

Representations and properties of a new family of ω -Caputo fractional derivatives

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Abstract: In the most general case of ω -weights, some normed functional spaces $X_{\omega}^p(a, b)$ ($1 \leq p \leq \infty$), $AC_{\gamma, \omega}^n[a, b]$ and a generalization of the fractional integro-differentiation operator are introduced and analyzed. The boundedness of the ω -weighted fractional operator over $X_{\omega}^p(a, b)$ is proved. Some theorems and lemmas on the properties of the inversions of the mentioned operator and several representations of functions from $AC_{\gamma, \omega}^n[a, b]$ are established. A general ω -weighted Caputo fractional derivative of order α is studied over $AC_{\gamma, \omega}^n[a, b]$. Some representations and other properties of this fractional derivative are proved. Some conclusions are presented.

Key words: ω -weighted fractional derivatives and integrals, functional spaces, representations, absolutely continuous functions

1. Introduction

Recently, many researchers are dealing with some well-known equations by means of some generalized fractional integro-differential operators, see, e.g., [5, 8, 10, 15]. These kind of extensions imply several expected and unexpected properties of the solutions of the considered equation, we refer to [3, 6, 18]. Besides, in many papers some special classes of functions have been introduced to apply the apparatus of the fractional integro-differential equations in different fields of knowledge: engineering, physics, chemistry, mathematics, etc., see, e.g., [4, 12, 17]. This is the base for studying in this paper more general fractional derivatives. These derivatives will be considered for functions belonging to several spaces $AC_{\gamma, \omega}^n[a, b]$, which are some subsets of the set $AC[a, b]$ of absolutely continuous functions on $[a, b]$, see, e.g., [13, 18]. Notice that that the space $AC[a, b]$ coincides with the space of primitives of Lebesgue summable functions, see, e.g., [18, p. 3], then absolutely continuous functions have a summable derivative $f'(z)$ almost everywhere. The converse is no true. Indeed, this is one of the most important facts of the theory of fractional calculus to define fractional integro-differentiation operators with good enough representations and properties. In general terms, this paper gives some ω -weighted extensions of the results in [18, Chapter 1. Sec. 2] and [12, 13]. Furthermore, we give some other new results that appear naturally while it was proving the ω -extension.

The paper is organized as follows: In Section 2 we give some definitions on the fractional integro-differentiation operators. Section 3 is devoted to the representation of any function $f \in AC_{\gamma, \omega}^n[a, b]$ in

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terms of ω -weighted fractional integrals. Besides, it shows that if $f \in AC_{\gamma,\omega}^n[a, b]$ then $I_{a,\omega}^\alpha f \in AC_{\gamma,\omega}^n[a, b]$. Section 4 gives the original contribution of the paper on representations and properties of some general ω -Caputo fractional derivatives over $AC_{\gamma,\omega}^n[a, b]$. Some theorems on inverse properties of the ω -Caputo fractional derivatives and fractional integrals are established. In Section 5 we give a boundedness result on ω -weighted fractional integral in a weighted space $X_\omega^p(a, b)$, while in Section 6 we finish the paper with some conclusions.

The goal of this paper is to present and study some properties of a Caputo fractional derivative with respect to another function. With this idea, we generalize some previous works dealing with the Caputo fractional derivatives. We study the main properties of this operator.

2. ω -weighted fractional integrals and derivatives

Below we recall some classical definitions on fractional integration and fractional derivative of a function.

Definition 2.1 Let $a, b \in \mathbb{R}$, $a < b$, $\alpha > 0$ and $f \in L^1(a, b)$, then

$$(J_{a+}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a, \tag{2.1}$$

and

$$(J_{b-}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad b > x. \tag{2.2}$$

These integrals are called right-sided and left-sided Riemann–Liouville fractional integrals respectively, see, e.g., [7, 15, 18].

Moreover, other kind of fractional integrals and derivatives of a function f with respect to another function h have been defined and studied in different articles and books, see, e.g., [3, 15, 18]. In this article, we will consider the same form of fractional integro-differentiation operators, but with respect to a weight ω that turns into those in [3, 7, 15, 17, 18] for some particular weights. The consideration of these weights ω will bring out more expected and unknown results. These kind of generalized fractional integrals and derivatives are partially studied in more general settings in [1, 3, 12, 16, 17] and the references therein.

From now on, we assume that Ω is the class of those absolutely continuous functions $\omega(x)$ on (a, b) , such that $\omega'(x) \neq 0$ for any $x \in (a, b)$ and $-\infty < a < b < +\infty$. Besides, \mathbb{N} is the set of natural numbers, \mathbb{C} is the set of complex numbers and $\lceil x \rceil$ is the ceiling function.

Below we introduce the ω -weighted fractional integro-differentiation operators and give some examples of them.

Definition 2.2 If $\omega \in \Omega$, $\alpha \in \mathbb{C}$ ($Re \alpha > 0$) and $f \in L(a, b)$, we define the left-sided (right-sided) fractional integrals of order α and the left-sided (right-sided) fractional derivatives of order α of a function f with respect to a weight ω as:

$$(I_{a,\omega}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\omega(x) - \omega(t))^{\alpha-1} \omega'(t) f(t) dt, \quad x \geq a, \tag{2.3}$$

$$(I_{\omega,b}^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\omega(t) - \omega(x))^{\alpha-1} \omega'(t) f(t) dt, \quad x \leq b, \tag{2.4}$$

and setting $\gamma = \frac{1}{\omega'(x)} \frac{d}{dx}$, $n = \lceil \operatorname{Re} \alpha \rceil$

$$(D_{a,\omega}^\alpha f)(x) = \gamma^n (I_{a,\omega}^{n-\alpha} f)(x) \quad \text{and} \quad (D_{\omega,b}^\alpha f)(x) = (-\gamma)^n (I_{\omega,b}^{n-\alpha} f)(x). \tag{2.5}$$

We assume that $I_{a,\omega}^0 f = I_{\omega,b}^0 f = f$.

Notice that in the above definition we are not taking $\alpha \in \mathbb{C}$ with $\operatorname{Re} \alpha \in \mathbb{N}$, since $n = \lceil \operatorname{Re} \alpha \rceil = \operatorname{Re} \alpha$ and hence $n - \alpha = -i \operatorname{Im} \alpha$, then the fractional integral $(I_{a,\omega}^{n-\alpha} f)(x)$ is not defined due to $\operatorname{Re} \alpha = 0$. While for the case $\alpha \in \mathbb{N}$ in (2.5) we simply get $(D_{a,\omega}^\alpha f)(x) = \gamma^n (I_{a,\omega}^{n-\alpha} f)(x) = \gamma^n (I_{a,\omega}^0 f)(x) = \gamma^n (f)(x)$ and $(D_{\omega,b}^\alpha f)(x) = (-\gamma)^n (I_{\omega,b}^{n-\alpha} f)(x) = (-\gamma)^n (I_{\omega,b}^0 f)(x) = (-\gamma)^n (f)(x)$. Hence, below we always distinguish both cases just setting $\alpha \in \mathbb{N}$ or $n = \lceil \operatorname{Re} \alpha \rceil$.

Semigroup and commutative properties hold for $\operatorname{Re} \alpha, \operatorname{Re} \beta > 0$:

$$I_{a,\omega}^\alpha I_{a,\omega}^\beta f(x) = I_{a,\omega}^{\alpha+\beta} f(x) \quad \text{and} \quad I_{\omega,b}^\alpha I_{\omega,b}^\beta f(x) = I_{\omega,b}^{\alpha+\beta} f(x),$$

$$I_{a,\omega}^\alpha I_{a,\omega}^\beta f(x) = I_{a,\omega}^\beta I_{a,\omega}^\alpha f(x) \quad \text{and} \quad I_{\omega,b}^\alpha I_{\omega,b}^\beta f(x) = I_{\omega,b}^\beta I_{\omega,b}^\alpha f(x).$$

For proving the above properties is just necessary to take into account the change of integration region and Beta's function representation.

Now some examples. First, we recall the Pochhammer k -symbol [9].

Definition 2.3 Let $x \in \mathbb{C}$, $k \in \mathbb{R}$ and $n \in \mathbb{N}^+$, the Pochhammer k -symbol is given by

$$(x)_{n,k} = x(x+k)(x+2k) \dots (x+(n-1)k).$$

Example 2.4 If $\omega \in \Omega$, $\alpha, \beta \in \mathbb{C}$ with $\operatorname{Re} \alpha > 0$, $\operatorname{Re} \beta > 0$ and $n = \lceil \operatorname{Re} \alpha \rceil$, then

$$(D_{a,\omega}^\alpha (\omega(t) - \omega(a))^{\beta-1})(x) = (\beta - \alpha - 1) \frac{\Gamma(\beta)(\beta - \alpha)_{n,1}}{\Gamma(n - \alpha - \beta)} (\omega(x) - \omega(a))^{\beta-\alpha-1}, \tag{2.6}$$

$$(D_{\omega,b}^\alpha (\omega(b) - \omega(t))^{\beta-1})(x) = (\beta - \alpha - 1) \frac{\Gamma(\beta)(\beta - \alpha)_{n,1}}{\Gamma(n - \alpha - \beta)} (\omega(b) - \omega(x))^{\beta-\alpha-1},$$

$$(D_{a,\omega}^\alpha (\omega(t) - \omega(a))^{\alpha-j})(x) = 0, \quad j = 1, \dots, n - 1, \tag{2.7}$$

$$(D_{\omega,b}^\alpha (\omega(b) - \omega(t))^{\alpha-j})(x) = 0, \quad j = 1, \dots, n - 1.$$

Proof By definition 2.2 and the substitution $u = \frac{\omega(t)-\omega(a)}{\omega(x)-\omega(a)}$, we have

$$(D_{a,\omega}^\alpha (\omega(t) - \omega(a))^{\beta-1})(x) = \frac{\gamma^n ((\omega(x) - \omega(a))^{n-\alpha+\beta-1})}{\Gamma(n - \alpha)} \int_0^1 (1 - u)^{n-\alpha-1} u^{\beta-1} du.$$

Thus, formula (2.6) follows straightforward by the calculus of $\gamma^n ((\omega(x) - \omega(a))^{n-\alpha+\beta-1})$. Indeed, $\gamma^n ((\omega(x) - \omega(a))^{n-\alpha+\beta-1}) = (\beta - \alpha - 1)(\beta - \alpha)_{n,1} (\omega(x) - \omega(a))^{\beta-\alpha-1}$. Besides, it is clear that

$$(D_{a,\omega}^\alpha (\omega(t) - \omega(a))^{\alpha-j})(x) = \frac{\gamma^n ((\omega(x) - \omega(a))^{n-j})}{\Gamma(n - \alpha)} \int_0^1 (1 - u)^{n-\alpha-1} u^{\alpha-j} du = 0.$$

The other formulas are proved in the same way.

Notice also that if $\operatorname{Re} \beta > \operatorname{Re} \alpha$ and the property $(x)_{n,1} = \frac{\Gamma(x+n)}{\Gamma(x)}$ [9, Prop. 6] for $\operatorname{Re} x > 0$ we get

$$(D_{a,\omega}^\alpha (\omega(t) - \omega(a))^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha - 1)} (\omega(x) - \omega(a))^{\beta-\alpha-1},$$

$$(D_{\omega,b}^\alpha (\omega(b) - \omega(t))^{\beta-1})(x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha - 1)} (\omega(b) - \omega(x))^{\beta-\alpha-1}.$$

□

Some classical weights can be taken to give illustrative operators. For instance, if we take $\omega(x) = x$, we get for any $f \in L(a, b)$

$$(I_{a,\omega}^\alpha f)(x) = (J_{a^+}^\alpha f)(x) \text{ and } (I_{\omega,b}^\alpha f)(x) = (J_{b^-}^\alpha f)(x).$$

Besides, if $\omega(x) = \frac{x^{\rho+1}}{\rho+1}$ for any $\rho \in \mathbb{R} \setminus \{-1\}$, we obtain

$$(I_{a,\omega}^\alpha f)(x) = (J_{a^+,\rho}^\alpha f)(x) \text{ and } (I_{\omega,b}^\alpha f)(x) = (J_{b^-,\rho}^\alpha f)(x).$$

Also,

$$(I_{0,\omega}^\alpha \omega)(x) = \frac{(\omega(x) - \omega(0))^\alpha}{\Gamma(\alpha + 2)} [\omega(x) + \alpha \omega(0)].$$

Furthermore setting $f(x) = x^\mu$, $g(x) = 1$ and $\omega(x) = \frac{x^{\rho+1}}{\rho+1}$ for $x > 0$, $\alpha > 0$, $\rho \geq 0$ and $\mu > -1$ we get

$$(I_{0,\omega}^\alpha x^\mu)(t) = \frac{(\rho+1)^{-\alpha} \Gamma(\frac{\rho+\mu+1}{\rho+1})}{\Gamma(\alpha + \frac{\rho+\mu+1}{\rho+1})} t^{\alpha(\rho+1)+\mu} \text{ and } (I_{0,\omega}^\alpha 1)(t) = \frac{(\rho+1)^{-\alpha}}{\Gamma(\alpha+1)} t^{\alpha(\rho+1)}.$$

3. Preliminary results

We begin this section introducing the spaces of functions that will be considered through this paper.

Definition 3.1 We define

$$AC_{\gamma,\omega}^n[a, b] := \left\{ f : [a, b] \rightarrow \mathbb{C} \mid \gamma^{n-1} f \in AC[a, b], \gamma = \frac{1}{\omega'(x)} \frac{d}{dx} \right\},$$

where $\omega \in \Omega$ and $n \in \mathbb{N}$.

It is easy to see that under some suitable weights ω , the above space coincide with those define and treat in [12, 13, 15, 18].

Below we give a characterization of any function over the space $AC_{\gamma,\omega}^n[a, b]$.

Theorem 3.2 *If $\omega \in \Omega$, then any function $f \in AC_{\gamma,\omega}^n[a, b]$ if and only if f can be represented as*

$$\begin{aligned} f(x) &= \frac{1}{(n-1)!} \int_a^x (\omega(x) - \omega(t))^{n-1} (\gamma^n f)(t) \omega'(t) dt \\ &\quad + \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{(n-k-1)!} (\omega(x) - \omega(a))^{n-k-1}, \\ &= I_{a,\omega}^n (\gamma^n f)(x) + \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{(n-k-1)!} (\omega(x) - \omega(a))^{n-k-1}, \end{aligned}$$

where $(\gamma^n f)(x)\omega'(x) \in L^1(a, b)$.

Proof Let $f \in AC_{\gamma,\omega}^n[a, b]$. Then $\gamma^{n-1} f \in AC[a, b]$; hence, the derivative of $\gamma^{n-1} f(x)$ exist almost everywhere on (a, b) ; thus, for any $x \in (a, b)$

$$\begin{aligned} \left(\frac{1}{\omega'(x)} \frac{d}{dx}\right)^{n-1} f(x) &= \int_a^x \frac{d}{dt} \left(\left(\frac{1}{\omega'(t)} \frac{d}{dt}\right)^{n-1} f(t) \right) dt + (\gamma^{n-1} f)(a) \\ &= \int_a^x \omega'(t) \left(\left(\frac{1}{\omega'(t)} \frac{d}{dt}\right)^n f(t) \right) dt + (\gamma^{n-1} f)(a) \\ &= \int_a^x (\gamma^n f)(t) \omega'(t) dt + A_0, \end{aligned} \tag{3.1}$$

where $A_0 = (\gamma^{n-1} f)(a)$. Multiplying both sides of (3.1) by $\omega'(x)$ and integrating over (a, x) , we obtain

$$\begin{aligned} \left(\frac{1}{\omega'(x)} \frac{d}{dx}\right)^{n-2} f(x) &= \int_a^x \omega'(t) \left(\int_a^t (\gamma^n f)(y) \omega'(y) dy + A_0 \right) dt \\ &= \int_a^x (\gamma^n f)(y) \omega'(y) \left(\int_y^x \omega'(t) dt \right) dy + A_0(\omega(x) - \omega(a)) + A_1 \\ &= \int_a^x (\omega(x) - \omega(y)) (\gamma^n f)(y) \omega'(y) dy + A_0(\omega(x) - \omega(a)) + A_1, \end{aligned}$$

where $A_1 = (\gamma^{n-2} f)(a)$. Again, multiplying both sides of the above equation by $\omega'(x)$ and integrating over

(a, x) , we get

$$\begin{aligned} & \left(\frac{1}{\omega'(x)} \frac{d}{dx}\right)^{n-3} f(x) = \\ &= \int_a^x \omega'(t) \left(\int_a^t (\omega(t) - \omega(y))(\gamma^n f)(y)\omega'(y) dy\right) dt \\ & \quad + A_0 \int_a^x (\omega(t) - \omega(a))\omega'(t) dt + A_1 \int_a^x \omega'(t) dt + A_2 \\ &= \int_a^x (\gamma^n f)(y)\omega'(y) \left(\int_y^x (\omega(t) - \omega(y))\omega'(t) dt\right) dy \\ & \quad + A_0 \frac{(\omega(x) - \omega(a))^2}{2} + A_1(\omega(x) - \omega(a)) + A_2 \\ &= \int_a^x \frac{(\omega(x) - \omega(y))^2}{2} (\gamma^n f)(y)\omega'(y) dy + A_0 \frac{(\omega(x) - \omega(a))^2}{2} \\ & \quad + A_1(\omega(x) - \omega(a)) + A_2, \end{aligned}$$

where $A_2 = (\gamma^{n-3} f)(a)$. If the same process is repeated $n - 3$ times, we arrive at

$$f(x) = \int_a^x \frac{(\omega(x) - \omega(y))^{n-1}}{(n-1)!} (\gamma^n f)(y)\omega'(y) dy + \sum_{k=0}^{n-1} A_k \frac{(\omega(x) - \omega(a))^{n-k-1}}{(n-k-1)!},$$

where $A_k = (\gamma^{n-k-1} f)(a)$. Another implication follows by the application of γ^{n-1} over the representation. It is just to remember the Leibniz's rule for differentiation under the integral sign and $\gamma^{n-1}(\omega(x) - \omega(a))^{n-k-1} = 0$ for any $k = 1, \dots, n - 1$. □

Remark 3.3 Under the same conditions of Theorem 3.2, it can be proved similarly that $f \in AC_{\gamma,\omega}^n[a, b]$ if and only if f is of the form

$$\begin{aligned} f(x) &= \frac{1}{(n-1)!} \int_x^b (\omega(t) - \omega(x))^{n-1} ((-\gamma)^n f)(t)\omega'(t) dt \\ & \quad + \sum_{k=0}^{n-1} \frac{((-\gamma)^{n-k-1} f)(b)}{(n-k-1)!} (\omega(b) - \omega(x))^{n-k-1}, \\ &= I_{\omega,b}^n ((-\gamma^n) f)(x) + \sum_{k=0}^{n-1} \frac{((-\gamma)^{n-k-1} f)(b)}{(n-k-1)!} (\omega(b) - \omega(x))^{n-k-1}, \end{aligned}$$

where $((-\gamma^n) f)(x)\omega'(x) \in L^1(a, b)$.

Theorem 3.4 If $\omega \in \Omega$, $\alpha \in \mathbb{C}$ with $Re \alpha > 0$, $f \in AC_{\gamma,\omega}^n[a, b]$ and $n = \lceil Re \alpha \rceil$, then $I_{a,\omega}^\alpha f \in AC_{\gamma,\omega}^n[a, b]$.

Proof By the representation of Theorem 3.2 we can write $I_{a,\omega}^\alpha f$ as follows

$$I_{a,\omega}^\alpha f(x) = I_{a,\omega}^\alpha (I_{a,\omega}^n (\gamma^n f))(x) + \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{(n-k-1)!} I_{a,\omega}^\alpha ((\omega(t) - \omega(a))^{n-k-1})(x).$$

By the semigroup properties of $I_{a,\omega}^\alpha$ and the calculation of the second fractional integral above by means of the substitution $u = \frac{\omega(t)-\omega(a)}{\omega(x)-\omega(a)}$ and taking into account that $1 - u = \frac{\omega(x)-\omega(t)}{\omega(x)-\omega(a)}$ and the properties of the Beta's function we obtain

$$I_{a,\omega}^\alpha f(x) = I_{a,\omega}^{\alpha+n}(\gamma^n f)(x) + \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{\Gamma(\alpha + n - k)} (\omega(x) - \omega(a))^{\alpha+n-k-1}. \tag{3.2}$$

Under the last representation it remains to prove that $\gamma^{n-1}(I_{a,\omega}^\alpha f)(x) \in AC[a, b]$. We now calculate the first term of (3.2). Indeed, by the Leibniz's rule for differentiation under the integral sign we have

$$\begin{aligned} \gamma^{n-1}(I_{a,\omega}^{\alpha+n}(\gamma^n f))(x) &= \gamma^{n-1} \left(\frac{1}{\Gamma(\alpha + n)} \int_a^x (\omega(x) - \omega(t))^{\alpha+n-1} \omega'(t)(\gamma^n f)(t) dt \right) \\ &= \gamma^{n-2} \left(\frac{1}{\Gamma(\alpha + n - 1)} \int_a^x (\omega(x) - \omega(t))^{\alpha+n-2} \omega'(t)(\gamma^n f)(t) dt \right) \\ &= \gamma^{n-3} \left(\frac{1}{\Gamma(\alpha + n - 2)} \int_a^x (\omega(x) - \omega(t))^{\alpha+n-3} \omega'(t)(\gamma^n f)(t) dt \right) \\ &\quad \vdots \\ &= \frac{1}{\Gamma(\alpha + 1)} \int_a^x (\omega(x) - \omega(t))^\alpha \omega'(t)(\gamma^n f)(t) dt. \end{aligned}$$

Notice that

$$\begin{aligned} &\left| \frac{1}{\Gamma(\alpha + 1)} \int_a^x (\omega(x) - \omega(t))^\alpha \omega'(t)(\gamma^n f)(t) dt \right| \\ &\leq \frac{1}{|\Gamma(\alpha + 1)|} \max_{a \leq x \leq b; a \leq s \leq b} |\omega(x) - \omega(s)|^{\operatorname{Re} \alpha} \int_a^x \frac{d}{dt}(\gamma^{n-1} f)(t) < \infty, \end{aligned}$$

since $f \in AC_{\gamma,\omega}^n[a, b]$ (i.e. $(\gamma^{n-1} f) \in AC[a, b]$). This implies that the first term of (3.2) belongs to $AC[a, b]$.

On the other hand, for the second term of (3.2) we just need to estimate $\gamma^{n-1}((\omega(x) - \omega(a))^{\alpha+n-k-1})$ for $k = 0, \dots, n - 1$ since the other terms are constants. Indeed,

$$\gamma^{n-1}((\omega(x) - \omega(a))^{\alpha+n-k-1}) = (\alpha - k)_{n,1} (\omega(x) - \omega(a))^{\alpha-k}$$

and $(\omega(x) - \omega(a))^{\alpha-k} \in AC[a, b]$ for any $k = 0, \dots, n - 1$ since $k \leq n - 1 \leq \operatorname{Re} \alpha$. □

4. Main results

We begin this section showing the representations of the fractional derivative $D_{a,\omega}^\alpha f(x)$ and $D_{\omega,b}^\alpha f(x)$ for any function in the space $AC_{\gamma,\omega}^n[a, b]$.

Theorem 4.1 *If $\omega \in \Omega$, $f \in AC_{\gamma,\omega}^n[a, b]$, $\alpha \in \mathbb{C}$ with $Re\alpha > 0$ and $n = \lceil Re\alpha \rceil$, then*

$$D_{a,\omega}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (\omega(x) - \omega(y))^{n-\alpha-1} \omega'(y) (\gamma^n f)(y) dy + \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a) (2n - \alpha - k - 1)_{n,-1}}{\Gamma(2n - k - \alpha)} (\omega(x) - \omega(a))^{n-k-\alpha-1}, \tag{4.1}$$

$$D_{\omega,b}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b (\omega(y) - \omega(x))^{n-\alpha-1} \omega'(y) (\gamma^n f)(y) dy + \sum_{k=0}^{n-1} \frac{((-\gamma)^{n-k-1} f)(b) (2n - \alpha - k - 1)_{n,-1}}{\Gamma(2n - k - \alpha)} (\omega(b) - \omega(x))^{n-k-\alpha-1}, \tag{4.2}$$

exist almost everywhere on $[a, b]$.

Proof Let $f \in AC_{\gamma,\omega}^n[a, b]$. By (2.5) and Theorem 3.2 we obtain

$$D_{a,\omega}^\alpha f(x) = \gamma^n \left(\int_a^x \int_a^t \frac{(\omega(x) - \omega(t))^{n-\alpha-1} (\omega(t) - \omega(y))^{n-1} (\gamma^n f)(y) \omega'(t) \omega'(y)}{(n-1)! \Gamma(n-\alpha)} dy dt \right) + \gamma^n \left(\sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{\Gamma(n-\alpha) (n-k-1)!} \int_a^x (\omega(x) - \omega(t))^{n-\alpha-1} (\omega(t) - \omega(a))^{n-k-1} \omega'(t) dt \right).$$

To calculate the first integral, we have to change the integration region and use the substitution $u = \frac{\omega(t)-\omega(y)}{\omega(x)-\omega(y)}$, after that we just need to use the property of Beta's function $B(r, s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)}$ for $Re r, Re s > 0$; while the second one follows by the substitution $s = \frac{\omega(t)-\omega(a)}{\omega(x)-\omega(a)}$ and we use the property of Beta's function. Thus,

$$D_{a,\omega}^\alpha f(x) = \frac{1}{\Gamma(2n-\alpha)} \gamma^n \left(\int_a^x (\omega(x) - \omega(y))^{2n-\alpha-1} \omega'(y) (\gamma^n f)(y) dy \right) + \gamma^n \left(\sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{\Gamma(2n-k-\alpha)} (\omega(x) - \omega(a))^{2n-k-\alpha-1} \right).$$

By applying γ^n to the above integral (Leibniz's rule for differentiation under the integral sign) and by the straightforward estimation $\gamma^n ((\omega(x) - \omega(a))^{2n-\alpha-k-1}) = (2n - \alpha - k - 1)_{n,-1} (\omega(x) - \omega(a))^{n-\alpha-k-1}$, we get the desired formula (4.1). In the same manner it is proved (4.2). \square

Now, we present a new family of ω -Caputo fractional derivatives of any order α , with $\alpha \in \mathbb{C}$ such that $Re\alpha > 0$.

Definition 4.2 *If $\omega \in \Omega$, $\alpha \in \mathbb{C}$ with $Re\alpha > 0$, $n = \lceil Re\alpha \rceil$, $f \in AC_{\gamma,\omega}^n[a, b]$ and $-\infty < a < b < +\infty$. Then we defined the left and right ω -Caputo fractional derivatives of f of order α as*

$$D_{a,\omega}^{\alpha,C} f(x) = D_{a,\omega}^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{(n-k-1)!} (\omega(t) - \omega(a))^{n-k-1} \right) (x), \tag{4.3}$$

and

$$D_{\omega,b}^{\alpha,C} f(x) = D_{b,\omega}^{\alpha} \left(f(t) - \sum_{k=0}^{n-1} \frac{((- \gamma)^{n-k-1} f)(b)}{(n-k-1)!} (\omega(b) - \omega(t))^{n-k-1} \right) (x).$$

It should be clear that for $\alpha \in \mathbb{N}$, $D_{a,\omega}^{\alpha,C} f(x) = (\gamma^\alpha f)(x)$ and $D_{\omega,b}^{\alpha,C} f(x) = ((-\gamma)^\alpha f)(x)$ since $\gamma^\alpha (\omega(t) - \omega(a))^{\alpha-k-1} = (-\gamma)^\alpha (\omega(b) - \omega(t))^{\alpha-k-1} = 0$ for any $k = 0, 1, \dots, \alpha - 1$, see definition 2.2 or formula (2.5). Notice that in many particular cases, taking suitable weights ω , the above derivatives generalize and extend those derivatives introduced in [12, 13, 15, 17, 18] and some others.

Below, we give a characterization of any ω -Caputo fractional derivative in terms of the fractional integrals $I_{a,\omega}^\alpha f$ or $I_{b,\omega}^\alpha f$.

Theorem 4.3 *If $\omega \in \Omega$, $\alpha \in \mathbb{C}$ with $Re \alpha > 0$ and $f \in AC_{\gamma,\omega}^n[a, b]$, then:*

1. *If $n = [Re \alpha]$,*

$$D_{a,\omega}^{\alpha,C} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (\omega(x) - \omega(y))^{n-\alpha-1} \omega'(y) (\gamma^n f)(y) dy, \tag{4.4}$$

$$D_{\omega,b}^{\alpha,C} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b (\omega(y) - \omega(x))^{n-\alpha-1} \omega'(y) ((-\gamma)^n f)(y) dy. \tag{4.5}$$

2. *If $\alpha \in \mathbb{N}$*

$$D_{a,\omega}^{\alpha,C} f(x) = (\gamma^\alpha f)(x) \quad \text{and} \quad D_{\omega,b}^{\alpha,C} f(x) = ((-\gamma)^\alpha f)(x). \tag{4.6}$$

Proof The representations (4.4) and (4.5) follow by Theorems 3.2 and 4.1. For the case $\alpha \in \mathbb{N}$, formulas in (4.6) are obvious. □

Remark 4.4 *Notice that representations (4.4), (4.5), and (4.6) can be written as*

$$D_{a,\omega}^{\alpha,C} f(x) = I_{a,\omega}^{n-\alpha} (\gamma^n f)(x) \quad \text{and} \quad D_{\omega,b}^{\alpha,C} f(x) = I_{\omega,b}^{n-\alpha} ((-\gamma)^n f)(x),$$

while for $\alpha \in \mathbb{N}$,

$$D_{a,\omega}^{\alpha,C} f(x) = I_{a,\omega}^0 (\gamma^n f)(x) = (\gamma^n f)(x) \quad \text{and} \quad D_{\omega,b}^{\alpha,C} f(x) = ((-\gamma)^n f)(x).$$

Now we show some classical examples of these derivatives.

Lemma 4.5 *If $\omega \in \Omega$, $\alpha, \beta \in \mathbb{C}$ with $Re \alpha > 0$, $n = [Re \alpha]$ and $Re \beta > n$, then*

$$(D_{a,\omega}^{\alpha,C} (\omega(t) - \omega(a))^{\beta-1})(x) = (\beta - 1)_{n,-1} \frac{\Gamma(\beta - n)}{\Gamma(\beta - \alpha)} (\omega(x) - \omega(a))^{\beta-\alpha-1}, \tag{4.7}$$

$$(D_{\omega,b}^{\alpha,C} (\omega(b) - \omega(t))^{\beta-1})(x) = (\beta - 1)_{n,-1} \frac{\Gamma(\beta - n)}{\Gamma(\beta - \alpha)} (\omega(b) - \omega(x))^{\beta-\alpha-1},$$

$$(D_{a,\omega}^{\alpha,C} (\omega(t) - \omega(a))^j)(x) = 0, \quad j = 1, \dots, n - 1, \tag{4.8}$$

$$(D_{\omega,b}^{\alpha,C} (\omega(b) - \omega(t))^j)(x) = 0, \quad j = 1, \dots, n - 1.$$

Proof By Theorem 4.3 and the substitution $u = \frac{\omega(t)-\omega(a)}{\omega(x)-\omega(a)}$, it follows (4.7). Moreover, as $\gamma^n((\omega(x)-\omega(a))^j) = 0$ for any $j = 0, 1, \dots, n-1$, and the equality (4.8) becomes obvious. Similarly, the other formulas can be proved. \square

In the Figure, we plot (4.7) for the various values of the parameters. Some theorems on inverse properties of the ω -Caputo fractional derivatives and fractional integrals.

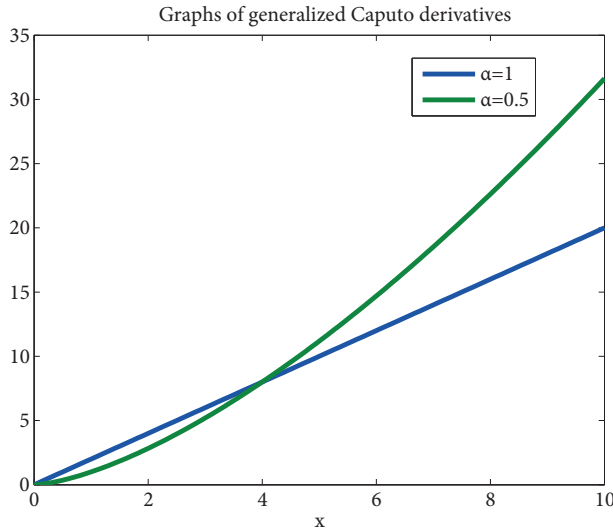


Figure . Graph of $D_{0^+,\omega}^{\alpha,C} f(x)$, for the kernels $\omega(x) = x$.

Theorem 4.6 If $\omega \in \Omega$, $\alpha \in \mathbb{C}$ with $Re \alpha > 0$, $n = \lceil Re \alpha \rceil$ and $f \in AC_{\gamma,\omega}^n[a, b]$. Then

$$D_{a,\omega}^{\alpha,C} I_{a,\omega}^\alpha f(x) = f(x) \quad \text{and} \quad D_{\omega,b}^{\alpha,C} I_{\omega,b}^\alpha f(x) = f(x).$$

Proof By Theorem 3.4 we have that $I_{a,\omega}^\alpha f \in AC_\gamma^n[a, b]$ since $f \in AC_\gamma^n[a, b]$. Hence, by Theorem 4.3, Remark 4.4, Leibniz’s rule for differentiation under the integral sign and the semigroup properties of $I_{a,\omega}^\alpha f$ we obtain

$$D_{a,\omega}^{\alpha,C} (I_{a,\omega}^\alpha f)(x) = I_{a,\omega}^{n-\alpha} (\gamma^n (I_{a,\omega}^\alpha f))(x) = I_{a,\omega}^{n-\alpha} (I_{a,\omega}^{\alpha-n} f)(x) = I_{a,\omega}^0 f(x) = f(x).$$

The other formula is proved in the same way. \square

Theorem 4.7 If $\omega \in \Omega$, $\alpha \in \mathbb{C}$ with $Re \alpha > 0$, $n = \lceil Re \alpha \rceil$ and $f \in AC_{\gamma,h}^n[a, b]$, then

$$I_{a,\omega}^\alpha (D_{a,\omega}^{\alpha,C} f)(x) = I_{a,\omega}^n (\gamma^n f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{(n-k-1)!} (\omega(x) - \omega(a))^{n-k-1},$$

and

$$I_{\omega,b}^\alpha (D_{\omega,b}^{\alpha,C} f)(x) = I_{\omega,b}^n ((-\gamma^n) f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{((-\gamma)^{n-k-1} f)(b)}{(n-k-1)!} (\omega(b) - \omega(x))^{n-k-1}.$$

Proof By Theorem 4.3, Remark 4.4, and the semigroup properties of $I_{a,\omega}^\alpha f$ we can arrive at

$$I_{a,\omega}^\alpha (D_{a,\omega}^{\alpha,C} f)(x) = I_{a,\omega}^\alpha (I_{a,\omega}^{n-\alpha}(\gamma^n f))(x) = I_{a,\omega}^n (\gamma^n f)(x).$$

To complete the proof we just have to recall that in virtue of Theorem 3.2 we have

$$f(x) = I_{a,\omega}^n (\gamma^n f)(x) + \sum_{k=0}^{n-1} \frac{(\gamma^{n-k-1} f)(a)}{(n-k-1)!} (\omega(x) - \omega(a))^{n-k-1}.$$

Analogously, the other formula can be proved. □

5. Further results

In this section we consider a ω -weighted space of $L^p(a, b)$, which will be denoted by $X_\omega^p(a, b)$ and we are going to prove boundedness of $(I_{a,\omega}^\alpha f)(x)$ as an operator from $X_\omega^p(a, b)$ to $X_\omega^p(a, b)$.

Definition 5.1 Let $1 \leq p < \infty$ and $\omega(x) \in \Omega$. We define the space $X_\omega^p(a, b)$ of those real-valued Lebesgue measurable functions f on (a, b) for which

$$\|f\|_{X_\omega^p} = \left(\int_a^b |f(t)|^p |\omega'(t)| dt \right)^{\frac{1}{p}} < \infty.$$

It is clear that this is a norm. Notice now that when $\omega(x) = x$ the space $X_\omega^p(0, \infty)$ coincides with the well-known $L_p(0, \infty)$ -space. Besides, if we take $\omega(x) = \frac{x^{k+1}}{k+1}$ ($1 \leq p < \infty, k \geq 0$) the space $X_\omega^p(0, \infty)$ becomes $L_{p,k}(0, \infty)$ -space. Moreover, if $\omega(x) = \frac{x^{cp}}{cp}$ for $c \in \mathbb{R}$ and $1 \leq p < \infty$, we get that space $X_\omega^p(a, b)$ coincides with the space $X_c^p(a, b)$ introduced in [14]. Moreover, under some suitable weights the considered space $X_\omega^p(a, b)$ turns into those spaces introduced and studied in [2, 13].

Now we prove the boundedness of the operator $I_{a,\omega}^\alpha$.

Theorem 5.2 Let $\alpha \in \mathbb{C}$ with $Re \alpha > 0, 1 \leq p < +\infty, -\infty < a < b < +\infty$. Then the operator $I_{a,\omega}^\alpha$ defined from $X_\omega^p(a, b)$ to $X_\omega^p(a, b)$, is bounded and

$$\frac{\|I_{a,\omega}^\alpha f\|_{X_\omega^p}}{\|f\|_{X_\omega^p}} \leq M < \infty,$$

where

$$M = \frac{\left((b-a)^{p-1} \max_{a \leq s \leq b} \left\{ |\omega'(s)|^{p-1} |\omega(b) - \omega(s)|^{(Re \alpha - 1)p + 1} \right\} \right)^{1/p}}{|\Gamma(\alpha)| ((Re \alpha - 1)p + 1)^{1/p}}.$$

Proof By definition 5.1, Jensen’s inequality and changing of integration region we get

$$\begin{aligned}
 & \|I_{a,\omega}^\alpha f\|_{X_\omega^p}^p \\
 & \leq \frac{1}{|\Gamma(\alpha)|^p} \int_a^b (t-a)^p \left(\frac{1}{t-a} \int_a^t |\omega(t) - \omega(\tau)|^{\operatorname{Re} \alpha - 1} |\omega'(\tau) f(\tau)| d\tau \right)^p |\omega'(t)| dt \\
 & \leq \frac{(b-a)^{p-1}}{|\Gamma(\alpha)|^p} \int_a^b \left(\int_a^t |\omega(t) - \omega(\tau)|^{(\operatorname{Re} \alpha - 1)p} |\omega'(\tau) f(\tau)|^p d\tau \right) |\omega'(t)| dt \\
 & \leq \frac{(b-a)^{p-1}}{|\Gamma(\alpha)|^p} \int_a^b |f(\tau) \omega'(\tau)|^p \left(\int_\tau^b |\omega(t) - \omega(\tau)|^{(\operatorname{Re} \alpha - 1)p} |\omega'(t)| dt \right) d\tau \\
 & \leq \frac{(b-a)^{p-1} \max_{a \leq s \leq b} \left\{ |\omega'(s)|^{p-1} |\omega(b) - \omega(s)|^{(\operatorname{Re} \alpha - 1)p + 1} \right\}}{|\Gamma(\alpha)|^p ((\operatorname{Re} \alpha - 1)p + 1)} \int_a^b |f(\tau)|^p |\omega'(\tau)| d\tau.
 \end{aligned}$$

Thus, the proof is completed. □

The following result on an inverse property involving the fractional integral operator $I_{a,\omega}^\alpha$ and the fractional derivative $D_{a,\omega}^\alpha$, over the space $X_\omega^p(a, b)$.

Theorem 5.3 *If $\omega \in \Omega$ and $0 < \operatorname{Re} \beta \leq \operatorname{Re} \alpha$. Then the following formulas hold:*

$$D_{a,\omega}^\beta I_{a,\omega}^\alpha f = I_{a,\omega}^{\alpha-\beta} f \quad \text{and} \quad D_{\omega,b}^\beta I_{\omega,b}^\alpha f = I_{\omega,b}^{\alpha-\beta} f, \tag{5.1}$$

$$D_{a,\omega}^\alpha I_{a,\omega}^\alpha f = f \quad \text{and} \quad D_{\omega,b}^\alpha I_{\omega,b}^\alpha f = f, \tag{5.2}$$

for any $f \in X_\omega^p(a, b)$.

Proof Below, we prove the first formula of (5.1) and the second one we leave to the reader. Hence, we sketch the proof into two cases. For $\beta = m$ a positive integer ($\alpha \geq m$), it follows by (2.5) and Leibniz integral rule that

$$\begin{aligned}
 D_{a,\omega}^\beta I_{a,\omega}^\alpha f(x) &= \left(\frac{1}{\omega'(x)} \frac{d}{dx} \right)^m \left(\frac{1}{\Gamma(\alpha)} \int_a^x (\omega(x) - \omega(t))^{\alpha-1} \omega'(t) f(t) dt \right) \\
 &= \gamma^m \left(\frac{1}{\Gamma(\alpha)} \int_a^x (\omega(x) - \omega(t))^{\alpha-1} \omega'(t) f(t) dt \right) \\
 &= \gamma^{m-1} \left(\frac{1}{\Gamma(\alpha-1)} \int_a^x (\omega(x) - \omega(t))^{\alpha-2} \omega'(t) f(t) dt \right) \\
 &= \gamma^{m-2} \left(\frac{1}{\Gamma(\alpha-2)} \int_a^x (\omega(x) - \omega(t))^{\alpha-3} \omega'(t) f(t) dt \right) \\
 &\quad \vdots \\
 &= \frac{1}{\Gamma(\alpha-m)} \int_a^x (\omega(x) - \omega(t))^{\alpha-m-1} \omega'(t) f(t) dt \\
 &= I_{a,\omega}^{\alpha-\beta} f(x).
 \end{aligned}$$

For the case $m = \lceil \beta \rceil$, we get by (2.5), the semigroup properties of $I_{a,\omega}^\alpha$ and the first case that

$$D_{a,\omega}^\beta (I_{a,\omega}^\alpha f)(x) = \gamma^m (I_{a,\omega}^{m-\beta} (I_{a,\omega}^\alpha f))(x) = \gamma^m (I_{a,\omega}^{\alpha+m-\beta} f)(x) = (I_{a,\omega}^{\alpha-\beta} f)(x).$$

Notice that formulas in (5.2) follow by (5.1) and the fact that $I_{a,\omega}^0 f = I_{\omega,b}^0 f = f$. \square

6. Conclusions

Most of the results of this article have been many special results of [11–13] in some particular cases. On the other hand, the consideration of some fractional derivative operators over good enough spaces of functions leads to some representations and properties, as that can be seen in the referred works. This paper gives some ideas for the future consideration of some initial value problems for nonlinear fractional differential equations involving the ω -Caputo fractional derivatives, i.e. the problem of the existence and uniqueness of the solutions of these equations, studied in terms of some standard fixed point theorems. For instance, the following Cauchy type initial value problem is natural to study under some initial conditions:

$$(D_{a,\omega}^{\alpha,C} g)(x) = f(x, g(x)).$$

It will be natural to consider the Picard iteration method for numerically solving the latter problem, as one can see in [3–5, 10, 15].

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Contribution of authors

All the authors contributed equally to the work.

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