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Wirtinger type inequalities for higher order differentiable functions

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Abstract: In this work, we establish a Wirtinger type inequality which gives the relation between the integral of square of a function and the integral of square of its any order derivative via Taylor's formula. Then, we provide a similar inequality for mappings that are elements of L_r space with $r > 1$. Also, we indicate that special cases of these inequalities give some results presented in the earlier works.

Key words: Wirtinger inequality, Taylor series

1. Introduction

In mathematics, Wirtinger's inequality for real functions was firstly used in Fourier analysis. This inequality, named after Wilhelm Wirtinger, was used in 1904 to prove the isoperimetric inequality. Over the years, a great number of authors have focused on Wirtinger type inequalities, because its theory plays an important role in many areas such as linear differential equations, Fourier analysis and differential geometry. What makes these inequalities so important is that there are integral inequalities involving a function and its derivative. The classical Wirtinger inequality gives the connection between the integral of square of a function and the integral of square of its first derivative. Also, this inequality [8] states that if $f \in C^1([a,b])$ satisfying $f(a) = f(b) = 0$, then

$$\int_a^b f^2(t)\,dt \le \int_a^b (f'(t))^2\,dt. \tag{1.1}$$

Afterwards, Beesack extended the inequality (1.1) as follows:

Theorem 1.1 *[3] For any $f \in C^2([a,b])$ satisfying $f(a) = f(b) = 0$, following inequality holds:*

$$\int_a^b f^4(t)\,dt \le \frac{4}{3}\int_a^b (f'(t))^4\,dt. \tag{1.2}$$

Wirtinger type inequalities, such as Bessel, Blaschke, Beesack, Poincare, Sobolev, have been presented by many researchers and they have been used in a large number of application areas such as the convergence of series, estimations of integrals and determination of the minimal eigenvalues of differential operators. For

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example, the best constant in the Poincare inequality that is a more general form of Wirtinger is known as the first eigenvalue of the Laplace operator and this inequality has been the inspiration of various geometric studies (see, e.g. [7],[9]). What's more, Böttcher and Widom [5] concerned with a sequence of constants that appear in some problems including the best constant in a Wirtinger-Sobole inequality comparing the integral of the square of a function with that of the square of its higher-order derivative. In [11], which is one of the most recent studies, Sarikaya examined some improved versions of Wirtinger type inequalities. The interested readers can also look over the references [1], [2], [4], [6], [10], [12], [13] and the references therein.

In this study, we investigate how to generalize Wirtinger type inequalities to higher order. For this, we firstly establish a generalized version of the classical Wirtinger inequality (1.1) via Cauchy-Schwarz inequality. Afterwards, a more general result is obtained by means of Hölder's inequality. We also note that the special cases of inequalities presented in this work give some results provided in the earlier works.

2. Some inequalities for high degree differentiable functions

We need the following identity, known as Taylor's formula in literature, to establish our main results. Let $f \in C^n([a,b])$, $n \in \mathbb{N}\backslash\{0\}$. Then, from Taylor's theorem, we have

$$f(x)=\sum_{k=0}^{n-1}\frac{f^{(k)}(c)}{k!}(x-c)^k+\frac{1}{(n-1)!}\int_c^x (x-t)^{n-1}f^{(n)}(t)dt \tag{2.1}$$

for $c \in [a,b]$.

Theorem 2.1 *Let $f \in C^n([a,b])$, $n \in \mathbb{N}\backslash\{0\}$, and $f^{(n)} \in L_2[a,b]$ such that $f^{(k)}(a)=f^{(k)}(b)=0$ for $k=0,1,2,...,n-1$. Then, we possess the inequality*

$$\int_a^b [f(x)]^2 dx \le \frac{(b-a)^{2n}}{[(n-1)!]^2(2n-1)(2n)(2n+1)}\int_a^b \left[f^{(n)}(x)\right]^2 dx. \tag{2.2}$$

Proof If we apply Cauchy-Shwarz inequality after taking absolute value of both sides of (2.1), owing to the hypotheses of the theorem, then we have the inequalities

$$\begin{aligned} |f(x)|^2 &= \left|\frac{1}{(n-1)!}\int_a^x (x-s)^{n-1}f^{(n)}(s)ds\right|^2 \\ &\le \frac{1}{[(n-1)!]^2}\frac{(x-a)^{2n-1}}{2n-1}\int_a^x \left|f^{(n)}(s)\right|^2 ds, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} |f(x)|^2 &= \left|\frac{1}{(n-1)!}\int_x^b (s-x)^{n-1}f^{(n)}(s)ds\right|^2 \\ &\le \frac{1}{[(n-1)!]^2}\frac{(b-x)^{2n-1}}{2n-1}\int_x^b \left|f^{(n)}(s)\right|^2 ds. \end{aligned} \tag{2.4}$$

Integrating both sides of (2.3) with respect to x from a to $\varphi a+(1-\varphi)b$ for $\varphi\in[0,1]$. After that, applying Dirichlet's formula to the double integral in the right-hand side of the resulting inequality, we find that

$$\begin{aligned}
&\int_a^{\varphi a+(1-\varphi)b} |f(x)|^2\,dx \qquad (2.5)\\
&\leq \frac{1}{[(n-1)!]^2}\int_a^{\varphi a+(1-\varphi)b}\frac{(x-a)^{2n-1}}{2n-1}\int_a^x\left|f^{(n)}(s)\right|^2 dsdx\\
&= \frac{1}{[(n-1)!]^2}\int_a^{\varphi a+(1-\varphi)b}\left[\frac{(1-\varphi)^{2n}(b-a)^{2n}-(s-a)^{2n}}{(2n-1)(2n)}\right]\left|f^{(n)}(s)\right|^2 ds.
\end{aligned}$$

If we follow the similar processes by considering the inequality (2.4), then we possess

$$\begin{aligned}
&\int_{\varphi a+(1-\varphi)b}^{b} |f(x)|^2\,dx \qquad (2.6)\\
&\leq \frac{1}{[(n-1)!]^2}\int_{\varphi a+(1-\varphi)b}^{b}\frac{(b-x)^{2n-1}}{2n-1}\int_x^b\left|f^{(n)}(s)\right|^2 dsdx\\
&= \frac{1}{[(n-1)!]^2}\int_{\varphi a+(1-\varphi)b}^{b}\frac{\varphi^{2n}(b-a)^{2n}-(b-s)^{2n}}{(2n-1)(2n)}\left|f^{(n)}(s)\right|^2 ds.
\end{aligned}$$

Applying the change of the variable $s=a\omega+(1-\omega)b$ to the resulting inequality after adding (2.5) and (2.6), we conclude that

$$\begin{aligned}
\int_a^b |f(x)|^2\,dx \leq\ & \frac{(b-a)^{2n+1}}{[(n-1)!]^2(2n-1)(2n)}\\
&\left\{\int_\varphi^1\left[(1-\varphi)^{2n}-(1-\omega)^{2n}\right]\left|f^{(n)}(a\omega+(1-\omega)b)\right|^2 d\omega\right.\\
&\left.+\int_0^\varphi\left[\varphi^{2n}-\omega^{2n}\right]\left|f^{(n)}(a\omega+(1-\omega)b)\right|^2 d\omega\right\}.
\end{aligned}$$

Integrating both sides of the above integral with respect to φ over $[0,1]$ and later changing the order of integration in order to calculate two double integrals in the right-hand side of the resulting inequality, we have

$$\begin{aligned}
&\int_a^b |f(x)|^2\,dx \\
&\leq \frac{(b-a)^{2n+1}}{[(n-1)!]^2\,(2n-1)\,(2n)} \int_0^1 h(\omega) \left| f^{(n)}(a\omega + (1-\omega)\,b) \right|^2 d\omega.
\end{aligned}$$

where

$$h(\omega) = \frac{2}{2n+1} - \frac{(1-\omega)^{2n+1}}{2n+1} - \frac{\omega^{2n+1}}{2n+1} - (1-\omega)^{2n}\,\omega - \omega^{2n}\,(1-\omega)\,.$$

It is easy to see that the maximum value of function $h(\omega)$ for $\omega \in [0,1]$ is $\frac{1}{2n+1}$. Finally, if we apply the change of the variable $t = \omega a + (1-\omega)\,b$, then we attain the desired inequality. The proof is thus completed. $\square$

Remark 2.2 *If we choose $n=1$ in (2.2), then we have*

$$\int_a^b |f(x)|^2\,dx \leq \frac{(b-a)^2}{6} \int_a^b |f'(x)|^2\,dx,$$

which was given by Sarikya in [11].

Now, we deal with a more general result obtained by using Hölder's inequality in the following theorem.

Theorem 2.3 *Let $f \in C^n([a,b])$, $n \in \mathbb{N}\setminus\{0\}$, and $f^{(n)} \in L_r[a,b]$ with $r>1$ such that $f^{(k)}(a) = f^{(k)}(b) = 0$ for $k = 0,1,2,...,n-1$. Then, we have the inequality*

$$\int_a^b |f(x)|^r\,dx \leq \frac{(b-a)^{nr}}{[(n-1)!]^r} \left(\frac{r-1}{nr-1}\right)^{r-1} \frac{1}{(nr)\,(nr+1)} \int_a^b \left| f^{(n)}(x) \right|^r dx. \tag{2.7}$$

Proof Applying Hölder's inequality with the indices r and $\frac{r}{r-1}$ after taking the absolute value of both sides of (2.1), from the hypotheses, then we have the inequalities

$$\begin{aligned}
|f(x)|^r &= \left| \frac{1}{(n-1)!} \int_a^x (x-s)^{n-1} f^{(n)}(s)ds \right|^r \\
&\leq \frac{1}{[(n-1)!]^r} \left(\int_a^x (x-s)^{\frac{nr-r}{r-1}}\,ds \right)^{r-1} \int_a^x \left| f^{(n)}(s) \right|^r ds \\
&= \frac{1}{[(n-1)!]^r} \left(\frac{r-1}{nr-1}\right)^{r-1} (x-a)^{nr-1} \int_a^x \left| f^{(n)}(s) \right|^r ds,
\end{aligned}$$

and

$$
\begin{aligned}
|f(x)|^r &= \left| \frac{1}{(n-1)!} \int_x^b (s-x)^{n-1} f^{(n)}(s) ds \right|^r \\
&\leq \frac{1}{[(n-1)!]^r} \left(\int_x^b (s-x)^{\frac{nr-r}{r-1}} ds \right)^{r-1} \left(\int_x^b \left| f^{(n)}(s) \right|^r ds \right) \\
&= \frac{1}{[(n-1)!]^r} \left(\frac{r-1}{nr-1} \right)^{r-1} (b-x)^{nr-1} \int_x^b \left| f^{(n)}(s) \right|^r ds.
\end{aligned}
$$

Afterwards, if we follow the same strategy which was used in the proof of theorem 2.1 by considering the above inequalities, then we obtain the inequality (2.7) which completes the proof. □

Corollary 2.4 *Under the same assumptions of Theorem 2.3 with* $n = 1$, *the following result holds*

$$
\int_a^b |f(x)|^r dx \leq (b-a)^r \frac{1}{r(r+1)} \int_a^b |f'(x)|^r dx,
$$

which is a new Wirtinger type inequality.

Remark 2.5 *If we choose* $r = 2$ *in (2.7), the inequality (2.7) reduce to the result (2.2).*

Corollary 2.6 *Under the same assumptions of Theorem 2.3 with* $r = 4$, *the following result holds*

$$
\int_a^b |f(x)|^4 dx \leq \frac{(b-a)^{4n}}{[(n-1)!]^4} \left(\frac{3}{4n-1} \right)^3 \frac{1}{(4n)(4n+1)} \int_a^b \left| f^{(n)}(x) \right|^4 dx.
$$

Also, If we take $n = 1$ *in the above inequality, then we have*

$$
\int_a^b |f(x)|^4 dx \leq \frac{(b-a)^4}{20} \int_a^b |f'(x)|^4 dx.
$$

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