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Existence of positive solutions for nonlinear multipoint p-Laplacian dynamic equations on time scales

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Abstract: In this paper, we investigate the existence of positive solutions for nonlinear multipoint boundary value problems for p-Laplacian dynamic equations on time scales with the delta derivative of the nonlinear term. Sufficient assumptions are obtained for existence of at least twin or arbitrary even positive solutions to some boundary value problems. Our results are achieved by appealing to the fixed point theorems of Avery-Henderson. As an application, an example to demonstrate our results is given.

Key words: Time scales, boundary value problem, p-Laplacian, positive solutions, Fixed point theorem

1. Introduction

The theory of dynamic equation on time scales was pioneered by Stefan Hilger in his Ph.D. thesis in 1988 [12] as a process of combining construction for the research of differential equations in the continuous situation and research of finite difference equations in the discontinuous situation. In recent years, it has found a considerable amount of attraction and captivated the concentration of numerous researchers. It is still a fresh field, and investigation in this field is speedily flourishing. The research of time scales has led to various crucial practices, e.g., in the research of insect population samples, heat transfer, neural systems, phytoremediation of metals, injury treating, and prevalent samples [3, 13, 21, 22]. The familiar symbols and phraseology for time scales can be found in [2, 3, 9].

In [6], Dogan investigated the following p-Laplacian multipoint boundary value problem (BVP) on time scales

$$(\varphi_p(u^\Delta(t)))^\nabla + a(t)f(t,u(t)) = 0, \qquad t \in (0,T)_{\mathbf{T}},$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \varphi_p(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \varphi_p(u^\Delta(\xi_i)).$$

We obtained the existence of at least three positive solutions by using a Krasnosel'skii's fixed point theorem.

In [19], Su and Li studied the following multipoint BVPs on time scales

$$(\varphi_p(u^\Delta(t)))^\nabla + a(t)f(u(t)) = 0, \qquad t \in [0,T]_{\mathbf{T}},$$

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subject to boundary conditions (BCs)

$$u(0) - B_0\left(\sum_{i=1}^{m-2} a_i u^{\Delta}(\xi_i)\right) = 0, \qquad u^{\Delta}(T) = 0$$

or

$$u^{\Delta}(0) = 0, \qquad u(T) + B_1\left(\sum_{i=1}^{m-2} b_i u^{\Delta}(\xi_i)\right) = 0.$$

By using the five functionals fixed-point theorem, they showed that the BVP has at least three positive solutions.

In [24], Zhu and Zhu studied the following p-Laplacian multipoint BVP on time scales

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t)) = 0, \qquad t \in (0, T)_{\mathbf{T}},$$

$$\varphi_p(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \varphi_p(u^{\Delta}(\xi_i)), \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i).$$

They obtained some new results for the existence of at least two positive solutions by using fixed point index.

Recently, there is an increasing attention paid to question of positive solution for multipoint BVPs on time scales [5, 6, 8, 11, 15–20, 23, 24]. However, little work has been done on the existence of positive solutions for p-Laplacian multipoint BVPs on time scales with the first-order derivative of nonlinear term [4, 7, 14].

In this paper, we study the following p-Laplacian multipoint BVPs on time scales

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t, u(t), u^{\Delta}(t)) = 0, \qquad t \in [0, T]_{\mathbf{T}}, \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \varphi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \varphi_p(u^{\Delta}(\xi_i)) \tag{1.2}$$

or

$$\varphi_p(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \varphi_p(u^{\Delta}(\xi_i^{\star})) \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i^{\star}), \tag{1.3}$$

where $\varphi_p(s) = |s|^{p-1}s,\ p > 1, (\varphi_p)^{-1} = \varphi_q, 1/p + 1/q = 1,\ \xi_i, \xi_i^{\star} \in [0, T]_{\mathbf{T}}$, and satisfy $0 \le \xi_1 < \xi_2 < \ldots < \xi_{m-2} < \rho(T),\ \sigma(0) < \xi_1^{\star} < \xi_2^{\star} < \ldots < \xi_{m-2}^{\star} \le T,\ a_i, b_i \in [0, \infty),\ \ 0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$.

The main assumptions in this paper are as follows.

(H1) $f : [0, T]_{\mathbf{T}} \times \mathbf{R}^+ \times \mathbf{R} \to \mathbf{R}^+$ is ld-continuous, and does not disappear similarly on any closed subinterval of $[0, T]_{\mathbf{T}}$.

(H2) $a : \mathbf{T} \to \mathbf{R}^+$ is left dense continuous (i.e., $a \in C_{ld}(\mathbf{T}, \mathbf{R}^+)$) and does not disappear similarly on any closed subinterval of $[0, T]_{\mathbf{T}}$. Here $C_{ld}(\mathbf{T}, \mathbf{R}^+)$ indicates the set of all left dense continuous functions from $\mathbf{T}$ to $\mathbf{R}^+$.

(H3) For the BVP (1.1) and (1.2), let us assume that if $\xi_{m-2} > 0$, then let $0 < \nu = \xi_{m-2}$; if $\xi_{m-2} = 0$, then let $\nu = \min\{t \in \mathbf{T} : t \ge \frac{T}{2}\}$, and there exists $q \in \mathbf{T}$ such that $\nu < q < T$ is satisfied. For the BVP (1.1) and (1.3), let us assume that if $\xi_1^{\star} < T$, then let $\xi = \xi_1^{\star}$; if $\xi_1^{\star} = T$, then let $\xi = \max\{t \in \mathbf{T} : 0 < t \le \frac{T}{2}\}$, and there exists $l \in \mathbf{T}$ such that $0 < l < \xi < T$ is satisfied.

Inspired by the conclusions communicated earlier, in this paper, we prove the existence of at least twin positive solutions to the BVPs (1.1), (1.2) and (1.1), (1.3). To the best of our comprehension, there appear to be no such results for the existence of positive solutions to BVPs (1.1), (1.2) and (1.1), (1.3) by using the fixed point theorem. Our results generalize the paper by Li, Su and Feng [14] and an example is also included to clarify the significance of the results obtained.

2. Preliminaries

We display some background materials from the theory of cones in Banach space and we express the fixed point theorems.

Definition 2.1 *Suppose that $\mathfrak{B}$ is a real Banach space. Recall that a nonempty closed convex set $\mathcal{K} \subset \mathfrak{B}$ is a cone if it satisfies the following assumptions:*

(a) $u \in \mathcal{K}$, $\lambda \geq 0$ *implies* $\lambda u \in \mathcal{K}$;

(b) $u \in \mathcal{K}$, $-u \in \mathcal{K}$ *implies* $u = 0$.

Suppose that $\mathfrak{B}$ is a real Banach space which is partially ordered by a cone $\mathcal{K} \subset \mathfrak{B}$, i.e. $u_1 \leq u_2$ if and only if $u_2 - u_1 \in \mathcal{K}$.

Definition 2.2 *Let $\mathcal{K}$ be a cone in a real Banach space $\mathfrak{B}$. A function $\psi : \mathcal{K} \to \boldsymbol{R}$ is called to be increasing on $\mathcal{K}$, if $\psi(u_1) \leq \psi(u_2)$ for all $u_1, u_2 \in \mathcal{K}$ with $u_1 \leq u_2$.*

Definition 2.3 *A map α is called to be a nonnegative continuous concave function on a cone $\mathcal{K}$ provided that $\alpha : \mathcal{K} \to [0, \infty)$ is continuous and*

$$\alpha(tx + (1-t)y) \geq t\alpha(x) + (1-t)\alpha(y),$$

for all $x, y \in \mathcal{K}$ and $0 \leq t \leq 1$. Correspondingly, we state the map β is a nonnegative continuous convex function on a cone $\mathcal{K}$ provided that $\beta : \mathcal{K} \to [0, \infty)$ is continuous and

$$\beta(tx + (1-t)y) \leq t\beta(x) + (1-t)\beta(y),$$

for all $x, y \in \mathcal{K}$ and $0 \leq t \leq 1$.

Definition 2.4 *Let $\mathfrak{D} \subset \mathfrak{B}$. If $r : \mathfrak{B} \to \mathfrak{D}$ is continuous with $r(x) = x$ for all $x \in \mathfrak{D}$, then we say that the set $\mathfrak{D}$ is a retract of $\mathfrak{B}$ and the map r a retraction.*

The convex hull of a subset $\mathfrak{D}$ of a real Banach space $\boldsymbol{X}$ can be written by

$$conv(\mathfrak{D}) = \left\{ \sum_{j=1}^{k} \lambda_j x_j : x_j \in \mathfrak{D},\ \lambda_j \in [0,1],\ \ \sum_{j=1}^{k} \lambda_j = 1,\ \text{and } k \in \boldsymbol{N} \right\}.$$

Furthermore, we shall give the three lemmas to confirm our main results.

Lemma 2.5 *([10]). Assume that $\mathcal{K}$ is a cone in a real Banach space $\mathfrak{B}$. Let $\mathfrak{B}$ and W be a bounded, relatively open subset of $\mathcal{K}$. If $A : \overline{W} \to \mathcal{K}$ is a completely continuous operator and there exits a u_0 such that $u - Au \neq \lambda u_o$ for all $u \in \partial W$, $\lambda \geq 0$, then $i(A, W, \mathcal{K}) = 0$.*

Let ψ be a nonnegative continuous function on a cone $\mathcal{K}$ of a real Banach space $\mathfrak{B}$. For each $r_4 > 0$, we describe

$$\mathcal{K}(\gamma, r_4) = \{u \in \mathcal{K} : \gamma(u) < r_4\}.$$

Lemma 2.6 *([1]).Assume that $\mathcal{K}$ is a cone in a real Banach space $\mathfrak{B}$. Let α, β be increasing, nonnegative, continuous functions on $\mathcal{K}$, and let ψ be a nonnegative continuous function on $\mathcal{K}$ with $\psi(0) = 0$ such that, for some $r_3 > 0$, $M > 0$,*

$$\alpha(u) \leq \psi(u) \leq \beta(u), \qquad \|u\| \leq M\alpha(u) \ \text{ for all } \ u \in \overline{\mathcal{K}(\alpha, r_3)}.$$

Assume that $A : \overline{\mathcal{K}(\alpha, r_3)} \to \mathcal{K}$ is completely continuous and $0 < r_1 < r_2 < r_3$ satisfy

$$\psi(\lambda u) \leq \lambda \psi(u) \ \text{ for } \ \lambda \in [0,1], \quad u \in \partial \mathcal{K}(\psi, r_2),$$

(a) $\alpha(Au) > r_3$, *for all* $u \in \partial\mathcal{K}(\alpha, r_3)$;

(b) $\psi(Au) < r_2$, *for all* $u \in \partial\mathcal{K}(\psi, r_2)$;

(c) $\mathcal{K}(\beta, r_1) \neq \emptyset$, $\quad \beta(Au) > r_1$ *for* $u \in \partial\mathcal{K}(\beta, r_1)$.

Then A has at least two fixed points $u_1, u_2 \in \overline{\mathcal{K}(\alpha, r_3)}$ such that

$$r_1 < \beta(u_1), \qquad \psi(u_1) < r_2, \qquad r_2 < \psi(u_2), \qquad \alpha(u_2) < r_3.$$

Lemma 2.7 *([14]).Assume that $\mathcal{K}$ is a cone in a real Banach space $\mathfrak{B}$.Let α, β be increasing, nonnegative, continuous functions on $\mathcal{K}$, and let ψ be a nonnegative continuous function on $\mathcal{K}$ with $\psi(0) = 0$ such that, for some $r_3 > 0$, $M > 0$,*

$$\alpha(u) \leq \psi(u) \leq \beta(u), \qquad \|u\| \leq M\alpha(u) \ \text{ for all } \ u \in \overline{\mathcal{K}(\alpha, r_3)}.$$

Assume that $A : \overline{\mathcal{K}(\alpha, r_3)} \to \mathcal{K}$ is completely continuous and $0 < r_1 < r_2 < r_3$ satisfy

$$\psi(\lambda u) \leq \lambda \psi(u) \ \text{ for } \ \lambda \in [0,1], \quad u \in \partial \mathcal{K}(\psi, r_2),$$

(a) $\alpha(Au) < r_3$, *for all* $u \in \partial\mathcal{K}(\alpha, r_3)$;

(b) $\psi(Au) > r_2$, *for all* $u \in \partial\mathcal{K}(\psi, r_2)$;

(c) $\mathcal{K}(\beta, r_1) \neq \emptyset$, $\quad \beta(Au) < r_1$ *for* $u \in \partial\mathcal{K}(\beta, r_1)$.

Then A has at least two fixed points $u_1, u_2 \in \overline{\mathcal{K}(\alpha, r_3)}$ such that

$$r_1 < \beta(u_1), \qquad \psi(u_1) < r_2, \qquad r_2 < \psi(u_2), \qquad \alpha(u_2) < r_3.$$

3. Positive solutions for BVP (1.1) and (1.2)

In this section, we will investigate the existence of at least twin or arbitrary even positive solutions of BVP (1.1) and (1.2) by applying Avery and Henderson fixed point theorems [1].

Let

$$\mathfrak{B} = C_{ld}([0,\sigma(T)],\mathbf{R}),$$

endowed with the norm

$$\|u\| = \max\left\{ \sup_{t\in[0,T]_{\mathbf{T}}} |u(t)|, \sup_{t\in[0,T]_{\mathbf{T}}} |u^{\Delta}(t)| \right\}.$$

Let us define the cone $\mathcal{K}\subset\mathfrak{B}$ as follows

$$\mathcal{K} = \left\{ u\in\mathfrak{B}: \ u(t)\geq 0, \ \text{ for } \ [0,\sigma(T)]_{\mathbf{T}}, \quad u^{\Delta\nabla}(t)\leq 0, u^{\Delta}(t)\geq 0, \quad t\in[0,T]_{\mathbf{T}} \right\}.$$

Lemma 3.1 *Suppose (H1) and (H2). Let* $1-\sum_{i=1}^{m-2} a_i \neq 0$ *and* $1-\sum_{i=1}^{m-2} b_i \neq 0$*. Then u is a unique solution of the BVP*

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t)f(t,u(t),u^{\Delta}(t)) = 0, \qquad t\in[0,T]_{\boldsymbol{T}}, \tag{3.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \varphi_p(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i\varphi_p(u^{\Delta}(\xi_i)) \tag{3.2}$$

if and only if

$$u(t) = \int_0^t \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1} \right)\Delta s + \tilde{C_2}, \qquad t\in[0,T], \tag{3.3}$$

where

$$\tilde{C_1} = \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i},$$

$$\tilde{C_2} = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right)\Delta s}{1-\sum_{i=1}^{m-2} a_i}.$$

Proof First, we prove the necessity. Integrating the dynamic equation (3.1) from t to T gives

$$\varphi_p(u^{\Delta}(t)) = \int_t^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}, \qquad t\in[0,T], \tag{3.4}$$

i.e.,

$$u^{\Delta}(t) = \varphi_q\left(\int_t^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1} \right), \qquad t\in[0,T]. \tag{3.5}$$

A final integration yields

$$u(t) = \int_0^t \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \tilde{C_1}\right)\Delta s + \tilde{C_2}, \qquad t\in[0,T]. \tag{3.6}$$

Setting $t=T$ and $t=\xi_i$ in (3.4) gives

$$\varphi_p(u^\Delta(T)) = \tilde{C_1}$$

and

$$\varphi_p(u^\Delta(\xi_i)) = \int_{\xi_i}^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \tilde{C_1}.$$

Setting $t=0$ and $t=\xi_i$ in (3.6), we have

$$u(0) = \tilde{C_2},$$

$$u(\xi_i) = \int_0^{\xi_i} \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \tilde{C_1}\right)\Delta s + \tilde{C_2}.$$

Applying BCs (3.2) gives

$$\begin{aligned}\tilde{C_1} &= \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i},\\ \tilde{C_2} &= \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \tilde{C_1}\right)\Delta s}{1-\sum_{i=1}^{m-2} a_i}.\end{aligned}$$

To prove sufficiency, let u be as in (3.3). Then

$$u^\Delta(t) = \varphi_q\left(\int_t^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \tilde{C_1}\right), \qquad t\in[0,T],$$

and

$$\varphi_p(u^\Delta(t)) = \int_t^T a(\tau)f(\tau,u(\tau),u^\Delta(\tau))\nabla\tau + \tilde{C_1}, \qquad t\in[0,T].$$

Taking the nabla derivative of this expression, we find

$$(\varphi_p(u^\Delta(t)))^\nabla = -a(t)f(t,u(t),u^\Delta(t)), \qquad t\in[0,T].$$

Standard calculations verify that u satisfies the BCs in (3.2), so that u given in (3.3) is a solution of BVP (3.1)-(3.2). It can be readily seen that the BVP

$$(\varphi_p(u^\Delta(t)))^\nabla = 0, \qquad u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \varphi_p(u^\Delta(T)) = \sum_{i=1}^{m-2} b_i \varphi_p(u^\Delta(\xi_i))$$

has only the trivial solution if

$$1-\sum_{i=1}^{m-2} a_i \neq 0 \ \text{ and } \ 1-\sum_{i=1}^{m-2} b_i \neq 0.$$

Thus, u in (3.3) is the unique solution of BVP (3.1)-(3.2), and this completes the proof of the lemma. □

Lemma 3.2 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$. *Then the solution of BVP* (1.1)-(1.2) *fulfills* $u(t) \geq 0$, *for* $t \in [0,T]_{\mathbf{T}}$.

Proof Set

$$\varphi(s) = \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\right).$$

Then, we have

$$\begin{aligned}&\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\\ =\ &\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i} \geq 0, \quad s\in[0,T].\end{aligned}$$

It follows that $\varphi(s) \geq 0$. According to Lemma 3.1, we obtain

$$u(0) = \tilde{C}_2 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0$$

and

$$u(T) = \int_0^T \varphi(s)\Delta s + \tilde{C}_2 = \int_0^T \varphi(s)\Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0.$$

If $t \in (0,T)$, we have

$$u(t) = \int_0^t \varphi(s)\Delta s + \tilde{C}_2 = \int_0^t \varphi(s)\Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \geq 0.$$

Therefore, $u(t) \geq 0$, $t \in [0,T]_{\mathbf{T}}$. □

Lemma 3.3 *([5]). If* $u \in \mathcal{K}$, *then*

(a) $u(t) \geq \frac{t}{T}u(T) = \frac{t}{T}\sup_{t\in[0,T]_{\mathbf{T}}} u(t)$ *for* $t \in [0,T]_{\mathbf{T}}$;

(b) $su(t) \geq tu(s)$ *for* $s,t \in [0,T]_{\mathbf{T}}$ *with* $t \leq s$.

Lemma 3.4 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$. *If* $u \in \mathcal{K}$, *then*

$$\sup_{t\in[0,T]_{\mathcal{T}}} u(t) \le L \sup_{t\in[0,T]_{\mathcal{T}}} u^{\Delta}(t),$$

where $L = \max\left\{1, \dfrac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} + T\right\}$.

Proof Because $u(t) = u(0) + \int_0^t u^{\Delta}(s)\Delta s$, one has

$$\sup_{t\in[0,T]_{\mathbf{T}}} u(t) \le u(0) + T \sup_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t).$$

On the other hand,

$$\begin{aligned}\left(1-\sum_{i=1}^{m-2} a_i\right)u(0) &= u(0) - \sum_{i=1}^{m-2} a_i u(0)\\ &= \sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i u(0)\\ &= \sum_{i=1}^{m-2} a_i\left[u(\xi_i) - u(0)\right]\\ &\le \sum_{i=1}^{m-2} a_i \xi_i u^{\Delta}(\mu_i),\end{aligned}$$

where $\mu_i \in (0, \xi_i)$, so

$$u(0) \le \frac{\sum_{i=1}^{m-2} a_i\xi_i u^{\Delta}(\mu_i)}{1-\sum_{i=1}^{m-2} a_i} \le \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} \sup_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t).$$

Thus, we have

$$\sup_{t\in[0,T]_{\mathbf{T}}} u(t) \le \left(\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i} + T\right) \sup_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t).$$

□

Define the operator $S : \mathcal{K} \to \mathfrak{B}$ as follows

$$Su(t) = \int_0^t \varphi_q\left(\int_s^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\right)\Delta s + \tilde{C}_2. \tag{3.7}$$

Lemma 3.5 *([7]). Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$. $S : \mathcal{K} \to \mathcal{K}$ *is completely continuous.*

Note that each fixed point of S is a solution of the BVP (1.1)-(1.2). For $u \in \mathcal{K}$, define the nonnegative, increasing, continuous functions α, ψ, β as follows

$$\begin{aligned}
\alpha(u) &= \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t) + \min_{t\in[\nu,T]_{\mathbf{T}}} u(t) = \epsilon u^{\Delta}(0) + u(\nu),\\
\psi(u) &= \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t) + \max_{t\in[0,\nu]_{\mathbf{T}}} u(t) = \epsilon u^{\Delta}(0) + u(\nu),\\
\beta(u) &= \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t) + \max_{t\in[0,q]_{\mathbf{T}}} u(t) = \epsilon u^{\Delta}(0) + u(q),
\end{aligned}$$

where ϵ is an arbitrary positive number.

It is clear that

$$\alpha(u) \le \psi(u) \le \beta(u) \text{ for each } u \in \mathcal{K}.$$

By Lemma 3.4, we can find

$$\|u\| \le Lu^{\Delta}(0) < \frac{L}{\epsilon}\epsilon u^{\Delta}(0) + \frac{L}{\epsilon}u(\nu) = \frac{L}{\epsilon}\alpha(u) \text{ for all } u \in \mathcal{K}.$$

Moreover, for the positive constant $r_2^{\star}$, one has

$$\psi(\lambda u) = \lambda\psi(u) \text{ for } 0 \le \lambda \le 1 \text{ and } u \in \partial\mathcal{K}(\psi, r_2^{\star}).$$

Introduce the following notations.

$$\begin{aligned}
A &= \left(\nu + \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\right)\varphi_q\left(\int_{\nu}^{T} a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^{T} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right),\\
B &= \left(1 + \nu + \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\right)\varphi_q\left(\int_{0}^{T} a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^{T} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right),\\
C &= \left(1 + q + \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\right)\varphi_q\left(\int_{q}^{T} a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^{T} a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right).
\end{aligned}$$

Theorem 3.6 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$*. Let* ϵ *be an arbitrary small positive number and there exist positive numbers* $r_1^{\star}, r_2^{\star}$*, and* $r_3^{\star}$ *with* $0 < r_1^{\star} < \frac{Cr_2^{\star}}{B} < \frac{r_3^{\star}\nu C}{TB}$ *such that the following conditions are satisfied:*

(H4) $f(t,h,k) > \varphi_p\left(\frac{r_3^{\star}}{A}\right)$, *for* $(t,h,k) \in [\nu,T]_{\mathbf{T}} \times [r_3^{\star}-\epsilon, \frac{T}{\nu}r_3^{\star}] \times [0, \frac{r_3^{\star}}{\epsilon}]$;

(H5) $f(t,h,k) < \varphi_p\left(\frac{r_2^{\star}}{B}\right)$, *for* $(t,h,k) \in [0,T]_{\mathbf{T}} \times [0, \frac{T}{\nu}r_2^{\star}] \times [0, \frac{r_2^{\star}}{\epsilon}]$;

(H6) $f(t,h,k) > \varphi_p\left(\frac{r_1^{\star}}{C}\right)$, *for* $(t,h,k) \in [q,T]_{\mathbf{T}} \times [0, \frac{T}{q}r_1^{\star}] \times [0, \frac{r_1^{\star}}{\epsilon}]$.

Then the BVP (1.1)-(1.2) has at least two positive solutions u_1, u_2 *satisfying*

$$r_1^\star < \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u_1^\Delta(t) + \max_{t\in[0,q]_{\mathbf{T}}} u_1(t), \qquad \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u_1^\Delta(t) + \max_{t\in[0,\nu]_{\mathbf{T}}} u_1(t) < r_2^\star; \tag{3.8}$$

$$r_2^\star < \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u_2^\Delta(t) + \max_{t\in[0,\nu]_{\mathbf{T}}} u_2(t), \qquad \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u_2^\Delta(t) + \min_{t\in[\nu,T]_{\mathbf{T}}} u_2^\Delta(t) < r_3^\star. \tag{3.9}$$

Proof We will show that the operator S satisfies all conditions of Lemma 2.7.

Firstly, we show that if $u \in \partial\mathcal{K}(\alpha, r_3^\star)$, then $\alpha(Su) > r_3^\star$.

If $u \in \partial\mathcal{K}(\alpha, r_3^\star)$, then

$$\alpha(u) = \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^\Delta(t) + \min_{t\in[\nu,T]_{\mathbf{T}}} u(t) = \epsilon u^\Delta(0) + u(\nu) = r_3^\star. \tag{3.10}$$

Because

$$u^\Delta(t) \geq 0 \ \text{ and } \ u(t) \geq 0 \ \text{ for } \ t \in [0,T]_{\mathbf{T}},$$

one has

$$0 \leq u^\Delta(t) \leq u^\Delta(0) \leq \frac{1}{\epsilon}\epsilon u^\Delta(0) + \frac{1}{\epsilon}u(\nu) \leq \frac{1}{\epsilon}\alpha(u) = \frac{r_3^\star}{\epsilon} \ \text{ for } \ t \in [0,T]_{\mathbf{T}}.$$

From Lemma 3.3, we get

$$\max_{t\in[0,T]_{\mathbf{T}}} u(t) \leq \frac{T}{\nu} u(\nu) \leq \frac{T}{\nu} r_3^\star.$$

Now, (3.10) implies

$$u(t) \geq r_3^\star - \epsilon, \ t \in [\nu, T]_{\mathbf{T}}.$$

Using assumption (H4) in Theorem 3.6, we find

$$\begin{aligned}
\alpha(Su) &= \epsilon(Su)^{\Delta}(0) + Su(\nu) \\
&= \epsilon\varphi_q\Bigg(\int_0^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg) \\
&\quad + \int_0^{\nu} \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\right)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\right)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&> \int_0^{\nu} \varphi_q\left(\int_0^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\right)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C}_1\right)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&\geq \left(\nu + \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\right)\varphi_q\Bigg(\int_{\nu}^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg) \\
&> \left(\nu + \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\right)\varphi_q\Bigg(\int_{\nu}^T a(\tau)\varphi_p\left(\frac{r_3^{\star}}{A}\right)\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\varphi_p\left(\frac{r_3^{\star}}{A}\right)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\Bigg) \\
&= \frac{r_3^{\star}}{A}\left(\nu + \frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\right)\varphi_q\left(\int_{\nu}^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right) \\
&= r_3^{\star}.
\end{aligned}$$

Secondly, we show that $\psi(Su) < r_2^{\star}$ for $u \in \partial\mathcal{K}(\psi, r_2^{\star})$. If $u \in \partial\mathcal{K}(\psi, r_2^{\star})$, then one has

$$\psi(u) = \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t) + \max_{t\in[0,\nu]_{\mathbf{T}}} u(t) = \epsilon u^{\Delta}(0) + u(\nu) = r_2^{\star},$$

which leads to

$$\max_{t\in[0,\nu]_{\mathbf{T}}} u(t) = u(\nu) \leq r_2^{\star}, \qquad \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^{\Delta}(t) \leq r_2^{\star}.$$

Therefore,

$$0 \le u^{\Delta}(t) \le \frac{r_2^{\star}}{\epsilon} \quad \text{for} \quad t \in [0,T]_{\mathbf{T}}.$$

From Lemma 3.3, one has

$$\max_{t\in[0,T]_{\mathbf{T}}} u(t) \le \frac{T}{\nu} u(\nu) \le \frac{T}{\nu} r_2^{\star}.$$

Hence, we deduce that

$$0 \le u(t) \ \le \frac{T}{\nu} r_2^{\star}, \quad t \in [0,T]_{\mathbf{T}}.$$

Using (H5), we find

$$\begin{aligned}
\psi(Su) &= (Su)^{\Delta}(0) + (Su)(\nu) \\
&= \varphi_q\left(\int_0^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right) \\
&\quad + \int_0^{\nu} \varphi_q\left(\int_s^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\left(\int_s^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i} \\
&< \varphi_q\left(\int_0^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right) \\
&\quad + \nu\varphi_q\left(\int_0^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right) \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \xi_i \varphi_q\left(\int_s^T a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\right)}{1 - \sum_{i=1}^{m-2} a_i} \\
&< \left(1 + \nu + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i}\right) \varphi_q\left(\int_0^T a(\tau)\varphi_p\left(\frac{r_2^{\star}}{B}\right)\nabla\tau\right. \\
&\quad \left. + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\varphi_p\left(\frac{r_2^{\star}}{B}\right)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right) \\
&= \frac{r_2^{\star}}{B}\left(1 + \nu + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i}\right) \varphi_q\left(\int_0^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right) \\
&= r_2^{\star}.
\end{aligned}$$

Finally, we show that

$$\mathcal{K}(\beta, r_1^{\star}) \neq \emptyset,$$

and

$$\beta(Su) > r_1^\star \quad \text{for all} \quad u \in \partial\mathcal{K}(\beta, r_1^\star).$$

Indeed, we have $\frac{r_1^\star}{2} \in \mathcal{K}(\beta, r_1^\star)$ and for $u \in \partial\mathcal{K}(\beta, r_1^\star)$, one has

$$\beta(u) = \epsilon \max_{t\in[0,T]_{\mathbf{T}}} u^\Delta(t) + \max_{t\in[0,q]_{\mathbf{T}}} u(t) = \epsilon u^\Delta(0) + u(q) = r_1^\star,$$

which leads to

$$0 \le u(t) \le r_1^\star \quad \text{for} \quad t \in [0,q]_{\mathbf{T}}, \qquad 0 \le u^\Delta(t) \le \frac{r_1^\star}{\epsilon} \quad \text{for} \quad t \in [0,T]_{\mathbf{T}}.$$

In light of Lemma 3.3, one has

$$\max_{t\in[0,T]_{\mathbf{T}}} u(t) \le \frac{T}{q} u(q) \le \frac{T}{q} r_1^\star,$$

whereby

$$0 \le u(t) \ \le \frac{T}{q} r_1^\star, \quad t \in [q,T]_{\mathbf{T}}.$$

Employing (H6), one has

$$
\begin{aligned}
\beta(Su) &= (Su)^{\Delta}(0) + (Su)(q) \\
&= \varphi_q\bigg(\int_0^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\bigg) \\
&\quad + \int_0^q \varphi_q\bigg(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\bigg)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\bigg(\int_s^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\bigg)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&> \varphi_q\bigg(\int_0^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\bigg) \\
&\quad + \int_0^q \varphi_q\bigg(\int_q^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\bigg)\Delta s \\
&\quad + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \varphi_q\bigg(\int_q^T a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau + \tilde{C_1}\bigg)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&> \bigg(1+q+\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\bigg)\varphi_q\bigg(\int_q^T a(\tau)\varphi_p\Big(\frac{r_1^\star}{C}\Big)\nabla\tau \\
&\quad + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\varphi_p\Big(\frac{r_1^\star}{C}\Big)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\bigg) \\
&= \frac{r_1^\star}{C}\bigg(1+q+\frac{\sum_{i=1}^{m-2} a_i\xi_i}{1-\sum_{i=1}^{m-2} a_i}\bigg)\varphi_q\bigg(\int_q^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\bigg) \\
&= r_1^\star.
\end{aligned}
$$

Hence, all the conditions of Lemma 2.6 hold. We conclude that the BVP (1.1)-(1.2) has at least twin positive solutions $u_1, u_2 \in \overline{\mathcal{K}(\alpha, r_3^\star)}$, and such that (3.8)-(3.9) are satisfied. Thus, this completes the proof of the lemma. □

Next, we will discuss the existence of arbitrary even positive solutions of BVP (1.1) and (1.2).

Theorem 3.7 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$. *If there exist positive numbers* $r_{1_i}^\star, r_{2_i}^\star$, *and* $r_{3_i}^\star$ *with*

$$
0 < r_{1_1}^\star < \frac{C}{B} r_{2_1}^\star < \frac{\nu C}{TB} r_{3_1}^\star < r_{1_2}^\star < \frac{C}{B} r_{2_2}^\star < \frac{\nu C}{TB} r_{3_2}^\star < \ldots < r_{1_n}^\star < \frac{C}{B} r_{2_n}^\star < \frac{\nu C}{TB} r_{3_n}^\star
$$

$(i = 1, 2, \ldots, n, \quad n \in \boldsymbol{N})$ *such that the following conditions are satisfied:*

(H7) $f(t,h,k) > \varphi_p\Big(\frac{r^\star_{3_i}}{A}\Big)$, *for* $(t,h,k) \in [\nu,T]_{\boldsymbol{T}} \times [r^\star_{3_i} - \epsilon, \frac{T}{\nu} r^\star_{3_i}] \times [0, \frac{r^\star_{3_i}}{\epsilon}]$;

(H8) $f(t,h,k) < \varphi_p\Big(\frac{r^\star_{2_i}}{B}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{\nu} r^\star_{2_i}] \times [0, \frac{r^\star_{2_i}}{\epsilon}]$;

(H9) $f(t,h,k) > \varphi_p\Big(\frac{r^\star_{1_i}}{C}\Big)$, *for* $(t,h,k) \in [q,T]_{\boldsymbol{T}} \times [0, \frac{T}{q} r^\star_{1_i}] \times [0, \frac{r^\star_{1_i}}{\epsilon}]$.

Then the BVP (1.1)-(1.2) has at least $2n$ positive solutions.

Proof Note that when $i = 1$, in view of Theorem 3.6 it is correct that BVP (1.1), (1.2) has at least twin positive solutions $u_1, u_2 \in \overline{\mathcal{K}(\alpha, r^\star_{3_1})}$. By induction, we conclude that the BVP (1.1), (1.2) has at least $2n$ positive solutions.This completes the proof.

Let

$$
\begin{aligned}
A^\star &= \left(1 + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i} + \nu\right)\varphi_q\left(\int_0^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right),\\
B^\star &= \left(\frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i} + \nu\right)\varphi_q\left(\int_\nu^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right),\\
C^\star &= \left(1 + \frac{\sum_{i=1}^{m-2} a_i \xi_i}{1 - \sum_{i=1}^{m-2} a_i} + q\right)\varphi_q\left(\int_0^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}\right).
\end{aligned}
$$

□

As we have proved Theorem 3.6 and Theorem 3.7, one can prove the following results.

Theorem 3.8 *Suppose (H1), (H2),* $1 - \sum_{i=1}^{m-2} a_i > 0$ *and* $1 - \sum_{i=1}^{m-2} b_i > 0$. *If there exist positive numbers* $r^\star_1, r^\star_2$, *and* $r^\star_3$ *with* $0 < r^\star_1 < \frac{q}{T} r^\star_2 < \frac{qB^\star}{TA^\star} r^\star_3$ *such that the following conditions are satisfied:*

(H10) $f(t,h,k) < \varphi_p\Big(\frac{r^\star_3}{A^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, r^\star_3] \times [0, \frac{r^\star_3}{\epsilon}]$;

(H11) $f(t,h,k) > \varphi_p\Big(\frac{r^\star_2}{B^\star}\Big)$, *for* $(t,h,k) \in [\nu,T]_{\boldsymbol{T}} \times [r^\star_2 - \epsilon, \frac{T}{\nu} r^\star_2] \times [0, \frac{r^\star_2}{\epsilon}]$;

(H12) $f(t,h,k) < \varphi_p\Big(\frac{r^\star_1}{C^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{q} r^\star_1] \times [0, \frac{r^\star_1}{\epsilon}]$.

Then the BVP (1.1), (1.2) has at least twin positive solutions u_1, u_2 such that (3.8) and (3.9) hold.

Theorem 3.9 *Suppose (H1), (H2),* $1 - \sum_{i=1}^{m-2} a_i > 0$ *and* $1 - \sum_{i=1}^{m-2} b_i > 0$. *If there exist positive numbers* $r^\star_{1_i}, r^\star_{2_i}$ *and* $r^\star_{3_i}$ *with*

$$0 < r^\star_{1_1} < \frac{q}{T} r^\star_{2_1} < \frac{qB^\star}{TA^\star} r^\star_{3_1} < r^\star_{1_2} < \frac{q}{T} r^\star_{2_2} < \frac{qB^\star}{TA^\star} r^\star_{3_2} < \ldots < r^\star_{1_n} < \frac{q}{T} r^\star_{2_n} < \frac{qB^\star}{TA^\star} r^\star_{3_n}$$

$(i = 1, 2, \ldots, n, \quad n \in \boldsymbol{N})$ *such that the following conditions are satisfied:*

(H13) $f(t,h,k) < \varphi_p\Big(\frac{r^\star_{3_i}}{A^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, r^\star_{3_i}] \times [0, \frac{r^\star_{3_i}}{\epsilon}]$;

(H14) $f(t,h,k) > \varphi_p\Big(\frac{r^\star_{2_i}}{B^\star}\Big)$, *for* $(t,h,k) \in [\nu,T]_{\boldsymbol{T}} \times [r^\star_{2_i} - \epsilon, \frac{T}{\nu} r^\star_{2_i}] \times [0, \frac{r^\star_{2_i}}{\epsilon}]$;

(H15) $f(t,h,k) < \varphi_p\Big(\frac{r^\star_{1_i}}{C^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{q} r^\star_{1_i}] \times [0, \frac{r^\star_{1_i}}{\epsilon}]$.

Then the BVP (1.1), (1.2) *has at least* $2n$ *positive solutions.*

4. Positive solutions for BVP (1.1) and (1.3)

In this section, one shall investigate the existence of at least twin or arbitrary even positive solutions of BVP (1.1)-(1.3) by using Lemma 2.6 and Lemma 2.7.

Define the cone $\mathcal{K}_1 \subset \mathfrak{B}$ as follows

$$\mathcal{K}_1 = \Big\{u \in \mathfrak{B}: \ u(t) \geq 0, \quad \text{for} \ \ [0, \sigma(T)]_{\mathbf{T}}, \quad u^{\Delta\nabla}(t) \leq 0, u^{\Delta}(t) \leq 0, \quad t \in [0,T]_{\mathbf{T}}\Big\}.$$

Lemma 4.1 *Suppose (H1), (H2). Let* $1 - \sum_{i=1}^{m-2} a_i \neq 0$ *and* $1 - \sum_{i=1}^{m-2} b_i \neq 0$. *Then* u *is a unique solution of the BVP*

$$(\varphi_p(u^{\Delta}(t)))^{\nabla} + a(t) f(t, u(t), u^{\Delta}(t)) = 0, \qquad t \in [0,T]_{\boldsymbol{T}}, \tag{4.1}$$

$$\varphi_p(u^{\Delta}(0)) = \sum_{i=1}^{m-2} a_i \varphi_p(u^{\Delta}(\xi_i^\star)), \qquad u(T) = \sum_{i=1}^{m-2} b_i u(\xi_i^\star) \tag{4.2}$$

if and only if

$$u(t) = \int_t^T \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau)) \nabla\tau + \tilde{C}_3\right) \Delta s + \tilde{C}_4, \tag{4.3}$$

where

$$\tilde{C}_3 = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i^\star} a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau)) \nabla\tau}{1 - \sum_{i=1}^{m-2} a_i},$$

$$\tilde{C}_4 = \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i^\star}^T \varphi_q\left(\int_0^s a(\tau) f(\tau, u(\tau), u^{\Delta}(\tau)) \nabla\tau + \tilde{C}_3\right) \Delta s}{1 - \sum_{i=1}^{m-2} b_i}.$$

Lemma 4.2 *Suppose (H1), (H2),* $1 - \sum_{i=1}^{m-2} a_i > 0$ *and* $1 - \sum_{i=1}^{m-2} b_i > 0$. *The solution of BVP* (1.1)-(1.3) *fulfills* $u(t) \geq 0$, *for* $t \in [0,T]_{\boldsymbol{T}}$.

Lemma 4.3 *([5]). If* $u \in \mathcal{K}_1$, *then*

(a) $u(t) \geq \frac{T-t}{T} \sup_{[0,T]_{\boldsymbol{T}}} u(t) = \frac{T-t}{T} u(0)$ *for* $t \in [0,T]_{\boldsymbol{T}}$;

(b) $(T-s)u(t) \geq (T-t)u(s)$ *for* $s,t \in [0,T]_{\boldsymbol{T}}$, *with* $s \leq t$.

Lemma 4.4 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2}a_i>0$ *and* $1-\sum_{i=1}^{m-2}b_i>0$. *If* $u\in\mathcal{K}_1$, *then*

$$\sup_{t\in[0,T]_{\mathbb{T}}}|u(t)|\leq L_1\sup_{t\in[0,T]_{\mathbb{T}}}|u^{\Delta}(t)|,$$

where $L_1=\max\left\{1,\dfrac{\sum_{i=1}^{m-2}b_i(T-\xi_i^{\star})}{1-\sum_{i=1}^{m-2}b_i}+T\right\}$.

Define the operator $S_1:\mathcal{K}_1\to\mathfrak{B}$ as follows

$$S_1u(t)=\int_t^T\varphi_q\left(\int_0^s a(\tau)f(\tau,u(\tau),u^{\Delta}(\tau))\nabla\tau+\tilde{C}_3\right)\Delta s+\tilde{C}_4. \tag{4.4}$$

Note that $S_1:\mathcal{K}_1\to\mathcal{K}_1$ is completely continuous and each fixed point of S_1 is a solution of the BVP (1.1), (1.3).

For $u\in\mathcal{K}_1$, define the nonnegative, increasing, continuous functionals α_1,ψ_1, and β_1 as follows

$$\begin{aligned}\alpha_1(u)&=\epsilon\Big|\max_{t\in[0,T]_{\mathbf{T}}}u^{\Delta}(t)\Big|+\min_{t\in[0,\xi]_{\mathbf{T}}}u(t)=\epsilon|u^{\Delta}(T)|+u(\xi),\\ \psi_1(u)&=\epsilon\Big|\max_{t\in[0,T]_{\mathbf{T}}}u^{\Delta}(t)\Big|+\max_{t\in[\xi,T]_{\mathbf{T}}}u(t)=\epsilon|u^{\Delta}(T)|+u(\xi),\\ \beta_1(u)&=\epsilon\Big|\max_{t\in[0,T]_{\mathbf{T}}}u^{\Delta}(t)\Big|+\max_{t\in[l,T]_{\mathbf{T}}}u(t)=\epsilon|u^{\Delta}(T)|+u(l).\end{aligned}$$

We have

$$\alpha_1(u)\leq\psi_1(u)\leq\beta_1(u)\quad\text{for each}\quad u\in\mathcal{K}_1.$$

By Lemma 4.4, we find

$$\|u\|\leq L_1|u^{\Delta}(T)|=\frac{L_1}{\epsilon}\epsilon|u^{\Delta}(T)|<\frac{L_1}{\epsilon}\alpha_1(u)\quad\text{for all}\quad u\in\mathcal{K}_1.$$

Moreover, we have

$$\psi_1(\lambda u)=\lambda\psi_1(u)\quad\text{for}\quad\lambda\in[0,1],u\in\partial\mathcal{K}(\psi,r_2^{\star}).$$

Set

$$\begin{aligned}A_1&=\left(T-\xi+\frac{\sum_{i=1}^{m-2}b_i(T-\xi_i^{\star})}{1-\sum_{i=1}^{m-2}b_i}\right)\varphi_q\left(\int_0^{\xi}a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i^{\star}}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right),\\ B_1&=\left(1+T-\xi+\frac{\sum_{i=1}^{m-2}b_i(T-\xi_i^{\star})}{1-\sum_{i=1}^{m-2}b_i}\right)\varphi_q\left(\int_0^{T}a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i^{\star}}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right),\\ C_1&=\left(1+T-l+\frac{\sum_{i=1}^{m-2}b_i(T-\xi_i^{\star})}{1-\sum_{i=1}^{m-2}b_i}\right)\varphi_q\left(\int_0^{l}a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2}a_i\int_0^{\xi_i^{\star}}a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2}a_i}\right).\end{aligned}$$

As we have proved Theorem 3.6 and Theorem 3.7, one has the following results.

Theorem 4.5 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$*. If there exist positive numbers* $r_1^\star, r_2^\star$ *and* $r_3^\star$ *with* $0 < r_1^\star < \frac{C_1}{B_1} r_2^\star < \frac{(T-\xi)C_1}{TB_1} r_3^\star$ *such that the following conditions are satisfied:*

(H16) $f(t,h,k) > \varphi_p\Big(\frac{r_3^\star}{A_1}\Big)$, *for* $(t,h,k) \in [0,\xi]_{\boldsymbol{T}} \times [r_3^\star - \epsilon, \frac{T}{T-\xi} r_3^\star] \times [0, \frac{r_3^\star}{\epsilon}]$;

(H17) $f(t,h,k) < \varphi_p\Big(\frac{r_2^\star}{B_1}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-\xi} r_2^\star] \times [0, \frac{r_2^\star}{\epsilon}]$;

(H18) $f(t,h,k) > \varphi_p\Big(\frac{r_1^\star}{C_1}\Big)$, *for* $(t,h,k) \in [l,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-l} r_1^\star] \times [0, \frac{r_1^\star}{\epsilon}]$.

Then the BVP (1.1), (1.3) *has at least twin positive solutions* u_1, u_2 *such that*

$$r_1^\star < \epsilon \Big| \max_{t\in[0,T]_{\boldsymbol{T}}} u_1^\Delta(t) \Big| + \max_{t\in[l,T]_{\boldsymbol{T}}} u_1(t), \qquad \epsilon \Big| \max_{t\in[0,T]_{\boldsymbol{T}}} u_1^\Delta(t) \Big| + \max_{t\in[\xi,T]_{\boldsymbol{T}}} u_1(t) < r_2^\star;$$

$$r_2^\star < \epsilon \Big| \max_{t\in[0,T]_{\boldsymbol{T}}} u_2^\Delta(t) \Big| + \max_{t\in[\xi,T]_{\boldsymbol{T}}} u_2(t), \qquad \epsilon \Big| \max_{t\in[0,T]_{\boldsymbol{T}}} u_2^\Delta(t) \Big| + \min_{t\in[0,\xi]_{\boldsymbol{T}}} u_2(t) < r_3^\star.$$

Theorem 4.6 *Suppose (H1), (H2),* $1-\sum_{i=1}^{m-2} a_i > 0$ *and* $1-\sum_{i=1}^{m-2} b_i > 0$*. If there exist positive numbers* $r_{1_i}^\star, r_{2_i}^\star$ *and* $r_{3_i}^\star$ *with*

$$0 < r_{1_1}^\star < \frac{C_1}{B_1} r_{2_1}^\star < \frac{(T-\xi)C_1}{TB_1} r_{3_1}^\star < r_{1_2}^\star < \frac{C_1}{B_1} r_{2_2}^\star < \frac{(T-\xi)C_1}{TB_1} r_{3_2}^\star$$

$$< \ldots < r_{1_n}^\star < \frac{C_1}{B_1} r_{2_n}^\star < \frac{(T-\xi)C_1}{TB_1} r_{3_n}^\star \quad (i = 1, 2, \ldots n, \quad n \in \boldsymbol{N})$$

such that the subsequent assumptions are fulfilled:

(H19) $f(t,h,k) > \varphi_p\Big(\frac{r_{3_i}^\star}{A_1}\Big)$, *for* $(t,h,k) \in [0,\xi]_{\boldsymbol{T}} \times [r_{3_i}^\star - \epsilon, \frac{T}{T-\xi} r_{3_i}^\star] \times [0, \frac{r_{3_i}^\star}{\epsilon}]$;

(H20) $f(t,h,k) < \varphi_p\Big(\frac{r_{2_i}^\star}{B_1}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-\xi} r_{2_i}^\star] \times [0, \frac{r_{2_i}^\star}{\epsilon}]$;

(H21) $f(t,h,k) > \varphi_p\Big(\frac{r_{1_i}^\star}{C_1}\Big)$, *for* $(t,h,k) \in [l,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-l} r_{1_i}^\star] \times [0, \frac{r_{1_i}^\star}{\epsilon}]$.

Then the BVP (1.1), (1.3) *has at least* $2n$ *positive solutions.*

Denote

$$A_1^\star = \left(1 + T - \xi + \frac{\sum_{i=1}^{m-2} b_i (T - \xi_i^\star)}{1 - \sum_{i=1}^{m-2} b_i}\right) \varphi_q\left(\int_0^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i^\star} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right),$$

$$B_1^\star = \left(T - \xi + \frac{\sum_{i=1}^{m-2} b_i (T - \xi_i^\star)}{1 - \sum_{i=1}^{m-2} b_i}\right) \varphi_q\left(\int_\xi^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i^\star} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right),$$

$$C_1^\star = \left(1 + T - l + \frac{\sum_{i=1}^{m-2} b_i (T - \xi_i^\star)}{1 - \sum_{i=1}^{m-2} b_i}\right) \varphi_q\left(\int_0^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i^\star} a(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} a_i}\right).$$

We have the following results.

Theorem 4.7 *Suppose (H1), (H2), $1-\sum_{i=1}^{m-2} a_i > 0$ and $1-\sum_{i=1}^{m-2} b_i > 0$. If there exist positive numbers $r_1^\star, r_2^\star$ and $r_3^\star$ with*

$$0 < r_1^\star < \frac{T-C_1^\star}{T} r_2^\star < \frac{(T-C_1^\star)B_1^\star}{TA_1^\star} r_3^\star$$

such that the following conditions are satisfied:

(H22) $f(t,h,k) < \varphi_p\Big(\frac{r_3^\star}{A_1^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-\xi} r_3^\star] \times [0, \frac{r_3^\star}{\epsilon}]$;

(H23) $f(t,h,k) > \varphi_p\Big(\frac{r_2^\star}{B_1^\star}\Big)$, *for* $(t,h,k) \in [\xi,T]_{\boldsymbol{T}} \times [r_2^\star - \epsilon, \frac{T}{T-\xi} r_2^\star] \times [0, \frac{r_2^\star}{\epsilon}]$;

(H24) $f(t,h,k) < \varphi_p\Big(\frac{r_1^\star}{C_1^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-l} r_1^\star] \times [0, \frac{r_1^\star}{\epsilon}]$.

Then the BVP (1.1), (1.3) has at least twin positive solutions u_1, u_2 satisfying

$$r_1^\star < \epsilon\Big|\max_{t\in[0,T]_{\boldsymbol{T}}} u_1^\Delta(t)\Big| + \max_{t\in[l,T]_{\boldsymbol{T}}} u_1(t), \qquad \epsilon\Big|\max_{t\in[0,T]_{\boldsymbol{T}}} u_1^\Delta(t)\Big| + \max_{t\in[\xi,T]_{\boldsymbol{T}}} u_1(t) < r_2^\star;$$

$$r_2^\star < \epsilon\Big|\max_{t\in[0,T]_{\boldsymbol{T}}} u_2^\Delta(t)\Big| + \max_{t\in[\xi,T]_{\boldsymbol{T}}} u_2(t), \qquad \epsilon\Big|\max_{t\in[0,T]_{\boldsymbol{T}}} u_2^\Delta(t)\Big| + \min_{t\in[0,\xi]_{\boldsymbol{T}}} u_2(t) < r_3^\star.$$

Theorem 4.8 *Suppose (H1), (H2), $1-\sum_{i=1}^{m-2} a_i > 0$ and $1-\sum_{i=1}^{m-2} b_i > 0$. If there exist positive numbers $r_{1_i}^\star, r_{2_i}^\star$ and $r_{3_i}^\star$ with*

$$r_{1_1}^\star < \frac{T-C_1^\star}{T} r_{2_1}^\star < \frac{(T-C_1^\star)B_1^\star r_{3_1}^\star}{TA_1^\star} < r_{1_2}^\star < \frac{T-C_1^\star}{T} r_{2_2}^\star < \frac{(T-C_1^\star)B_1^\star r_{3_2}^\star}{TA_1^\star}$$

$$< \ldots < r_{1_n}^\star < \frac{T-C_1^\star}{T} r_{2_n}^\star < \frac{(T-C_1^\star)B_1^\star r_{3_n}^\star}{TA_1^\star} \quad (i = 1,2,\ldots n, \quad n \in \boldsymbol{N})$$

such that the following conditions are satisfied:

(H25) $f(t,h,k) < \varphi_p\Big(\frac{r_{3_i}^\star}{A_1^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-\xi} r_{3_i}^\star] \times [0, \frac{r_{3_i}^\star}{\epsilon}]$;

(H26) $f(t,h,k) > \varphi_p\Big(\frac{r_{2_i}^\star}{B_1^\star}\Big)$, *for* $(t,h,k) \in [\xi,T]_{\boldsymbol{T}} \times [r_{2_i}^\star - \epsilon, \frac{T}{T-\xi} r_{2_i}^\star] \times [0, \frac{r_{2_i}^\star}{\epsilon}]$;

(H27) $f(t,h,k) < \varphi_p\Big(\frac{r_{1_i}^\star}{C_1^\star}\Big)$, *for* $(t,h,k) \in [0,T]_{\boldsymbol{T}} \times [0, \frac{T}{T-l} r_{1_i}^\star] \times [0, \frac{r_{1_i}^\star}{\epsilon}]$.

Then the BVP (1.1), (1.3) has at least $2n$ positive solutions.

5. An example

In this section, we present an example to explain our results. Let $\mathbf{T} = \Big\{2-(\frac{1}{3})^{\mathbf{N}_0}\Big\} \cup \Big\{0, \frac{1}{8}, \frac{1}{4}, \frac{1}{6}, \frac{1}{2}, 1, \frac{5}{4}, \frac{3}{2}, \frac{7}{4}, 2\Big\} \cup [\frac{1}{20}, \frac{1}{10}]$, and $T = 2$. Let us consider the following BVP on time scales with $k \in \mathbf{N}_0$ and $p = 7$:

$$(\varphi_p(u^\Delta(t)))^\nabla + \Bigg\{\sum_{k=0}^{7} t^k(\rho(t))^{7-k}\Bigg\} t^\nabla f(t,u(t),u^\Delta(t)) = 0, \qquad t \in [0,2]_{\mathbf{T}}, \tag{5.1}$$

$$u(0) = \frac{1}{2}u\Big(\frac{1}{4}\Big) + \frac{1}{6}u\Big(\frac{3}{4}\Big), \qquad \varphi_7(u^\Delta(2)) = \frac{1}{3}\varphi_7\Big(u^\Delta\Big(\frac{1}{4}\Big)\Big) + \frac{1}{6}\varphi_7\Big(u^\Delta\Big(\frac{3}{4}\Big)\Big), \tag{5.2}$$

where

$$f(t,h,k)=\begin{cases} t+2+\frac{\epsilon}{10}k, & t\in[0,2]_{\mathbf{T}} \quad 0\le h<24, \quad 0\le k<\infty,\\ t+p(h,k), & t\in[0,2]_{\mathbf{T}}, \quad 24\le h<40, \quad 0\le k<\infty,\\ t+5\times 10^5+k, & t\in[0,2]_{\mathbf{T}}, \quad 40\le h<80, \quad 0\le k<\infty,\\ t+s(h,k), & t\in[0,2]_{\mathbf{T}}, \quad h\ge 80, \quad 0\le k<\infty,\end{cases}$$

where $p(h,k)$ and $s(h,k)$ satisfy the following conditions.

$$p(24,k)=2+\frac{\epsilon}{10}k, \quad p(40,k)=5\times 10^5+k, \quad s(80,k)=5\times 10^5+k,$$

$$p(h,k), s(h,k):\mathbf{R}^+\times\mathbf{R}^+\to\mathbf{R}^+ \ \text{ are continuous.}$$

Let $a(t)=\sum_{k=0}^{7}t^k(\rho(t))^{7-k}$. Let $g(t)=t^8$. Then one has $g^{\nabla}(t)=\sum_{k=0}^{7}t^k(\rho(t))^{7-k}$. Let us take $\nu=1,\ a_1=\frac{1}{2},\ a_2=\frac{1}{6},\ b_1=\frac{1}{3},\ b_2=\frac{1}{6},\ \xi_1=\frac{1}{4},\ \xi_2=\frac{3}{4},\ q=\frac{3}{2}$. By direct computation, we obtain

$$\begin{aligned} A &= \left(1+\frac{\frac{1}{2}\times\frac{1}{4}+\frac{1}{6}\times\frac{3}{4}}{1-\left(\frac{1}{2}+\frac{1}{6}\right)}\right)\left(\int_1^2\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t\right.\\ &\quad \left.+\frac{\frac{1}{3}\int_{\frac{1}{4}}^{2}\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t+\frac{1}{6}\int_{\frac{3}{4}}^{2}\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t}{1-\left(\frac{1}{3}+\frac{1}{6}\right)}\right)^{\frac{1}{6}}\\ &= \frac{7}{4}\left(2^8-1+\frac{\frac{1}{3}\left(2^8-(\frac{1}{4})^8\right)+\frac{1}{6}\left(2^8-(\frac{3}{4})^8\right)}{\frac{1}{2}}\right)^{\frac{1}{6}}\approx 4.94813,\\ B &= \left(1+1+\frac{\frac{1}{2}\times\frac{1}{4}+\frac{1}{6}\times\frac{3}{4}}{1-\left(\frac{1}{2}+\frac{1}{6}\right)}\right)\left(\int_0^2\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t\right.\\ &\quad \left.+\frac{\frac{1}{3}\int_{\frac{1}{4}}^{2}\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t+\frac{1}{6}\int_{\frac{3}{4}}^{2}\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t}{1-\left(\frac{1}{3}+\frac{1}{6}\right)}\right)^{\frac{1}{6}}\approx 7.77813,\\ C &= \left(1+\frac{3}{2}+\frac{\frac{1}{2}\times\frac{1}{4}+\frac{1}{6}\times\frac{3}{4}}{1-\left(\frac{1}{2}+\frac{1}{6}\right)}\right)\left(\int_{\frac{3}{2}}^2\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t\right.\\ &\quad \left.+\frac{\frac{1}{3}\int_{\frac{1}{4}}^{2}\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t+\frac{1}{6}\int_{\frac{3}{4}}^{2}\left\{\sum_{k=0}^{7}t^k(\rho(t))^{7-k}\right\}\nabla t}{1-\left(\frac{1}{3}+\frac{1}{6}\right)}\right)^{\frac{1}{6}}\approx 9.11404. \end{aligned}$$

Now, we choose $r_1^\star=2, r_2^\star=12$ and $r_3^\star=40$. Then we have that

$$0<r_1^\star<\frac{C}{B}r_2^\star<\frac{\nu C}{TB}r_3^\star.$$

By the definition of f, one has that

(1) $f(t,h,k) = t + 2 + \frac{\epsilon}{10}k < \varphi_p\left(\frac{r_2^\star}{B}\right) \approx 13.4845,$

for $t \in [0,2]_{\mathbf{T}}$, $\quad 0 \le h \le \frac{Tr_2^\star}{\nu} = 24, \quad 0 \le k \le \frac{r_2^\star}{\epsilon} = \frac{12}{\epsilon}$;

(2) $f(t,h,k) = t + 5 \times 10^5 + k > \varphi_p\left(\frac{r_3^\star}{A}\right) \approx 2.79 \times 10^5,$

for $t \in [1,2]_{\mathbf{T}}$, $\quad 40 - \epsilon \le h \le \frac{Tr_3^\star}{\nu} = 80, \quad 40 \le k \le \frac{r_3^\star}{\epsilon} + \infty$;

(3) $f(t,h,k) = t + 2 + \frac{\epsilon}{10}k > \varphi_p\left(\frac{r_1^\star}{C}\right) \approx 1.1166 \times 10^{-4},$

for $t \in [\frac{3}{2},2]_{\mathbf{T}}$, $\quad 0 \le h \le \frac{Tr_1^\star}{q} = 2.667, \quad 0 \le k \le \frac{r_1^\star}{\epsilon} = \frac{2}{\epsilon}$.

Hence, the conditions of Theorem 3.6 hold. By Theorem 3.6, the BVP (5.1) and (5.2) has at least twin positive solutions u_1 and u_2 that satisfy

$$2 < \epsilon \max_{t \in [0,T]_{\mathbf{T}}} u_1^\Delta(t) + \max_{t \in [0,\frac{3}{2}]_{\mathbf{T}}} u_1(t), \quad \epsilon \max_{t \in [0,2]_{\mathbf{T}}} u_1^\Delta(t) + \max_{t \in [0,1]_{\mathbf{T}}} u_1(t) < 12;$$

$$12 < \epsilon \max_{t \in [0,2]_{\mathbf{T}}} u_2^\Delta(t) + \max_{t \in [0,1]_{\mathbf{T}}} u_2(t), \quad \epsilon \max_{t \in [0,2]_{\mathbf{T}}} u_2^\Delta(t) + \min_{t \in [1,2]_{\mathbf{T}}} u_2^\Delta(t) < 40,$$

some values for ϵ.

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