_\infty = 0, i = 0; 1; 2.$$

Then, by Theorem 3.2, we have

$$\lim_{k \to \infty} \|L_{n_k} f - f\|_{p)} = 0, \forall f \in G^{p)}(0;1).$$

Consequently, by Theorem 2.5, for $\forall f \in G^{p)}(0;1)$ the sequence $\{L_n f\}_{n \in N}$ statistically converges to f in $G^{p)}(0;1)$. The theorem is proved. □

Corollary 4.3 *Let $\{K_n\}_{n \in N}$ be a sequence of positive linear operators defined by (2.1). Then in $G^{p)}(0;1)$*

$$\exists st - \lim_{n \to \infty} K_n f = f, \forall f \in G^{p)}(0;1).$$

Theorem 4.4 *Let $\{L_n\}_{n \in N}$ be a sequence of positive linear operators in $G^{p)}(-\pi;\pi)$ such that $L_n : C_{2\pi}(R) \to C_{2\pi}(R)$ and $\sup_n \|L_n\| = c < +\infty$. If in $C_{2\pi}(R)$*

$$\exists st - \lim_{n \to \infty} L_n g = g, \forall g \in \left\{1, \sin, \cos\right\},$$

then in $G^{p)}(-\pi;\pi)$

$$\exists st - \lim_{n \to \infty} L_n f = f, \forall f \in G^{p)}(-\pi;\pi).$$

Proof   Since there exists $st - \lim\limits_{n\to\infty} L_n g = g$, $\forall g \in \{1, t, t^2\}$, by Corollary 2.6 and Lemma 4.1 we can find $\{n_k\}_{k\in N} \in K$ $(n_1 < n_2 < ... < n_k < ...)$ such that

$$\lim_{k\to\infty} \|L_{n_k} g - g\|_\infty = 0, \forall g \in \{1, \sin, \cos\}.$$

Consequently, by Theorem 3.4, we have

$$\lim_{k\to\infty} \|L_{n_k} f - f\|_{p)} = 0, \forall f \in G^{p)}(-\pi; \pi).$$

Thus, by Theorem 2.5, for $\forall f \in G^{p)}(0; 1)$ the sequence $\{L_n f\}_{n\in N}$ statistically converges to f in $G^{p)}(-\pi; \pi)$. The theorem is proved.   □

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