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The Bochner-convolution integral for generalized functional-valued functions of discrete-time normal martingales

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Abstract: Let M be a discrete-time normal martingale satisfying some mild conditions, $\mathcal{S}(M) \subset L^2(M) \subset \mathcal{S}^*(M)$ be the Gel'fand triple constructed from the functionals of M. As is known, there is no usual multiplication in $\mathcal{S}^*(M)$ since its elements are continuous linear functionals on $\mathcal{S}(M)$. However, by using the Fock transform, one can introduce convolution in $\mathcal{S}^*(M)$, which suggests that one can try to introduce a type of integral of an $\mathcal{S}^*(M)$-valued function with respect to an $\mathcal{S}^*(M)$-valued measure in the sense of convolution. In this paper, we just define such type of an integral. First, we introduce a class of $\mathcal{S}^*(M)$-valued measures and examine their basic properties. Then, we define an integral of an $\mathcal{S}^*(M)$-valued function with respect to an $\mathcal{S}^*(M)$-valued measure and, among others, we establish a dominated convergence theorem for this integral. Finally, we also prove a Fubini type theorem for this integral.

Key words: Normal martingale, Fubini theorem, Bochner integral, vector measure

1. Introduction

Integration theory was originally developed for real-valued functions with respect to real-valued measures. In the first half of the last century, real variable theory was extended to functions taking values in vector spaces [4]. Nowadays, integrals of vector-valued functions with respect to scalar measures, and integrals of scalar-valued functions with respect to vector measures have been explored extensively using different approaches. For example, Mitter and Young [6] developed an integration theory with respect to operator-valued measures which is required in the study of certain convex optimization problems. Rybakov [9] presented a generalization of the Bochner integral to locally convex spaces. Cao [1] defined the Henstock–Kurzweil integral for Banach space-valued functions. Sokol [10] defined the Hake–Henstock–Kurzweil and the Hake–McShane integrals of Banach space- valued functions defined on an open and bounded subset of m-dimensional Euclidean space. However, integral of vector-valued functions with respect to vector-valued measures is seldom appreciated because of its abstract nature. The purpose of this paper is to develop an integration theory for the case of vector-valued functions with respect to vector-valued measures on the space of generalized functionals of discrete-time normal martingale.

Discrete-time normal martingales [8] play an important role in many theoretical and applied fields. In recent years, functionals of discrete-time normal martingales have attracted much attention ([3, 5, 7, 11–13]). Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale satisfying some mild conditions. Then, by using a

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specific orthonormal basis for the space $L^2(M)$ of square integrable functionals of M, Gel'fand triple

$$\mathcal{S}(M) \subset L^2(M) \subset \mathcal{S}^*(M),$$

can be constructed [15], where elements of $\mathcal{S}(M)$ are called testing functionals of M, while elements of $\mathcal{S}^*(M)$ are called generalized functionals of M. In paper [11], a transform, called Fock transform has been introduced for generalized functionals of M. It has been shown that the generalized functionals of M can be characterized only by growth condition on their Fock transforms. As is known, there is no usual multiplication in $\mathcal{S}^*(M)$ since its elements are continuous linear functionals on $\mathcal{S}(M)$. However, by using the Fock transform, one can introduce convolution in $\mathcal{S}^*(M)$, which suggests that one can try to introduce a type of integral of an $\mathcal{S}^*(M)$-valued function with respect to an $\mathcal{S}^*(M)$-valued measure in the sense of convolution. In this paper, we just define such a type of integral. The main work therein is as follows. In Section 3, $\mathcal{S}^*(M)$-valued measures are introduced and their basic properties are examined. In Section 4, an integral of an $\mathcal{S}^*(M)$-valued function with respect to an $\mathcal{S}^*(M)$-valued measure is defined and, among others, a dominated convergence theorem is established for this integral. In Section 5, a Fubini type theorem is also proved for integral of an $\mathcal{S}^*(M)$-valued function with respect to an $\mathcal{S}^*(M)$-valued measure.

Throughout this paper, $\mathbb{N}$ denotes the set of all nonnegative integers and Γ the finite power set of $\mathbb{N}$, namely

$$\Gamma = \{\, \sigma \mid \sigma \subset \mathbb{N} \text{ and } \#(\sigma) < \infty \,\}, \tag{1.1}$$

where $\#(\sigma)$ means the cardinality of σ as a set.

2. Preliminaries

In this section, we briefly recall some notions and results on generalized functionals of discrete time normal martingales. For details, see [2, 11, 13, 14] and references therein.

Let $(\Sigma, \mathscr{A}, P)$ be a given probability space with $\mathbb{E}$ denoting the expectation with respect to P. We use $L^2(\Sigma, \mathscr{A}, P)$ denote the usual Hilbert space of square integrable complex-valued functions on $(\Sigma, \mathscr{A}, P)$ and use $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ to mean its inner product and norm, respectively. By convention, $\langle \cdot, \cdot \rangle$ is conjugate-linear in its first argument and linear in its second argument.

Let $M = (M_n)_{n\in\mathbb{N}}$ be a discrete-time normal martingale on $(\Sigma, \mathscr{A}, P)$ that has the chaotic representation property. For brevity, we use $L^2(M)$ to mean the space of square integrable functionals of M, namely

$$L^2(M) = L^2(\Sigma, \mathscr{A}_\infty, P), \tag{2.1}$$

which shares the same inner product and norm with $L^2(\Sigma, \mathscr{A}, P)$, namely $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$. It is known that $\{Z_\sigma \mid \sigma \in \Gamma\}$ forms a countable orthonormal basis for $L^2(M)$, where $Z_\emptyset = 1$ and

$$Z_\sigma = \prod_{i\in\sigma} Z_i, \quad \sigma \in \Gamma,\ \sigma \neq \emptyset. \tag{2.2}$$

where $Z = (Z_n)_{n\in\mathbb{N}}$ is the discrete-time normal noise associated with M (see [11] for details).

Lemma 2.1 *[15] Let $\sigma \mapsto \lambda_\sigma$ be the $\mathbb{N}$-valued function on Γ given by*

$$\lambda_\sigma = \begin{cases} \prod_{k\in\sigma}(k+1), & \sigma \neq \emptyset,\ \sigma \in \Gamma; \\ 1, & \sigma = \emptyset,\ \sigma \in \Gamma. \end{cases} \tag{2.3}$$

Then, for $p > 1$*, the positive term series* $\sum_{\sigma\in\Gamma}\lambda_\sigma^{-p}$ *converges; moreover,*

$$\sum_{\sigma\in\Gamma}\lambda_\sigma^{-p} \le \exp\left[\sum_{k=1}^{\infty} k^{-p}\right] < \infty. \tag{2.4}$$

Using the orthonormal basis $\{Z_\sigma \mid \sigma \in \Gamma\}$ and $\mathbb{N}$-valued function defined by (2.3), one can construct a chain of Hilbert spaces $\mathcal{S}_p(M)$ $(p \ge 0)$ of functionals of M, put

$$\mathcal{S}(M) = \bigcap_{p=0}^{\infty} \mathcal{S}_p(M), \tag{2.5}$$

it is a dense linear subspace of $L^2(M)$, which itself is a countable Hilbert nuclear space and continuously contained in $L^2(M)$ [11]. For $p \ge 0$, we denote by $\mathcal{S}_p^*(M)$ the dual of $\mathcal{S}_p(M)$ and $\|\cdot\|_{-p}$ the norm of $\mathcal{S}_p^*(M)$. Let $\mathcal{S}^*(M)$ be the dual of $\mathcal{S}(M)$ and endow it with the strong topology, then

$$\mathcal{S}^*(M) = \bigcup_{p=0}^{\infty} \mathcal{S}_p^*(M). \tag{2.6}$$

We mention that, by identifying $L^2(M)$ with its dual, the Gel'fand triple

$$\mathcal{S}(M) \subset L^2(M) \subset \mathcal{S}^*(M),$$

can be constructed, which is the framework where we will work. Elements of $\mathcal{S}^*(M)$ are called generalized functionals of M, while elements of $\mathcal{S}(M)$ are called testing functionals of M.

Definition 2.2 *[11] For* $\Phi \in \mathcal{S}^*(M)$*, its Fock transform is the function* $\widehat{\Phi}$ *on* Γ *given by*

$$\widehat{\Phi}(\sigma) = \langle\!\langle \Phi, Z_\sigma \rangle\!\rangle, \quad \sigma \in \Gamma, \tag{2.7}$$

where $\langle\!\langle \cdot,\cdot \rangle\!\rangle$ *is the canonical bilinear form.*

In general, the usual product of two generalized functionals of $\mathcal{S}^*(M)$ is no longer a generalized functional of $\mathcal{S}^*(M)$. This means that the usual product is not a multiplication in $\mathcal{S}^*(M)$. However, by using the Fock transform, one can introduce convolution in $\mathcal{S}^*(M)$.

For two generalized functionals $\Phi,\ \Psi \in \mathcal{S}^*(M)$, denote by $\Phi * \Psi$ their convolution. Recall that

$$\widehat{\Phi * \Psi}(\sigma) = \widehat{\Phi}(\sigma)\widehat{\Psi}(\sigma),$$

for any $\sigma \in \Gamma$.

The following lemma characterizes the generalized functionals of M through their Fock transforms.

Lemma 2.3 *[11] Let* F *be a function on* Γ*. Then* F *is the Fock transform of an element* Φ *of* $\mathcal{S}^*(M)$ *if and only if it satisfies*

$$|F(\sigma)| \le C\lambda_\sigma^p, \quad \sigma \in \Gamma \tag{2.8}$$

for some constants $C \geq 0$ *and* $p \geq 0$. *In that case, for* $q > p + \frac{1}{2}$, *one has*

$$\|\Phi\|_{-q} \leq C\Big[\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-p)}\Big]^{\frac{1}{2}} \tag{2.9}$$

and in particular $\Phi \in \mathcal{S}_q^*(M)$.

The next lemma offers a criterion in terms of the Fock transform for checking whether or not a sequence in $\mathcal{S}^*(M)$ is convergent.

Lemma 2.4 *[14] Let* $\Phi \in \mathcal{S}^*(M)$, $(\Phi_n)_{n\geq 1}$ *be a sequence of generalized functionals in* $\mathcal{S}^*(M)$. *Suppose* $\widehat{\Phi_n}(\sigma)$ *converges to* $\widehat{\Phi}(\sigma)$ *for all* $\sigma \in \Gamma$; *moreover, there exist constants* $C \geq 0$ *and* $p \geq 0$ *such that*

$$\sup_{n\geq 1}|\widehat{\Phi_n}(\sigma)| \leq C\lambda_\sigma^p, \quad \sigma \in \Gamma. \tag{2.10}$$

Then $(\Phi_n)_{n\geq 1}$ *converges weakly to* Φ *in* $\mathcal{S}^*(M)$.

3. Generalized functional-valued measures

Throughout this section we assume that $\mathcal{F}$ is the σ-algebra of subsets of a set Ω. In this section, we introduce a class of $\mathcal{S}^*(M)$-valued measures and examine their basic properties.

Definition 3.1 *Let* $(\Omega, \mathcal{F})$ *be a measure space. A set function* $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ *is called a generalized functional-valued measure if it is countably additive on* $\mathcal{F}$, *i.e. for every countable disjoint sequence* (E_n) *in* $\mathcal{F}$, *we have*

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty}\mu(E_n), \tag{3.1}$$

where the sum is convergent in the weak topology on $\mathcal{S}^*(M)$.

Definition 3.2 *Let* $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ *be a generalized functional-valued measure. For* $p \geq 0$, *the* p*-variation* $|\mu|_p$ *of* μ *is defined by*

$$|\mu|_p(E) = \sup_{\pi}\sum_{B\in\pi}\|\mu(B)\|_{-p}, \quad E \in \mathcal{F},$$

where the supremum is taken over all finite disjoint measurable partition π *of* E. *If* $|\mu|_p(\Omega) < \infty$, *then* μ *will be called a generalized functional-valued measure of* p*-bounded variation.*

Theorem 3.3 *Let* $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ *be a generalized functional-valued measure of* p*-bounded variation, then the* p*-variation* $|\mu|_p$ *is a monotone and finitely additive function on* $\mathcal{F}$.

Proof Let E_1 and E_2 be disjoint members of $\mathcal{F}$, then for any finite disjoint measurable partition π of $E_1 \cup E_2$, $\{B \cap E_1 | B \in \pi\}$ and $\{B \cap E_2 | B \in \pi\}$ are partitions of E_1 and E_2, respectively. Thus,

$$\begin{aligned}\sum_{B\in\pi} \|\mu(B)\|_{-p} &= \sum_{B\in\pi} \|\mu(B\cap E_1) + \mu(B\cap E_2)\|_{-p} \\ &\leq \sum_{B\in\pi} \|\mu(B\cap E_1)\|_{-p} + \sum_{B\in\pi} \|\mu(B\cap E_2)\|_{-p} \\ &\leq |\mu|_p(E_1) + |\mu|_p(E_2)\end{aligned}$$

Taking the supremum over all finite disjoint measurable partition π of $E_1 \cup E_2$ yields

$$|\mu|_p(E_1 \cup E_2) \leq |\mu|_p(E_1) + |\mu|_p(E_2). \tag{3.2}$$

On the other hand, $|\mu|_p(E_1) \leq |\mu|_p(\Omega)$, so there exists a finite disjoint measurable partition π_1 of E_1 such that for any $\varepsilon > 0$, we have

$$|\mu|_p(E_1) - \varepsilon < \sum_{B\in\pi_1} \|\mu(B)\|_{-p},$$

similarly, there exists a finite disjoint measurable partition π_2 of E_2 such that

$$|\mu|_p(E_2) - \varepsilon < \sum_{B\in\pi_2} \|\mu(B)\|_{-p}.$$

It is easy to see that $\pi_1 \cup \pi_2$ be a finite disjoint measurable partition of $E_1 \cup E_2$. Then

$$\begin{aligned}&|\mu|_p(E_1) + |\mu|_p(E_2) - 2\varepsilon \\ \leq &\sum_{B\in\pi_1} \|\mu(B)\|_{-p} + \sum_{B\in\pi_2} \|\mu(B)\|_{-p} \\ \leq &|\mu|_p(E_1 \cup E_2).\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we have the reverse inequality

$$|\mu|_p(E_1) + |\mu|_p(E_2) \leq |\mu|_p(E_1 \cup E_2),$$

which together with (3.2) means that $|\mu|_p$ is finitely additive. It is immediate that $|\mu|_p$ is monotone. □

Theorem 3.4 *A generalized functional-valued measure μ of p-bounded variation is countably additive with respect to $\|\cdot\|_{-p}$ if and only if its p-variation $|\mu|_p$ is also countably additive.*

Proof Suppose $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ be a generalized functional-valued measure of p-bounded variation. Since $\|\mu(E)\|_{-p} \leq |\mu|_p(E)$ for each $E \in \mathcal{F}$, it is plain that μ is countably additive with respect to $\|\cdot\|_{-p}$ if $|\mu|_p$ is countably additive.

Conversely, suppose that $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ is a countably additive measure with respect to $\|\cdot\|_{-p}$. Let $(E_n) \subset \mathcal{F}$ be a sequence of disjoint sets such $\cup_{n=1}^{\infty} E_n \in \mathcal{F}$ and let π be a finite disjoint measurable partition of $\cup_{n=1}^{\infty} E_n$. Then

$$\begin{aligned}\sum_{B\in\pi} \|\mu(B)\|_{-p} &= \sum_{B\in\pi} \|\mu(B\cap(\cup_{n=1}^{\infty} E_n))\|_{-p} \\ &\leq \sum_{n=1}^{\infty}\sum_{B\in\pi} \|\mu(B\cap E_n)\|_{-p} \\ &\leq \sum_{n=1}^{\infty} |\mu|_p(E_n).\end{aligned}$$

Since this holds for any partition π, we have the inequality

$$|\mu|_p(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} |\mu|_p(E_n).$$

By Theorem 3.3, $|\mu|_p$ is a finitely additive and monotone on $\mathcal{F}$. Thus, for any n, we have

$$\sum_{k=1}^{n} |\mu|_p(E_k) = |\mu|_p(\bigcup_{k=1}^{n} E_k) \leq |\mu|_p(\bigcup_{n=1}^{\infty} E_n).$$

This proves the reverse inequality $\sum_{n=1}^{\infty} |\mu|_p(E_n) \leq |\mu|_p(\bigcup_{n=1}^{\infty} E_n)$ and shows that $|\mu|_p$ is countably additive on $\mathcal{F}$. □

Proposition 3.5 *Let $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ be a generalized functional-valued measure of p-bounded variation. Then for any $p \geq 0$ and $E \in \mathcal{F}$, we have*

$$|\widehat{\mu(E)}(\sigma)| \leq \lambda_\sigma^p |\mu|_p(E),$$

for all $\sigma \in \Gamma$.

Proof Let $p \geq 0$ and $E \in \mathcal{F}$. If $|\mu|_p(\Omega) < \infty$, then we have

$$\|\mu(E)\|_{-p} \leq |\mu|_p(E) \leq |\mu|_p(\Omega) < \infty.$$

This means $\mu(E) \in \mathcal{S}_p^*(M)$. By $\|Z_\sigma\|_p = \lambda_\sigma^p < \infty$ for $\sigma \in \Gamma$, we have

$$|\widehat{\mu(E)}(\sigma)| = |\langle\langle \mu(E), Z_\sigma\rangle\rangle| \leq \lambda_\sigma^p \|\mu(E)\|_{-p} \leq \lambda_\sigma^p |\mu|_p(E).$$

This completes the proof. □

4. Bochner-convolution integral

Let $(\Omega, \mathcal{F})$ be a measure space as in Section 3, $\mu : \mathcal{F} \to \mathcal{S}^*(M)$ be a generalized functional-valued measure. In this section we define an integral of an $\mathcal{S}^*(M)$-valued function with respect to μ and establish a dominated convergence theorem for this integral. The integral will be defined by following a approach, that is, we first define the integral for a class of simple integrands and then we extend it to a larger class of integrands which are strongly measurable functions on $\mathcal{S}^*(M)$.

Definition 4.1 *A function* $\Phi : \Omega \to \mathcal{S}^*(M)$ *is called simple if there exists* $\Phi_1, \Phi_2, \cdots, \Phi_n \in \mathcal{S}^*(M)$ *such that*

$$\Phi(\omega) = \sum_{i=1}^{n} \Phi_i I_{E_i}(\omega), \quad \omega \in \Omega,$$

where $E_1, E_2, \cdots, E_n$ *is a finite disjoint measurable partition of* Ω. *A function* $\Phi : \Omega \to \mathcal{S}^*(M)$ *is called strongly measurable if there exists* $p \geq 0$ *and a sequence of simple functions* $\{\Psi_n\}_{n\geq 0}$ *such that*

$$\lim_{n\to\infty} \|\Phi(\omega) - \Psi_n(\omega)\|_{-p} = 0, \qquad |\mu|_p - \text{a.e.}$$

Definition 4.2 *For a simple function* $\Phi(\omega) = \sum_{i=1}^{n} \Phi_i I_{E_i}(\omega)$, $\omega \in \Omega$, *the Bochner-convolution integral* $\int_\Omega \Phi(\omega) * d\mu$ *is defined by*

$$\int_\Omega \Phi(\omega) * d\mu = \sum_{i=1}^{n} \Phi_i * \mu(E_i).$$

From the definition it is clear that Bochner-convolution integrals have the linear operational property, i.e. for any $\alpha, \beta \in \mathbb{R}$ and simple functions Φ, Ψ, we have

$$\int_\Omega [\alpha\Phi(\omega) + \beta\Psi(\omega)] * d\mu = \alpha \int_\Omega \Phi(\omega) * d\mu + \beta \int_\Omega \Psi(\omega) * d\mu.$$

Theorem 4.3 *Let* $\Phi : \Omega \to \mathcal{S}^*(M)$ *be a simple function, then for* $q > 2p + \frac{1}{2}$, *we have*

$$\|\int_\Omega \Phi(\omega) * d\mu\|_{-q} \leq [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} \int_\Omega \|\Phi(\omega)\|_{-p} d|\mu|_p.$$

Proof Let $\Phi(\omega) = \sum_{i=1}^{n} \Phi_i I_{E_i}(\omega)$. We denote by $Y = \int_\Omega \Phi(\omega) * d\mu$, then for any $\sigma \in \Gamma$,

$$\widehat{Y}(\sigma) = [\widehat{\sum_{i=1}^{n} \Phi_i * \mu(E_i)}](\sigma) = \sum_{i=1}^{n} \widehat{\Phi_i}(\sigma)\widehat{\mu(E_i)}(\sigma),$$

by Lemma 2.3 and Proposition 3.5, we have

$$\begin{aligned}
|\widehat{Y}(\sigma)| \leq \sum_{i=1}^{n} |\widehat{\Phi_i}(\sigma)||\widehat{\mu(E_i)}(\sigma)| &\leq \sum_{i=1}^{n} \|\Phi_i\|_{-p} |\mu|_p(E_i)\lambda_\sigma^{2p} \\
&\leq \lambda_\sigma^{2p} \int_\Omega \sum_{i=1}^{n} \|\Phi_i\|_{-p} I_{E_i}(\omega) d|\mu|_p \\
&= \lambda_\sigma^{2p} \int_\Omega \|\Phi(\omega)\|_{-p} d|\mu|_p.
\end{aligned}$$

Then by Lemma 2.3, for $q > 2p + \frac{1}{2}$,

$$\|Y\|_{-q} \leq [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} \int_\Omega \|\Phi(\omega)\|_{-p} d|\mu|_p.$$

This completes the proof. □

Theorem 4.4 *Let $\Phi : \Omega \to \mathcal{S}^*(M)$ be a strongly measurable function, if there exists $p \geq 0$ and a sequence of simple functions $\{\Phi_n\}_{n\geq 0}$ such that*

$$\lim_{n\to\infty} \int_\Omega \|\Phi(\omega) - \Phi_n(\omega)\|_{-p} d|\mu|_p = 0, \qquad |\mu|_p - \text{a.e.}.$$

Then for $q > 2p + \frac{1}{2}$, $\{\int_\Omega \Phi_n(\omega) * d\mu\}_{n\geq 1}$ converges with respect to $\|\cdot\|_{-q}$, that is, there exists a $\Psi \in \mathcal{S}_q^*(M)$, such that

$$\lim_{n\to\infty} \|\Psi - \int_\Omega \Phi_n(\omega) * d\mu\|_{-q} = 0.$$

Proof By the linear operational property of Bochner-convolution integrals and Theorem 4.3

$$\begin{aligned}
&\|\int_\Omega \Phi_m(\omega) * d\mu - \int_\Omega \Phi_n(\omega) * d\mu\|_{-q} \\
&= \|\int_\Omega [\Phi_m(\omega) - \Phi_n(\omega)] * d\mu\|_{-q} \\
&\leq [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} \int_\Omega \|\Phi_m(\omega) - \Phi_n(\omega)\|_{-p} d|\mu|_p \\
&\to 0 \ (n, m \to \infty).
\end{aligned}$$

Thus, $\{\int_\Omega \Phi_n(\omega) * d\mu\}_{n\geq 1}$ is a Cauchy sequence with respect to $\|\cdot\|_{-q}$, so there exist $\Psi \in \mathcal{S}_q^*(M)$ such that

$$\lim_{n\to\infty} \|\Psi - \int_\Omega \Phi_n(\omega) * d\mu\|_{-q} = 0.$$

This completes the proof. □

Definition 4.5 *The generalized functional Ψ in Theorem 4.4 is called Bochner-convolution integral of Φ, we denote it by*

$$\int_\Omega \Phi(\omega) * d\mu$$

In this case, Φ is said to Bochner-convolution integrable.

Theorem 4.6 *A function $\Phi : \Omega \to \mathcal{S}^*(M)$ is Bochner-convolution integrable if and only if Φ is strongly measurable and there exists $p \geq 0$ such that*

$$\int_\Omega \|\Phi(\omega)\|_{-p} d|\mu|_p < +\infty.$$

In that case, for $q > 2p + \frac{1}{2}$, one has

$$\|\int_\Omega \Phi(\omega) * d\mu\|_{-q} \leq [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} \int_\Omega \|\Phi(\omega)\|_{-p} d|\mu|_p.$$

Proof The necessity is obvious, we only show the sufficiency. Take a sequence of simple functions $\{\Phi_n\}_{n\geq 0}$ such that

$$\lim_{n\to\infty}\|\Phi(\omega)-\Phi_n(\omega)\|_{-p}=0, \qquad |\mu|_p - \text{a.e.}.$$

Denote $E_n=\{\omega:\|\Phi_n(\omega)\|_{-p}\geq \frac{3}{2}\|\Phi(\omega)\|_{-p}\}$. Let $\Psi_n=\Phi_n I_{E_n}, n\geq 1$, it is easy to see that Ψ_n is a sequence of simple functions, a direct computation gives

$$\|\Psi_n(\omega)-\Phi(\omega)\|_{-p}\leq \|\Psi_n(\omega))\|_{-p}+\|\Phi(\omega)\|_{-p}\leq \frac{5}{2}\|\Phi(\omega)\|_{-p}$$

and

$$\begin{aligned}&\|\Psi_n(\omega)-\Phi(\omega)\|_{-p}\\ \leq\ &\|\Psi_n(\omega)-\Phi_n(\omega)\|_{-p}+\|\Phi_n(\omega)-\Phi(\omega)\|_{-p}\\ \longrightarrow\ &0(n\to\infty),\ |\mu|_p - a.e.\end{aligned}$$

By the dominated convergence theorem, we have

$$\int_\Omega \|\Psi_n(\omega)-\Phi(\omega)\|_{-p}d|\mu|_p\longrightarrow 0(n\to\infty), \qquad |\mu|_p - a.e.$$

by Theorem 4.4, Φ is Bochner-convolution integrable.

For $q>2p+\frac{1}{2}$, from the definition and Theorem 4.4 it is clear that

$$\|\int_\Omega \Phi(\omega)*d\mu\|_{-q}\leq [\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}}\int_\Omega \|\Phi(\omega)\|_{-p}d|\mu|_p.$$

This completes the proof. □

Theorem 4.7 *(Dominated convergence theorem) Let $\{\Phi_n\}_{n\geq 1}$ be a sequence of Bochner-convolution integrable functions on Ω. If for $p\geq 0$, $\lim_{n\to\infty}|\mu|_p\{\omega\in\Omega \mid \|\Phi_n-\Phi\|_{-p}>\varepsilon\}=0$ for any $\varepsilon>0$ and if there exists a real-valued integrable function $g(\cdot)$ on Ω with $\|\Phi_n(\omega)\|_{-p}\leq g(\omega)$ $|\mu|_p$-a.e., then Φ is Bochner-convolution integrable and for $q>2p+\frac{1}{2}$*

$$\lim_{n\to\infty}\|\int_\Omega \Phi_n(\omega)*d\mu-\int_\Omega \Phi(\omega)*d\mu\|_{-q}=0.$$

Proof Just apply the scalar Dominate convergence theorem to $\|\Phi(\omega)-\Phi_n(\omega)\|_{-p}$ with dominating function $2g$, Φ is Bochner-convolution integrable. For $q>2p+\frac{1}{2}$, by Theorem 4.6

$$\begin{aligned}&\|\int_\Omega \Phi(\omega)*d\mu-\int_\Omega \Phi_n(\omega)*d\mu\|_{-q}\\ \leq\ &\|\int_\Omega [\Phi(\omega)-\Phi_n(\omega)]*d\mu\|_{-q}\\ \leq\ &[\sum_{\sigma\in\Gamma}\lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}}\int_\Omega \|\Phi(\omega)-\Phi_n(\omega)\|_{-p}d|\mu|_p\\ \to\ &0\ (n\to\infty)\end{aligned}$$

This completes the proof. □

Theorem 4.8 *Let $\Phi : \Omega \to \mathcal{S}^*(M)$ be a strongly measurable function, if Φ is Bochner-convolution integrable, then for any $\sigma \in \Gamma$, the function $\omega \to \widehat{\Phi(\omega)}(\sigma)$ is integrable with respect to $\widehat{\mu}(\sigma)$ and*

$$[\widehat{\int_\Omega \Phi(\omega) * d\mu}](\sigma) = \int_\Omega \widehat{\Phi(\omega)}(\sigma) d\widehat{\mu}(\sigma).$$

Proof It is easy to see that the function $\omega \to \widehat{\Phi(\omega)}(\sigma)$ is measurable and

$$|\widehat{\Phi(\omega)}(\sigma)| \leq \|\Phi(\omega)\|_{-p} \lambda_\sigma^p, \ \omega \in \Omega,$$

by Proposition 3.5,

$$|\int_\Omega \widehat{\Phi(\omega)}(\sigma) d\widehat{\mu}(\sigma)| \leq \lambda_\sigma^{2p} \int_\Omega \|\Phi(\omega)\|_{-p} d|\mu|_p, \tag{4.1}$$

which together with Φ Bochner-convolution integrable means that $\widehat{\Phi(\omega)}(\sigma)$ is integrable with respect to $\widehat{\mu}(\sigma)$.

On the other hand, Φ is strongly measurable, so there exists a sequence of simple functions $\{\Phi_n\}_{n\geq 0}$ such that

$$\lim_{n\to\infty} \|\Phi_n(\omega) - \Phi(\omega)\|_{-p} = 0, \quad |\mu|_p - \text{a.e.}.$$

by Theorem 4.4, for $q > 2p + \frac{1}{2}$

$$\lim_{n\to\infty} \| \int_\Omega \Phi_n(\omega) * d\mu - \int_\Omega \Phi(\omega) * d\mu\|_{-q} = 0, \ |\mu|_p - \text{a.e.}.$$

And also

$$|\widehat{\Phi_n(\omega)}(\sigma)| \leq \|\Phi_n(\omega)\|_{-p} \lambda_\sigma^p, \ \omega \in \Omega,$$

by the dominated convergence theorem, we have

$$[\widehat{\int_\Omega \Phi(\omega) * d\mu}](\sigma) = \lim_{n\to\infty} [\widehat{\int_\Omega \Phi_n(\omega) * d\mu}](\sigma)$$
$$= \lim_{n\to\infty} \int_\Omega \widehat{\Phi_n(\omega)}(\sigma) d\widehat{\mu}(\sigma) = \int_\Omega \widehat{\Phi(\omega)}(\sigma) d\widehat{\mu}(\sigma).$$

This completes the proof. □

5. The Fubini theorem

In the present section, we will prove a Fubini type theorem for generalized functional-valued measure using the Fock transforms of generalized functionals in $\mathcal{S}^*(M)$.

Throughout this section, we suppose that μ is a generalized functional-valued measure defined on a measurable space $(\Omega_1, \mathcal{F}_1)$, υ is a generalized functional-valued measure defined on another measurable space $(\Omega_2, \mathcal{F}_2)$. We also suppose that there exists $p \geq 0$ such that $|\mu|_p$ and $|\upsilon|_p$ are countably additive measures.

In the following, we first prove that there exists a convolution measure $\mu * \upsilon$ on the product measurable space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ which satisfies

$$\mu * \upsilon(A \times B) = \mu(A) * \upsilon(B),$$

for any $A \in \mathcal{F}_1, B \in \mathcal{F}_2$. Therefore, we consider a function $F : \Gamma \to \mathbb{R}$ defined by $F(\sigma) = \widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma)(A \times B)$ for $A \times B \in \mathcal{F}_1 \times \mathcal{F}_2$ is the Fock transform of an element of $S^*(M)$.

Theorem 5.1 *Let $F(\sigma) = \widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma)(A \times B)$ for $A \times B \in \mathcal{F}_1 \times \mathcal{F}_2$, then F is the Fock transform of an element of $S^*(M)$.*

Proof According to the suppose of μ, υ, we have $|\widehat{\mu(\cdot)}(\sigma)| \leq \lambda_\sigma^p |\mu|_p(\cdot)$ and $|\widehat{\upsilon(\cdot)}(\sigma)| \leq \lambda_\sigma^p |\upsilon|_p(\cdot)$, then

$$\begin{aligned} |F(\sigma)| = |\widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma)(A \times B)| &= |\int_{\Omega_1 \times \Omega_2} I_{A \times B} d(\widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma))| \\ &\leq \lambda_\sigma^{2p} \int_{\Omega_1 \times \Omega_2} I_{A \times B} d(|\mu|_p \times |\upsilon|_p) \\ &= \lambda_\sigma^{2p} (|\mu|_p \times |\upsilon|_p)(A \times B) \end{aligned}$$

By Lemma 2.3, $F(\sigma)$ is the Fock transform of an element of $S^*(M)$. □

Theorem 5.2 *Let $A \times B \in \mathcal{F}_1 \times \mathcal{F}_2$ define $\mu * \upsilon$ as $[\mu * \widehat{\upsilon(A \times} B)](\sigma) = \widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma)(A \times B)$, then $\mu * \upsilon$ is a unique generalized functional-valued measure of p-bounded variation defined on the product measurable space $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$ and satisfies*

$$\mu * \upsilon(A \times B) = \mu(A) * \upsilon(B).$$

Proof We first prove that $\mu * \upsilon$ is countably additive on $\mathcal{F}_1 \times \mathcal{F}_2$. Let $(A_n \times B_n)_{n \geq 1} \subset \mathcal{F}_1 \times \mathcal{F}_2$ be a disjoint sequence, then

$$\begin{aligned} [\mu * \upsilon(\widehat{\bigcup_{n=1}^{\infty} (A_n} \times B_n))](\sigma) &= \sum_{n=1}^{\infty} [\mu * \widehat{\upsilon(A_n \times} B_n)](\sigma) \\ &= \lim_{n \to \infty} \sum_{k=1}^{n} [\mu * \widehat{\upsilon(A_k \times} B_k)](\sigma) \\ &= \lim_{n \to \infty} [\sum_{k=1}^{n} \mu * \widehat{\upsilon(A_k \times B_k)}](\sigma) \end{aligned}$$

On the other hand,

$$
\begin{aligned}
|[\sum_{k=1}^{n} \widehat{\mu * \upsilon(A_k \times B_k)}](\sigma)| &\leq \sum_{k=1}^{n} |[\mu * \upsilon\widehat{(A_k \times} B_k)](\sigma)| \\
&= \sum_{k=1}^{n} |\widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma)(A_k \times B_k)| \\
&\leq \sum_{k=1}^{n} |\widehat{\mu(A_k)}(\sigma)||\widehat{\upsilon(B_k)}(\sigma)| \\
&\leq \lambda_\sigma^{2p} \sum_{k=1}^{n} (|\mu|_p \times |\upsilon|_p)(A_k \times B_k)) \\
&= \lambda_\sigma^{2p} (|\mu|_p \times |\upsilon|_p)(\bigcup_{k=1}^{n} (A_k \times B_k))
\end{aligned}
$$

by Lemma 2.4, $\sum_{k=1}^{n} \mu * \upsilon(A_k \times B_k)$ converges weakly to $\mu * \upsilon(\bigcup_{n=1}^{\infty}(A_n \times B_n))$. By Definition 3.1, $\mu * \upsilon$ is a generalized functional-valued measure. By Lemma 2.3, for $q > 2p + \frac{1}{2}$

$$
\|\mu * \upsilon(A \times B)\|_{-q} \leq [\sum_{\sigma \in \Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} (|\mu|_p \times |\upsilon|_p)(A \times B),
$$

which means that $\mu * \upsilon$ is a p-bounded variation measure. Finally, the uniqueness of $\mu * \upsilon$ is immediate from the uniqueness of $\widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma)$. □

Definition 5.3 *A function $\Phi : \Omega_1 \times \Omega_2 \to \mathcal{S}^*(M)$ is called strongly measurable if for $p \geq 0$, there exists a sequence of simple functions $\{\Phi_n\}_{n \geq 0}$ with*

$$
\lim_{n \to \infty} \|\Phi(\omega_1, \omega_2) - \Phi_n(\omega_1, \omega_2)\|_{-p} = 0, \qquad |\mu|_p \times |\upsilon|_p - \text{a.e.}
$$

Theorem 5.4 *Suppose $\Phi(\omega_1, \omega_2)$ is a strongly measurable function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$. Then both $\int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon$ and $\int_{\Omega_1} \Phi(\cdot, \omega_2) * d\mu$ are strongly measurable, and in addition, they are both Bochner-convolution integrable.*

Proof We only prove the statement for $\int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon$. Let

$$
\mathfrak{M} = \{H \in \mathcal{F}_1 \times \mathcal{F}_2, \int_{\Omega_2} I_H(\omega_1, \cdot) * d\upsilon \text{ be strongly measurable}\}
$$

and

$$
\mathfrak{S} = \{A \times B \mid A \in \mathcal{F}_1, B \in \mathcal{F}_2\}.
$$

We can easily prove that $\mathfrak{M}$ is a monotone class and then the Dynkin's monotone class theorem tells us that $\mathfrak{M} \supset \sigma(\mathfrak{S}) = \mathcal{F}_1 \times \mathcal{F}_2$, so $\int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon$ is strongly measurable for any simple function Φ, choose a sequence of simple functions Φ_n such that $\Phi_n \uparrow \Phi$. Then for any $\omega_1 \in \Omega_1$ and $q > 2p + \frac{1}{2}$,

$$\begin{aligned}&\| \int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon - \int_{\Omega_2} \Phi_n(\omega_1, \cdot) * d\upsilon \|_{-q} \\ &\leq [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} \int_{\Omega_2} \|\Phi(\omega_1, \cdot) - \Phi_n(\omega_1, \cdot)\|_{-p} d|\upsilon|_p \\ &\to 0 (n \to \infty)\end{aligned}$$

which means that $\int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon$ is strongly measurable.

On the other hand,

$$\begin{aligned}&\int_{\Omega_1} \| \int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon \|_{-q} d|\mu|_p \\ &\leq \int_{\Omega_1} [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} [\int_{\Omega_2} \|\Phi(\omega_1, \cdot)\|_{-p} d|\upsilon|_p] d|\mu|_p \\ &\leq [\sum_{\sigma\in\Gamma} \lambda_\sigma^{-2(q-2p)}]^{\frac{1}{2}} \int_{\Omega_1\times\Omega_2} \|\Phi(\omega_1, \omega_2)\|_{-p} d(|\mu|_p \times |\upsilon|_p) \\ &< \infty\end{aligned}$$

By Theorem 4.6, $\int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon$ is Bochner-convolution integrable. □

Theorem 5.5 *Suppose $\Phi(\omega_1, \omega_2)$ is a strongly measurable function on $(\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2)$. Then the three integrals in the following equation exists and satisfies*

$$\int_{\Omega_1\times\Omega_2} \Phi(\omega_1, \omega_2) * d(\mu * \upsilon) = \int_{\Omega_1} [\int_{\Omega_2} \Phi(\omega_1, \cdot) * d\upsilon] * d\mu = \int_{\Omega_2} [\int_{\Omega_1} \Phi(\cdot, \omega_2) * d\mu] * d\upsilon \tag{5.1}$$

Proof The existence of the integrals are guaranteed by the preceding theorem. We only need to prove the second half of (5.1). By Theorem 4.8, we have

$$\begin{aligned}[\widehat{\int_{\Omega_1\times\Omega_2} \Phi(\omega_1, \omega_2) * d(\mu * \upsilon)}](\sigma) &= \int_{\Omega_1\times\Omega_2} \widehat{\Phi(\omega_1, \omega_2)}(\sigma) d\widehat{\mu}(\sigma) \times \widehat{\upsilon}(\sigma) \\ &= \int_{\Omega_2} [\int_{\Omega_1} \widehat{\Phi(\omega_1, \omega_2)}(\sigma) d\widehat{\mu}(\sigma)] d\widehat{\upsilon}(\sigma) \\ &= \int_{\Omega_2} [\widehat{\int_{\Omega_1} \Phi(\omega_1, \omega_2) * d\mu}](\sigma) d\widehat{\upsilon}(\sigma) \\ &= [\widehat{\int_{\Omega_2} (\int_{\Omega_1} \Phi(\omega_1, \omega_2) * d\mu) * d\upsilon}](\sigma)\end{aligned}$$

By Theorem 13 in paper [11], we have

$$\int_{\Omega_1\times\Omega_2} \Phi(\omega_1, \omega_2) * d(\mu * \upsilon) = \int_{\Omega_2} [\int_{\Omega_1} \Phi(\omega_1, \omega_2) * d\mu] * d\upsilon.$$

This completes the proof. □

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