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Digital Hurewicz theorem and digital homology theory

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Abstract: In this paper, we develop homology groups for digital images based on cubical singular homology theory for topological spaces. Using this homology, we obtain two main results that make this homology different from already-existing homologies of digital images. We prove digital analog of Hurewicz theorem for digital cubical singular homology. We also show that the homology functors developed in this paper satisfy properties that resemble the Eilenberg–Steenrod axioms of homology theory, in particular, the homotopy and the excision axioms. We finally define axioms of digital homology theory.

Key words: Digital topology, digital homology theory, digital Hurewicz theorem, cubical singular homology for digital images, digital excision

1. Introduction

Digital topology is a developing research area, where topological properties of digital images are explored. In this area, digital images are mostly defined as subsets of $\mathbb{Z}^d$, equipped with certain adjacency relations. Though digital images are discrete in nature, they model continuous objects of the real world. Researchers are trying to understand whether or not digital images show similar properties as their continuous counterparts. The main motivation behind such studies is to develop a theory for digital images that is similar to the theory of topological spaces in classical topology. Due to discrete nature of digital images, it is difficult to get results that are analogous to those in classical topology.

Several notions that are well-studied in general topology and algebraic topology have been developed for digital images, which include continuity of functions [2, 19], Jordan curve theorem [18, 21], covering spaces [14], fundamental group [3, 12], homotopy (see [4, 5]), homology groups [1, 6, 7, 11, 15], cohomology groups [7], H-spaces [8], and fibrations [9].

The idea of digital fundamental group was first introduced by Kong [12]. Boxer [3] adopted a classical approach to define and study digital fundamental group, which was closer to the methods of algebraic topology. Digital simplicial homology groups were introduced by Arslan et al. [1] and extended by Boxer et al. [6]. Eilenberg–Steenrod axioms for digital simplicial homology groups of digital images were investigated by Ege and Karaca in [7], where it was claimed that homotopy and excision axioms do not hold in digital simplicial setting. They demonstrate using an example that Hurewicz theorem does not hold in case of digital simplicial homology groups.

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Karaca and Ege [11] developed the digital cubical homology groups in a similar way to the cubical homology groups of topological spaces in algebraic topology. Unlike the case of algebraic topology, digital cubical homology groups are in general not isomorphic to digital simplicial homology groups studied in [6]. Furthermore, Mayer–Vietoris theorem fails for cubical homology on digital images, which is another contrast to the case of algebraic topology. In [15], singular homology group of digital images were developed, where digital analogs of Hurewicz theorem and Eilenberg–Steenrod axioms of homology theory were not proven.

In this paper, we introduce digital cubical singular homology groups for digital images, based on cubical singular homology groups of topological spaces [16]. We found out that unlike digital simplicial homology group, digital cubical homology group, and digital singular homology group, the digital cubical singular homology group relates to the digital fundamental group [3], in the same way as in algebraic topology. Furthermore, the digital cubical singular homology groups also satisfy digital analogs the Eilenberg–Steenrod axioms of homology theory, which makes it different from other three homologies developed for digital images.

This paper is organized as follows. We review some of the basic concepts of digital topology in Section 2. We develop homology groups of digital images based on cubical singular homology of topological spaces as given in [16] in Section 3, and give some basic results including the functoriality, additivity, and homotopy invariance of cubical singular homology groups. In Section 4, we show that the digital fundamental group (given by [3]) is related to our first homology group, and obtain a result that is analogous to Hurewicz theorem of algebraic topology. In Section 5, we prove a result for cubical singular homology on digital images (Theorem 5.6) similar to the excision theorem of algebraic topology except that our result holds only in dimensions less than 3. This result is then generalized and we call this generalization 'Excision-like property' for cubical singular homology on digital images (Theorem 5.10). Cubical singular homology groups satisfy properties that are much similar to the Eilenberg–Steenrod axioms of homology theory.

We define digital homology theory in Section 6, the axioms of which can be regarded as digital version of Eilenberg–Steenrod axioms in algebraic topology and show that digital cubical singular homology is a digital homology theory. Throughout this paper, we consider finite binary digital images, though most of the results also hold for infinite case.

2. Preliminaries

2.1. Basic concepts of digital topology

Let $\mathbb{Z}^d$ be the Cartesian product of d copies of set of integers $\mathbb{Z}$, for a positive integer d. A relation that is symmetric and irreflexive is called an adjacency relation. A digital image is a subset of $\mathbb{Z}^d$, with an adjacency relation.

In digital images, adjacency relations give a concept of proximity or closeness among its elements, which allows some constructions in digital images that closely resemble those in topology and algebraic topology. The adjacency relations on digital images used in this paper are defined below.

Definition 2.1 [4] *Consider a positive integer l, where $1 \leq l \leq d$. The points $p, q \in \mathbb{Z}^d$ are said to be c_l-adjacent if they are different and there are at most l coordinates of p and q that differ by one unit, while the rest of the coordinates are equal.*

Usually the notation c_l is replaced by number of points κ that are c_l-adjacent to a point. For $\mathbb{Z}^2$, there are 4 points that are c_1-adjacent to a point and there are 8 points that are c_2-adjacent to a point, thus $c_1 = 4$ and

$c_2 = 8$. Two points that are κ-adjacent to each other are said to be κ-neighbors of each other. For $a, b \in \mathbb{Z}$, $a < b$, a *digital interval* denoted as $[a,b]_{\mathbb{Z}}$ is a set of integers from a to b, including a and b. The digital image $X \subseteq \mathbb{Z}^d$ equipped with adjacency relation κ is represented by the ordered pair (X,κ).

Definition 2.2 [12][3] *Let (X,κ) and (Y,λ) be digital images.*

(i) *The function $f : X \to Y$ is (κ,λ)-continuous if for every pair of κ-adjacent points x_0 and x_1 in X, either the images $f(x_0)$ and $f(x_1)$ are equal or λ-adjacent.*

(ii) *Digital image (X,κ) is said to be (κ,λ)-homeomorphic to (Y,λ) , if there is a (κ,λ)-continuous bijection $f : X \to Y$, which has a (λ,κ)-continuous inverse $f^{-1} : Y \to X$.*

(iii) *A κ-path in (X,κ) is a $(2,\kappa)$-continuous function $f : [0,m]_{\mathbb{Z}} \to X$. We say f is κ-path of length m from $f(0)$ to $f(m)$. For a given κ-path f of length m, we define reverse κ-path $\overline{f} : [0,m]_{\mathbb{Z}} \to X$ defined by $\overline{f}(t) = f(m-t)$. A κ-loop is a κ-path $f : [0,m]_{\mathbb{Z}} \to X$, with $f(0) = f(m)$.*

(iv) *A subset $A \subset X$ is κ-connected if and only if for all $x, y \in A$, $x \neq y$, there is a κ-path from x to y. A κ-component of a digital image is the maximal κ-connected subset of the digital image.*

Definition 2.3 *Consider digital images (X,κ) and (Y,κ) with $X, Y \subset \mathbb{Z}^d$.*

- *We say that (X,κ) is κ-connected with (Y,κ) if there is $x \in X$ and $y \in Y$ such that x and y are κ-adjacent in $\mathbb{Z}^d$.*
- *If (X,κ) is not κ-connected with (Y,κ), we say that (X,κ) is κ-disconnected with (Y,κ).*

Proposition 2.4 [3] *If $f : X \to Y$ is a (κ,λ)-continuous function, with $A \subset X$ a κ-connected subset, then $f(A)$ is λ-connected in Y.*

Definition 2.5 [3]

(i) *Let $f, g : X \to Y$ be (κ,λ)-continuous functions. Suppose there is a positive integer m and a function $H : [0,m]_{\mathbb{Z}} \times X \to Y$ such that:*

- *for all $x \in X$, $H(0,x) = f(x)$ and $H(m,x) = g(x)$,*
- *for all $x \in X$, the function $H_x : [0,m]_{\mathbb{Z}} \to Y$ defined by $H_x(t) = H(t,x)$ for all $t \in [0,m]_{\mathbb{Z}}$ is $(2,\lambda)$-continuous,*
- *for all $t \in [0,m]_{\mathbb{Z}}$, the function $H_t : X \to Y$ defined by $H_t(x) = H(t,x)$ for all $x \in X$ is (κ,λ)-continuous.*

Then H is called (κ,λ)-homotopy from f to g and f and g are said to be (κ,λ)-homotopic, denoted as $f \simeq_{(\kappa,\lambda)} g$. If g is a constant function, H is a null-homotopy and f is null-homotopic.

(ii) *Two digital images (X,κ) and (Y,λ) are homotopically equivalent, if there is a (κ,λ)-continuous function $f : X \to Y$ and (λ,κ)-continuous function $g : Y \to X$ such that $g \circ f \simeq_{(\kappa,\lambda)} 1_X$ and $f \circ g \simeq_{(\lambda,\kappa)} 1_Y$, where 1_X and 1_Y are identity functions on X and Y, respectively.*

(iii) *Let* $H : [0,m]_{\mathbb{Z}} \times [0,n]_{\mathbb{Z}} \to X$ *be a homotopy between* κ*-paths* $f, g : [0,n]_{\mathbb{Z}} \to X$ *in* (X,κ)*. The homotopy* H *is said to hold the end-points fixed if* $f(0) = H(t,0) = g(0)$ *and* $f(n) = H(t,n) = g(n)$ *for all* $t \in [0,m]_{\mathbb{Z}}$*.*

2.2. Digital fundamental group

The concept of digital fundamental group of a digital image was first given by [12], but a more classical approach to define and study digital fundamental group was adopted by Boxer [3]. We briefly explain digital fundamental group as defined in the latter paper.

Definition 2.6 [3]

(i) *A pointed digital image is a pair* (X,p)*, where* X *is a digital image and* $p \in X$*. A pointed digital image* (X,p) *can be represented as* $((X,p),\kappa)$*, if one wishes to emphasize the adjacency relation of the digital image* X*.*

(ii) *Let* f *and* g *be* κ*-paths of lengths* m_1 *and* m_2*, respectively, in the pointed digital image* (X,p)*, such that* g *starts where* f *ends, i.e.* $f(m_1) = g(0)$*. The 'product'* $f * g$ *of two paths is defined as follows:*

$$(f * g)(t) = \begin{cases} f(t), & \text{if } t \in [0,m_1]_{\mathbb{Z}} \\ g(t-m_1), & \text{if } t \in [m_1, m_1+m_2]_{\mathbb{Z}}. \end{cases}$$

The concept of trivial extension allows stretching the domain of a loop, without changing its homotopy class and thus allows to compare homotopy properties of paths even when the cardinalities of their domain differ.

Definition 2.7 [3]

(i) *Let* f *and* f' *be* κ*-paths in a pointed digital image* (X,p)*. We say that* f' *is a trivial extension of* f*, if there exist sets of* κ*-paths* $\{f_1, f_2, \ldots, f_k\}$ *and* $\{f'_1, f'_2, \ldots, f'_n\}$ *in* X *such that*

- $0 < k \le n$
- $f = f_1 * f_2 * \cdots * f_k$
- $f' = f'_1 * f'_2 * \cdots * f'_n$
- *there are indices* $1 \le i_1 < i_2 < \cdots < i_k \le n$ *such that:*
  - $f'_{i_j} = f_j, 1 \le j \le k$ *and*
  - $i \notin \{i_1, i_2, \ldots, i_k\}$ *implies* f'_i *is a constant* κ*-path.*

(ii) *Two* κ*-loops* f *and* g *with the same basepoint* $p \in X$ *belong to the same loop class, if there exist trivial extensions of* f *and* g*, which have homotopy between them that holds the end-points fixed.*

Definition 2.8 [3] *Let* $\Pi_1^\kappa(X,p)$ *be the set of loop classes in* (X,p) *with basepoint* p*. Let* $[f]_\Pi$ *denote the loop class of* κ*-loop* f *in* (X,κ)*. The product operation* $*$ *defined as:*

$$[f]_\Pi * [g]_\Pi = [f * g]_\Pi$$

is well defined on $\Pi_1^\kappa(X,p)$ *as well as associative [3]. The loop class* $[c]_\Pi$ *of the constant loop is identity in* $\Pi_1^\kappa(X,p)$ *with respect to taking product. For every loop class* $[f]_\Pi$ *the loop class* $[\overline{f}]_\Pi$*, where* $\overline{f}$ *is the reverse path of* f*, is the inverse of* $[f]_\Pi$ *with respect to taking product* $*$*. Thus,* $\Pi_1^\kappa(X,p)$ *is a group under* $*$ *and called the digital fundamental group of the pointed digital image* (X,p)*.*

3. Cubical singular homology on digital images

Consider digital interval $I = [0,1]_{\mathbb{Z}}$. Let I^n be the Cartesian product of n copies of I for $n > 0$. We shall consider I^n as a digital image $(I^n, 2n)$. By definition, I^0 is a digital image consisting of single point. For an integer $n \geq 0$, a digitally singular n-cube or briefly a digital n-cube in (X,κ) is a $(2n,\kappa)$-continuous map $T : I^n \to X$.

For an integer $n \geq 0$, let $dQ_{n,\kappa}(X)$ denote the free Abelian group generated by the set of all digitally singular n-cubes in (X,κ). We write $dQ_n(X)$ for $dQ_{n,\kappa}(X)$, when the adjacency relation is clear from the context. An element of $dQ_n(X)$ is a finite formal linear combination of digital n-cubes. The basis of the group $dQ_0(X)$ can be identified with X itself, and one can denote the elements of $dQ_0(X)$ as $\sum_i m_i x_i$, where $x_i \in X$. A digitally singular n-cube $T : I^n \to X$ is degenerate if there is an integer i, $1 \leq i \leq n$ such that $T(t_1, t_2, \ldots, t_n)$ does not depend on t_i. Let $dD_{n,\kappa}(X)$, or simply $dD_n(X)$, denote the subgroup of $dQ_n(X)$ generated by the set of all degenerate digitally singular n-cubes in (X,κ). Let $dC_{n,\kappa}(X)$, or simply $dC_n(X)$, denote the quotient group $dQ_n(X)/dD_n(X)$. We say $dC_n(X)$ is the group of digitally cubical singular n-chains in (X,κ) and the elements of $dC_n(X)$ are n-chains in (X,κ). For any digital image X, $dC_n(X)$ can be shown as free Abelian group generated by nondegenerate digital n-cubes in X.

We define faces of a digitally singular n-cube as follows: For a digital n-cube $T : I^n \to X$ and $i = 1, 2, \ldots, n$, we define digital $(n-1)$-cubes $A_iT, B_iT : I^{n-1} \to X$ as

$$\begin{aligned} & A_iT(t_1, t_2, \ldots, t_{n-1}) = T(t_1, t_2, \ldots, t_{i-1}, 0, t_i, \ldots, t_{n-1}), \\ \text{and} \quad & B_iT(t_1, t_2, \ldots, t_{n-1}) = T(t_1, t_2, \ldots, t_{i-1}, 1, t_i, \ldots, t_{n-1}). \end{aligned}$$

A_iT and B_iT are called front i-face and back i-face of T, respectively.

We define the boundary operator ∂_n on the basis element of $dQ_n(X)$ as $\partial_n(T) = \sum_{i=1}^{n} (-1)^i (A_iT - B_iT)$ and extend it by linearity (see [20], for the definition of extension by linearity) to get the homomorphism $\partial_n : dQ_n(X) \to dQ_{n-1}(X)$, $n \geq 1$. One may write ∂ for ∂_n if n is clear from the context. For $n < 0$, let $dQ_n(X) = dC_n(X) = 0$ and for $n \leq 0$, let $\partial_n = 0$. It can be shown that $\partial_{n-1}\partial_n = 0$, for all integers n (see [16] for details). A cubical singular complex of the digital image (X,κ), denoted as $(C_{\bullet,\kappa}(X), \partial)$ or $(dC_\bullet(X), \partial)$, is the following chain complex:

$$\cdots \xrightarrow{\partial_{n+1}} dC_n(X) \xrightarrow{\partial_n} dC_{n-1}(X) \xrightarrow{\partial_{n-1}} \cdots$$

Let $dZ_n(X)$ denote the kernel of ∂_n and $dB_n(X)$ denote the image of ∂_{n+1}, for all integers n. The elements of $dZ_n(X)$ and $dB_n(X)$ are called n-cycles and n-boundaries of (X,κ), respectively. We define n^{th} cubical singular homology group of the digital image (X,κ), as $dH_{n,\kappa}(X) = H_n(dC_\bullet, \partial) = dZ_n(X)/dB_n(X)$, for all non-negative integers n. If the adjacency relation κ is clear from context, we shall simply write $dH_n(X)$ for $dH_{n,\kappa}(X)$.

κ-path and digital 1-cubes: A digital 1-cube $T : I \to X$ in a digital image (X, κ) can be considered a κ-path of length 1. A κ-path f of length m can be "subdivided" into smaller paths of length 1 or digital 1-cubes. For a κ-path f of length m, we can associate an element $\sum_{j=1}^{m} f_j$ of $dQ_1(X)$ to f, where $f_j : I \to X$ as $f_j(t) = f(j+t-1)$. We say that the element $\sum_{j=1}^{m} f_j$ is subdivision of f. The following are some properties of subdivision $\sum_{j=1}^{m} f_j$ of f:

1. f_j are degenerate, whenever $f(j-1) = f(j)$
2. If f is a nonconstant path then $\sum_{j=1}^{m} f_j$ is not degenerate, and so $\sum_{j=1}^{m} f_j$ is a nontrivial element in $dC_1(X)$, where some f_j might be 0 in $dC_1(X)$.
3. $\partial \left(\sum_{j=1}^{m} f_j \right) = f(m) - f(0)$.
4. If f is a κ-loop then $\sum_{j=1}^{m} f_j$ is a 1-cycle.

Proposition 3.1 *If (X, κ) be a nonempty κ-connected digital image, then $dH_0(X) \approx \mathbb{Z}$.*

Proof Consider the map $\varepsilon : dC_0(X) \to \mathbb{Z}$ defined as $\sum_i m_i x_i \mapsto \sum_i m_i$. Now for $\sum_i n_i T_i \in dC_1(X)$, we have $\varepsilon \circ \partial(\sum_i n_i T_i) = \varepsilon(\sum_i n_i(B_1 T - A_1 T)) = \sum_i (n_i - n_i) = 0$. Thus, $dB_0(X) \subset ker(\varepsilon)$. The reverse relation also holds for the following reason. Consider $\sum_i m_i x_i \in ker(\varepsilon)$. We have $\sum_i m_i = 0$. Consider $x \in X$ (X is non-empty) and κ-paths f_i (X is κ-connected) from x to x_i. These paths can be subdivided to form elements $\sum_j f_{ij} \in dC_1(X)$ for each i. It can be verified that $\partial(\sum_j f_{ij}) = x_i - x$. Thus, $\partial(\sum_{i,j} m_i f_{ij}) = \sum_i m_i x_i - (\sum_i m_i)x = \sum_i m_i x_i$, implying $\sum_i m_i x_i \in dB_0(X)$. From first isomorphism theorem of groups $dH_0(X) = dZ_0(X)/dB_0(X) = dC_0(X)/dB_0(X) \approx \mathbb{Z}$. □

Proposition 3.2 *Let $\{X_\alpha | \alpha \in \Lambda\}$ be the set of κ-components of the digital image (X, κ). Then $dH_n(X) \approx \bigoplus_\alpha dH_n(X_\alpha)$.*

Proof The groups $dQ_n(X)$, $dD_n(X)$ and $dC_n(X)$ break up to $\bigoplus_\alpha dQ_n(X_\alpha)$, $\bigoplus_\alpha dD_n(X_\alpha)$ and $\bigoplus_\alpha dC_n(X_\alpha)$, respectively, because the image of each digital n-cube T lies entirely in one κ-component of (X, κ) (see Section 2). We also have $dZ_n(X) = \bigoplus_\alpha dZ_n(X_\alpha)$ and $dB_n(X) = \bigoplus_\alpha dB_n(X_\alpha)$; hence, $dH_n(X) = \bigoplus_\alpha dH_n(X_\alpha)$, because the boundary map $\partial_n : dC_n(X) \to dC_{n-1}(X)$ maps $dC_n(X_\alpha)$ to $dC_{n-1}(X_\alpha)$. □

Proposition 3.3 *For any digital image (X, κ), $dH_0(X)$ is a free Abelian group with rank equal to the number of κ-components of (X, κ).*

Proof Follows from Propositions 3.1 and 3.2. □

Proposition 3.4 *The cubical singular homology group $dH_n(-)$ is a functor from Dig to Ab.*

Proof We define $dH_n(-)$ on morphisms of Dig as follows: Consider a (κ, λ) -continuous function $f : X \to Y$ from digital image (X, κ) to digital image (Y, λ). For a digital n-cube $T : I^n \to X$ in $dQ_n(X)$, we have $f \circ T \in dQ_n(Y)$. We define functions $f_\# : dQ_n(X) \to dQ_n(Y)$ as $T \mapsto f \circ T$ and extending by linearity, for integers $n \geq 0$. Since $f_\#(T)$ is degenerate, if $T \in dD_n(X)$, the map $f_\#$ induces $f_\# : dC_n(X) \to dC_n(Y)$, for integers $n \geq 0$. It can be shown that $f_\#$ is a chain map that sends n-cycles to n-cycles and n-boundaries to n-boundaries, and therefore induces a map $f_* = dH_n(f) : dH_n(X) \to dH_n(Y)$ defined as $[T] \mapsto [f_\#(T)]$.

Furthermore, it can be easily shown that for an identity map $id : X \to X$ the induced map $id_* = dH_n(id) : dH_n(X) \to dH_n(X)$ is an identity map. Also for functions $f : X \to Y$ and $g : Y \to Z$, which are (κ, λ)- and (λ, γ)-continuous, we have $(g \circ f)_* = g_* \circ f_* : dH_n(X) \to dH_n(Z)$, because $(g \circ f)_\# = g_\# \circ f_\# : dQ_n(X) \to dQ_n(Z)$. □

The following can be easily proved.

Proposition 3.5 *Let (X, κ) and (Y, λ) be (κ, λ)-homeomorphic digital images, then $dH_n(X) = dH_n(Y)$, for all n.*

Proposition 3.6 *If $X = \{x_0\}$ is a one-point digital image, then*

$$dH_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 3.7 *Let $f, g : X \to Y$ be (κ, λ)-homotopic maps from digital image (X, κ) to the digital image (Y, λ). Then f and g induce the same maps on homology group $dH_n(X)$, i.e. $f_* = g_*$.*

Proof Let $F : [0, m]_\mathbb{Z} \times X \to Y$ be the homotopy from f to g. The homotopy F can be subdivided into functions $F_j : I \times X \to Y$ defined as $F_j(t, x) = F(j + t - 1, x)$ for $j \in [1, m]_\mathbb{Z}$. Observe that $F_1(0, x) = f(x)$ and $F_m(1, x) = g(x)$. In order to show that $f_* = g_*$, we follow the standard method of algebraic topology, which is, to construct a map $\Phi_n : dQ_n(X) \to dQ_{n+1}(Y)$ that contains similar information as the Homotopy F, and satisfies:

$$g_\# - f_\# = \partial_{n+1}\Phi_n + \Phi_{n-1}\partial_n \tag{3.1}$$

Define $\Phi_n : dQ_n(X) \to dQ_{n+1}(Y)$ as $T \mapsto \sum_{j=1}^{m} F_j(id \times T)$ and extending by linearity, where $id : [0, 1]_\mathbb{Z} \to [0, 1]_\mathbb{Z}$ is identity function. We need to compute the boundary $\partial\Phi$ to verify eq. 3.1. One can observe the following:

$$A_1\Phi_n T = f_\#(T) + \sum_{j=2}^{m} F_j(0, T) \quad \text{and} \quad B_1\Phi_n(T) = \sum_{j=1}^{m-1} F_j(1, T) + g_\#(T) \tag{3.2}$$

$$A_i\Phi_n(T) = \Phi_{n-1}A_{i-1}T, \quad \text{and} \quad B_i\Phi_n(T) = \Phi_{n-1}B_{i-1}T, \quad i \in [2, n+1]_\mathbb{Z} \tag{3.3}$$

$$F_j(1, T) = F_{j+1}(0, T), \quad j \in [1, m-1]_\mathbb{Z} \tag{3.4}$$

Using these equations we can calculate the boundary of Φ:

$$\partial\Phi_n(T) = \sum_{i=1}^{n+1}(-1)^i(A_i\Phi_n(T) - B_i\Phi_n(T))$$

$$= g_{\#}(T) - f_{\#}(T) + \sum_{i=2}^{n+1}(-1)^i(A_i\Phi_n(T) - B_i\Phi_n(T))$$

using eqs. 3.2 and 3.4, for $i = 1$, and using eqs. 3.3

and substituting $j = i - 1$ for $i > 1$

$$= g_{\#}(T) - f_{\#}(T) - \Phi_{n-1}\partial T \quad \text{by definition of } \partial(T)$$

It can be shown that Φ maps degenerate digital n-cubes in (X,κ) to degenerate digital $(n+1)$-cubes in (Y,λ), inducing a homomorphism $\varphi_n : dC_n(X) \to dC_{n+1}(Y)$. If we choose T to be a nondegenerate n-cycle, i.e. $T \in dZ_n(X)$, then we get $g_{\#}(T) - f_{\#}(T) \in dB_n(Y)$. Therefore, in $dH_n(Y)$ we have,

$$[g_{\#}(T) - f_{\#}(T)] = g_*([T]) - f_*([T]) = 0 \quad \Rightarrow \quad g_* = f_*$$

□

Corollary 3.8 *If (X,κ) and (Y,λ) be homotopically equivalent digital images, then $dH_n(X) \approx dH_n(Y)$.*

Proof Follows from Proposition 3.7, and functoriality of dH_n. □

Example 3.9 *A digital image is said to be κ-contractible [3], if its identity map is (κ,κ)-homotopic to a constant function c_p for some $p \in X$. For a κ-contractible digital image (X,κ), one can compute the homology groups using Propositions 3.6 and 3.7 as $dH_n(X) = \begin{cases} \mathbb{Z}, & \text{if } n = 0 \\ 0, & \text{otherwise,} \end{cases}$ because a κ-contractible digital image is homotopy equivalent to a point [3].*

4. Digital Hurewicz theorem

Lemma 4.1 *Let (X,p,κ) be a digital image with basepoint p and κ-adjacency relation and $\Pi_1^\kappa(X,p)$ be the digital fundamental group. Then there is a homomorphism $\varphi : \Pi_1^\kappa(X,p) \to dH_1(X)$ given by $[f]_\Pi \mapsto \left[\sum_{j=1}^m f_j\right]$, where $\sum_{j=1}^m f_j$ is the subdivision of κ-loop f.*

Proof *Well-defined*: We need to show that φ is a well-defined. Consider κ-loops f and g of lengths m_1 and m_2, respectively, both based at point $p \in X$ such that $[f]_\Pi = [g]_\Pi \in \Pi_1^\kappa(X,p)$. Now f and g are in the same loop class implies that there are trivial extensions f' and g' of f and g, respectively such that there exists a homotopy $H : [0,m]_\mathbb{Z} \times [0,M]_\mathbb{Z} \to X$ from f' to g' that holds the end points fixed. Subdivide H into digital 2-cubes $j,k : I^2 \to X$ defined as $(s,t) \mapsto H(j+s-1, k+t-1)$, for $j \in [1,m]_\mathbb{Z}$ and $k \in [1,M]_\mathbb{Z}$ (see Figure 1). We shall show that the boundary $\partial\left(\sum_{j,k} H_{j,k}\right)$ is equal to the difference of $\sum_{j=1}^{m_1} f_j$ and $\sum_{j=1}^{m_2} g_j$, which implies that the classes of these subdivisions are equal in the homology group $dH_1(X)$.

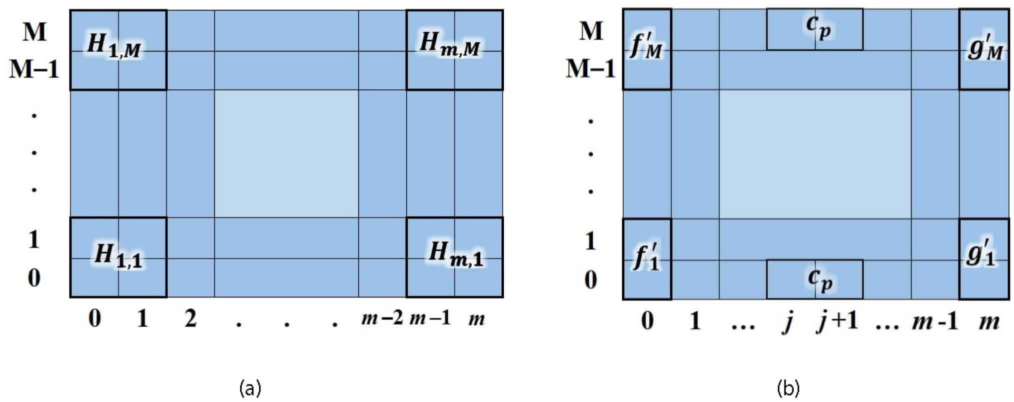


Figure 1. Domain of H (proof of Lemma 4.1). (a) Subdivision of H into digital 2-cubes H_{jk} (b) digital 1-cubes involved in $\partial(H_{jk})$.

Before computing $\partial\left(\sum_{j,k} H_{j,k}\right)$, note that the following equations hold:

$$\sum_{k=1}^{M} A_1 H_{1,k} = \sum_{j=1}^{M} f'_j = \sum_{j=1}^{m_1} f_j \quad \text{and} \quad \sum_{k=1}^{M} B_1 H_{m,k} = \sum_{j=1}^{M} g'_j = \sum_{j=1}^{m_2} g_j \tag{4.1}$$

The only difference between f and its trivial extension f' is that f' pauses more frequently for rest than f and whenever a path pauses for rest, its subdivision is trivial at that point in $dC_1(X)$ (being degenerate in $dQ_1(X)$). Further, it can be noted that:

$$\begin{aligned} &A_1 H_{j,k} = B_1 H_{j-1,k}, \;\; j \in [2,m]_{\mathbb{Z}}, k \in [1,M]_{\mathbb{Z}} \\ \text{and} \quad &A_2 H_{j,k} = B_2 H_{j,k-1}, \;\; j \in [1,m]_{\mathbb{Z}}, k \in [2,M]_{\mathbb{Z}} \end{aligned} \tag{4.2}$$

$$A_2 H_{j,1} = B_2 H_{j,M} = c_p, \quad j \in [1,m]_{\mathbb{Z}}, \tag{4.3}$$

where c_p is the constant path of length 1 at basepoint $p \in X$. Using eqs. 4.1 to 4.3, it can be shown that $\partial\left(\sum_{j,k} H_{j,k}\right) = \sum_{j=1}^{m_2} g_j - \sum_{j=1}^{m_1} f_j \in dC_1(X)$. This proves that φ is well-defined.

Homomorphism: Consider κ-loops f and g of lengths m_1 and m_2, respectively, both based at point $p \in X$. Then

$$\varphi([f]_\Pi * [g]_\Pi) = \varphi([f * g]_\Pi) = \left[\sum_{j=1}^{m_1+m_2} (f*g)_j\right] = \left[\sum_{j=1}^{m_1} (f*g)_j + \sum_{j=m_1+1}^{m_1+m_2} (f*g)_j\right]$$
$$= \left[\sum_{j=1}^{m_1} f_j + \sum_{j=1}^{m_2} g_j\right] = \left[\sum_{j=1}^{m_1} f_j\right] + \left[\sum_{j=1}^{m_2} g_j\right] = \varphi([f]_\Pi) + \varphi([g]_\Pi)$$

□

We say that the map φ defined in Lemma 4.1 is *Digital Hurewicz map*.

Lemma 4.2 *Let (X, κ) be a digital image.*

1. *Consider a digital 1-cube $T \in dC_1(X)$ and let $\overline{T}$ denote the 'reverse' of T, i.e. $\overline{T} \in dC_1(X)$, $\overline{T}(t) = T(1-t)$. Then class of $T + \overline{T}$ is trivial in $dH_1(X)$.*

2. *Consider digital 2-cube $T \in dC_n(X)$ and define κ-paths T_0, T_1, T_2 and T_3 to be A_1T, A_2T, B_1T and B_2T, respectively. Then there is a trivial extension of T_0 homotopic to $T_1 * T_2 * \overline{T_3}$.*

Proof

1. Let $S : I^2 \to X$ be a basis element of $dC_2(X)$ defined as $S(t,0) = T(t)$ and $S(t,1) = T(0)$, for $t = 0,1$ (see Figure 2a). Note that the back 1-face $B_1S = \overline{T}$ (see Figure 2a) and thus the boundary $\partial S = T + \overline{T}$ in $dC_1(X)$ making the class of $T + \overline{T}$ trivial in $dH_1(X)$.

2. Consider the homotopy H defined as $H : [0,3]_{\mathbb{Z}} \times I \to X$ as
$H(0,0) = T_1(0)$, $H(1,0) = T_2(0)$, $H(2,0) = T_3(1)$, $H(3,0) = T_3(0)$,
$H(0,1) = H(1,1) = T_0(0)$, $H(2,1) = H(3,1) = T_0(1)$ (see Figures 2b and 2c).
Clearly, $H(t,0) = T_1 * T_2 * \overline{T_3}(t)$ and $H(t,1)$ is a trivial extension of T_0.

□

The following Lemma (quoted from [20] with some minor changes) is required in the proof of digital Hurewicz theorem (Theorem 4.4).

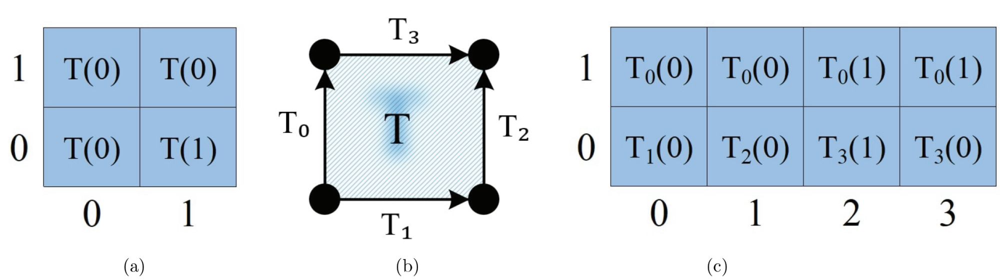


Figure 2. (a) Digital 2-cube S, and (b) faces of digital 2-cube T, (c) Homotopy H (proof of Lemma 4.2). (a) Domain of S with images labeled on each pixel (b) Schematic representation of T (c) Domain of H with images labeled on each pixel.

Lemma 4.3 Substitution principle
Let F be a free Abelian group with basis B, let $x_0, x_1, \ldots, x_N$ be a list of elements in B, possibly with repetitions and assume that $\sum_{i=0}^{k} m_i x_i = \sum_{i=k+1}^{N} m_i x_i$, where $m_i \in \mathbb{Z}$ and $0 \le k < N$. If G is any Abelian group and $y_0, y_1, \ldots, y_N$ is a list of elements in G such that $x_i = x_j \Rightarrow y_i = y_j$, then $\sum_{i=0}^{k} m_i y_i = \sum_{i=k+1}^{N} m_i y_i$ in G.

Proof Define a function $\eta : B \to G$ with $\eta(x_i) = y_i$ for all $i = 1, 2, \ldots, N$ and $\eta(x) = 0$, otherwise (η is well-defined because of the given hypothesis). Extend the map η by linearity to $\eta : F \to G$. Thus, $0 = \eta\left(\sum_{i=0}^{k} m_i x_i - \sum_{i=k+1}^{N} m_i x_i\right) = \sum_{i=0}^{k} m_i y_i - \sum_{i=k+1}^{N} m_i y_i$. □

Theorem 4.4 Digital Hurewicz theorem

If (X,κ) is a κ-connected digital image with $p \in X$ then the digital Hurewicz map (defined in Lemma 4.1) is surjective with $ker\varphi$ as commutator subgroup of the digital fundamental group $\Pi_1^\kappa(X,p)$. Hence, Abelianized digital fundamental group is isomorphic to $dH_1(X)$.

Proof *Surjectivity*: Consider $[z] \in dH_1(X)$, with $z = \sum_{i=0}^m n_i T_i$, where $T_i : I \to X$ is a nondegenerate digital 1-cube, for all i. Though $n_i \in \mathbb{Z}$, we can assume, without loss of generality, that $n_i = 1, \forall i$, for the following reason: If $n_i = 0$, no contribution is made to z by $n_i T_i$ and if $n_i < 0$ then we can replace $n_i T_i$ by $-n_i \overline{T_i}$ without changing the class $[z]$, using Lemma 4.2(1). Thus, we can assume $n_i > 0, \forall i$, but then each $n_i T_i$ can be written as $T_i + T_i + \cdots + T_i$ (n_i terms). Therefore, $z = \sum_{i=0}^m T_i$. Since z is a cycle, we have

$$\partial z = \partial \left(\sum_{i=0}^m T_i \right) = 0 \quad \Rightarrow \quad \sum_{i=0}^m (B_1 T_i - A_1 T_i) = 0. \tag{4.4}$$

For every $i \in [0,m]_\mathbb{Z}$, there exists $j \in [0,m]_\mathbb{Z}$ and $B_1 T_i = A_1 T_j$, so that the sum in eq. 4.4 is 0, but $i \neq j$, because in case $i = j$, T_i would be degenerate. Let ρ be the permutation on elements of $[0,m]_\mathbb{Z}$, satisfying the condition that $A_1 T_{\rho(i+1)} = B_1 T_{\rho(i)}$ for all $i \in [0,M]_\mathbb{Z}$, where arguments of ρ are read $\mod (M+1)$. We can take product of κ-paths $T_{\rho(i)}$ to get a κ-loop $\prod_{i=0}^m T_{\rho(i)}$ based at point $T_{\rho(0)}(0) \in X$. Since the digital image (X,κ) is κ-connected, we can take κ-path σ from p to $T_{\rho(0)}(0)$. We get:

$$\begin{aligned} \varphi\left(\left[\sigma * \prod_{i=0}^m T_{\rho(i)} * \overline{\sigma} \right]_\Pi \right) &= \left[\sum_{l=1}^M \sigma_l + \sum_{i=0}^m T_{\rho(i)} + \sum_{l=1}^M \overline{\sigma}_l \right] \\ &= \left[\sum_{l=1}^M \sigma_l + \sum_{i=0}^m T_{\rho(i)} - \sum_{l=1}^M \sigma_l \right], \text{ using Lemma 4.2(1)} \\ &= \left[\sum_{i=0}^m T_i \right] = [z]. \end{aligned}$$

Kernel of φ: Let Π' denote the commutator subgroup of $\Pi_1^\kappa(X,p)$ and $\overline{\Pi}$ denote the Abelianized digital fundamental group, i.e. $\overline{\Pi}$ is the quotient group $\Pi_1^\kappa(X,p)$ modulo the commutator subgroup Π'. Since $dH_1(X)$ is an Abelian group, $\Pi' \subset \ker\varphi$. We claim that the reverse inequality also holds. Consider a κ-loop f of length m such that $[f]_\Pi \in ker\varphi$. It suffices to show that $[\![f]\!]$ is identity in $\overline{\Pi}$, where $[\![f]\!] \in \overline{\Pi}$. Since $\varphi([f]_\Pi) = 0$, the cycle $\sum_{j=1}^m f_j$ lies in the boundary group $dB_1(X)$, i.e. there is $\sum_{i=1}^N n_i T_i \in dC_2(X)$ such that $\sum_{j=1}^m f_j = \partial(\sum_{i=1}^N n_i T_i)$, where $n_i \in \mathbb{Z}$ and $T_i : I^2 \to X$ are digital 2-cubes. We assume without loss of generality that $n_i = 1, \forall i$. Let us denote $A_1 T_i$, $A_2 T_i$, $B_1 T_i$, and $B_2 T_i$ as T_{i0}, T_{i1}, T_{i2}, and T_{i3}, respectively, for $i \in [1,N]_\mathbb{Z}$. We get

$$\sum_{j=1}^m f_j = \sum_{i=1}^M (-T_{i0} + T_{i2} + T_{i1} - T_{i3}) \tag{4.5}$$

This equation has basis elements of the free Abelian group $dC_1(X)$ on both sides. We shall apply substitution principle (Lemma 4.3) to obtain an analogous equation in $\overline{\Pi}$. We need for each term in eq. 4.5, an element in $\overline{\Pi}$, satisfying the hypothesis of substitution principle. For each $x \in X$, choose a κ-path from p to x, denoted by β_x, such that for the base point p, $\beta_p = c_p$ is a constant κ-path at p. For each $j \in [0, m]_{\mathbb{Z}}$, define κ-loops, $L'_j = \beta_{f(j-1)} * f_j * \overline{\beta_{f(j)}}$ based at p corresponding to each f_j (see Figure 3a). Similarly, define κ-loops $L_{iq} = \beta_{T_{iq}(0)} * T_{iq} * \overline{\beta_{T_{iq}(1)}}$ based at p, corresponding to each T_{iq} (see Figure 3b). We get the following in $\Pi_1^\kappa(X, p)$:

$$\begin{aligned}
&\left[\overline{L_{i0}} * L_{i1} * L_{i2} * \overline{L_{i3}}\right]_\Pi \\
&= [\beta_{T_{i0}(1)} * \overline{T_{i0}} * \overline{\beta_{T_{i0}(0)}} * \beta_{T_{i1}(0)} * T_{i1} * \overline{\beta_{T_{i1}(1)}} * \beta_{T_{i2}(0)} * T_{i2} * \overline{\beta_{T_{i2}(1)}} * \beta_{T_{i3}(1)} * \overline{T_{i3}} * \overline{\beta_{T_{i3}(0)}}]_\Pi \\
&= \left[\beta_{T_{i0}(1)} * \overline{T_{i0}} * T_{i1} * T_{i2} * \overline{T_{i3}} * \overline{\beta_{T_{i3}(0)}}\ \right]_\Pi \\
&= \left[\beta_{T_{i0}(1)} * \overline{T_{i0}} * T_{i0} * \overline{\beta_{T_{i3}(0)}}\right]_\Pi, \quad \text{using Lemma 4.2(2)} \\
&= \left[\beta_{T_{i0}(1)} * \overline{\beta_{T_{i3}(0)}}\right]_\Pi = [c_p]_\Pi
\end{aligned} \tag{4.6}$$

Second equality above follows because $T_{i0}(0) = T_{i1}(0) \Rightarrow \beta_{T_{i0}(0)} = \beta_{T_{i1}(0)}$, $T_{i1}(1) = T_{i2}(0) \Rightarrow \beta_{T_{i1}(1)} = \beta_{T_{i2}(0)}$, $T_{i2}(1) = T_{i3}(1) \Rightarrow \beta_{T_{i2}(1)} = \beta_{T_{i3}(1)}$ and $T_{i3}(0) = T_{i0}(1) \Rightarrow \beta_{T_{i3}(0)} = \beta_{T_{i0}(1)}$ (see Figure 3b) and for any κ-path ϱ, the loop $\varrho * \overline{\varrho}$ is homotopic to constant loop at $\varrho(0)$ (see Theorem 4.13 in [3]).

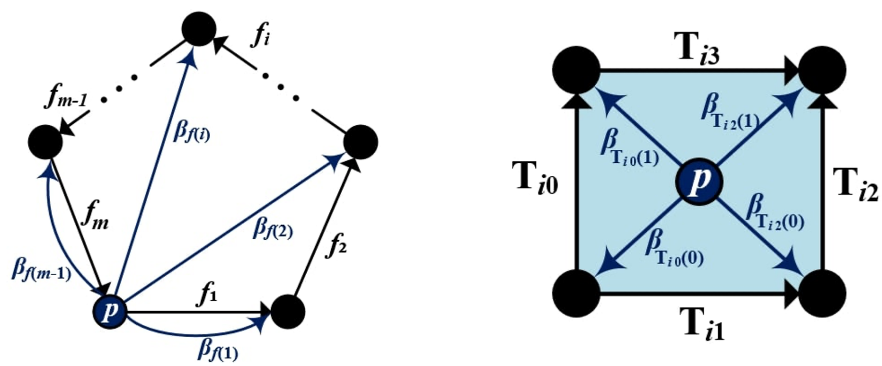


Figure 3. Schematic representation of paths β_x (proof of Theorem 4.4). Paths β_x are shown in blue color (a) from p to $T_{i0}(1)$, $T_{i1}(0)$, $T_{i2}(0)$ and $T_{i3}(1)$, and (b) from p to $f(j), j \in [0, m-1]_{\mathbb{Z}}$.

Similarly, $\left[\prod_{j=1}^m \beta_{f(j-1)} * f_j * \overline{\beta_{f(j)}}\right]_\Pi = \left[\prod_{j=1}^m f_j\right]_\Pi = [f]_\Pi$ in $\Pi_1^\kappa(X, p)$, because $\beta_{f(0)} = \overline{\beta_{f(m)}}$ is the constant path c_p at p. Therefore, we get the following in $\overline{\Pi}$,

$$\begin{aligned}
[\![f]\!] = \left[\!\!\left[\prod_{j=1}^m f_j\right]\!\!\right] &= \left[\!\!\left[\prod_{j=1}^m \beta_{f(j-1)} * f_j * \overline{\beta_{f(j)}}\right]\!\!\right] \\
&= \left[\!\!\left[\prod_{i=1}^M \overline{L_{i0}} * L_{i1} * L_{i2} * \overline{L_{i3}}\right]\!\!\right],
\end{aligned}$$

by applying substitution principle (Lemma 4.3) to eq. 4.5 for the free Abelian group $dC_1(X)$ and the multiplicative Abelian group $\overline{\Pi}$. Using eq. 4.6, $[\![f]\!]$ is trivial in $\overline{\Pi}$ and $[f]_\Pi \in \Pi'$. Therefore, the kernel of the digital Hurewicz map is the commutator of $\Pi_1^\kappa(X,p)$, and $\overline{\Pi} \approx dH_1(X)$, using first isomorphism theorem of groups. □

5. Relative homology and excision

For a digital image (X,κ) and $A \subset X$, (A,κ) is a digital image in its own right. Let $((X,A),\kappa)$ or briefly, (X,A) denote digital image pair with κ-adjacency. A map of pairs $f:(X,A)\to(Y,B)$ between digital image pairs $((X,A),\kappa)$ and $((Y,B),\lambda)$ is a map $f:X\to Y$, with $f(A)\subset B$. We say that $f:(X,A)\to(Y,B)$ is (κ,λ)-continuous if $f:X\to Y$ is (κ,λ)-continuous. It can be verified that $\partial_n : dC_n(X)\to dC_{n-1}(X)$ maps $dC_n(A)$ to $dC_{n-1}(A)$. If $dC_n(X,A)$ denotes the quotient group $dC_n(X)/dC_n(A)$, then ∂_n induces homomorphism $\partial_n : dC_n(X,A)\to dC_{n-1}(X,A)$ satisfying $\partial_{n-1}\circ\partial_n = 0$, and making up a chain complex $(dC_\bullet(X,A),\partial)$, given as:

$$\cdots \xrightarrow{\partial_{n+1}} dC_n(X,A) \xrightarrow{\partial_n} dC_{n-1}(X,A) \xrightarrow{\partial_{n-1}} \cdots$$

Let us denote the homology of this chain complex as $dH_n(X,A)$, i.e.

$$dH_n(X,A) = \frac{ker(\partial_n : dC_n(X,A)\to dC_{n-1}(X,A))}{Im(\partial_{n+1} : dC_{n+1}(X,A)\to dC_n(X,A))}.$$

We say that $dH_n(X,A)$ is n^{th}*-relative cubical singular homology group of the digital image pair* (X,A). Clearly, $dH_n(X) = dH_n(X,\emptyset)$.

Definition 5.1 *Let (X,κ) be a digital image. We define operators $Int_\kappa : \mathcal{P}(X)\to\mathcal{P}(X)$ and $Cl_\kappa : \mathcal{P}(X)\to\mathcal{P}(X)$ as follows:*

$$\begin{aligned} Int_\kappa(A) &= \{x\in A \mid N_\kappa(x,X)\subset A\},\\ Cl_\kappa(A) &= \{x\in X \mid N_\kappa(x,X)\cap A\neq\emptyset\},\\ \text{where}\quad N_\kappa(x,X) &= \{y\in X \mid x \text{ is } \kappa\text{-adjacent or equal to } y\}.\end{aligned}$$

We say that $Int_\kappa(A)$ is κ-interior of A in (X,κ) and $Cl_\kappa(A)$ is κ-closure of A in (X,κ) and the set $N_\kappa(x,X)$ is neighborhood of x in (X,κ).

Notions similar to above appear in [13] and [10] and also, the κ-interior and κ-closure operators defined above are very closely related to dilation and erosion operators, respectively, used in [10]. The following proposition shows that these operators satisfy many relations that are similar to those satisfied by their counterparts in topology.

Proposition 5.2 *Let (X,κ) be a digital image, $A,B\subset X$ and $x,y\in X$. Then:*

(i) $A\subset Cl_\kappa(A)$, $Int_\kappa(A)\subset A$

(ii) $Int_\kappa(X-A) = X - Cl_\kappa(A)$, $X - Int_\kappa(A) = Cl_\kappa(X-A)$

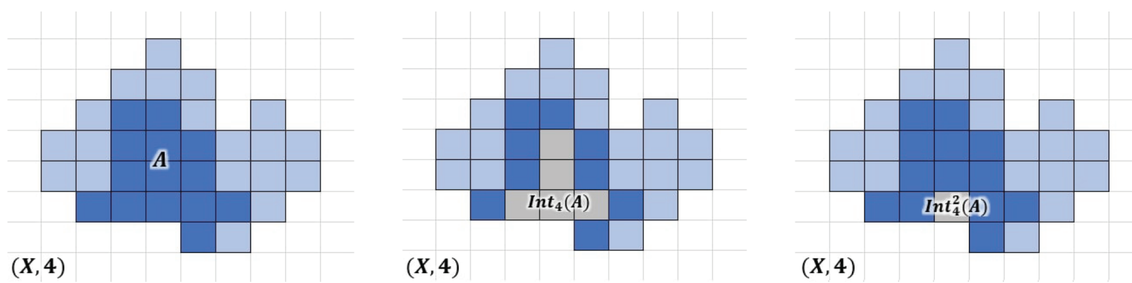
(iii) $A\subset B \Rightarrow Cl_\kappa(A)\subset Cl_\kappa(B)$ *and* $Int_\kappa(A)\subset Int_\kappa(B)$

(iv) $X = Int_\kappa(A) \cup Int_\kappa(B) \Leftrightarrow Cl_\kappa(X - B) \subset Int_\kappa(A)$

Proof The proofs are simple and follow easily from Definitions 5.1. □

The κ-interior and κ-closure operators for digital images are not idempotent, i.e. $Int_\kappa \circ Int_\kappa \neq Int_\kappa$ and $Cl_\kappa \circ Cl_\kappa \neq Cl_\kappa$, unlike interior and closure operators in topology, as shown in the following example.

Example 5.3 *Consider the digital image* $(X, 4)$ *and* $A \subset X$ *shown in Figure 4a. The interiors* $Int_4(A)$ *and* $Int_4^2(A) = Int_4(Int_4(A))$ *are shown in Figures 4b and 4c, respectively, and the closures* $Cl_4(A)$ *and* $Cl_4^2(A) = Cl_4(Cl_4(A))$ *in* X *in Figures 5a and 5b, respectively. Clearly,* $Int_4 \circ Int_4(A) \neq Int_4(A)$ *and* $Cl_4 \circ Cl_4(A) \neq Cl_4(A)$.



(a) (b) (c)

Figure 4. (a) Digital image $(X, 4)$, its subset A and (b) $Int_4(A)$, (c) $Int_4^2(A)$. Digital image X, A and interiors are shown in blue, dark blue and gray color, respectively.

(a) (b)

Figure 5. (a) $Cl_4(A)$ (b) $Cl_4^2(A)$ in $(X, 4)$ Closures are shown in dark blue color, where digital image $(X, 4)$ and $A \subset X$ are shown in Figure 4a.

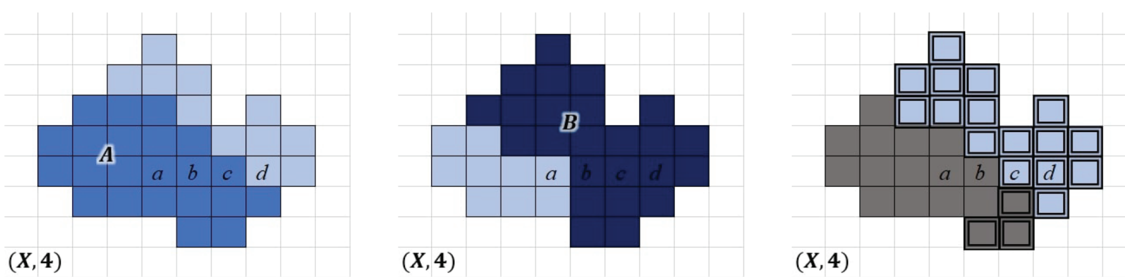
Lemma 5.4 *Let* (X, κ) *be a digital image, with subsets* A *and* B *such that* $X = Int_\kappa(A) \cup Int_\kappa(B)$. *Then for* $n \in \{0, 1, 2\}$ *and for every digital* n*-cube* T, *either* $Im(T) \subset A$ *or* $Im(T) \subset B$.

Proof Consider a digital n-cube $T : I^n \to X$ and the following cases for $n \in \{0, 1, 2\}$:
Case: $n = 0$ In this case $Im(T)$ consists of single element, say x_0, of X. Thus, $x_0 \in Int_\kappa(A)$ or $x_0 \in Int_\kappa(B)$, implying $Im(T) \subset A$ or $Im(T) \subset B$.

Case: $n = 1$ In this case, the set $Im(T) \subset X$ comprises two elements, namely, $T(0)$ and $T(1)$. We can assume without loss of generality that the element $T(0) \in Int_\kappa(A)$. By definition of Int_κ operator, κ-neighbors of $T(0)$ are in A, which implies $T(1) \in A$. Since $Int_\kappa(A) \subset A$ (Proposition 5.2 (i)), we get $Im(T) \subset A$.
Case: $n = 2$ In this case, the set $Im(T) \subset X$ comprises at most four distinct elements, namely, $T(0,0)$, $T(0,1)$, $T(1,0)$ and $T(1,1)$. We can assume without loss of generality that the element $T(0,0) \in Int_\kappa(A)$. By definition of Int_κ operator, κ-neighbors of $T(0,0)$ are in A, which implies $T(0,1), T(1,0) \in A$. Now $T(1,1)$ may or may not lie in A. If $T(1,1) \in A$, then $Im(T) \subset A$. If $T(1,1) \in X - A$, then we claim that $Im(T) \subset B$. Our claim follows from the following argument: From the definition of Cl_κ operator, $T(1,1) \in X - A$ implies that $T(0,1)$ and $T(1,0)$ both lie in $Cl_\kappa(X - A)$, which is a subset of $Int_\kappa(B)$ by Proposition 5.2 (iv). Therefore, $T(0,1), T(1,0) \in Int_\kappa(B) \Rightarrow T(0,0) \in B \Rightarrow Im(T) \subset B$. □

We show in the following example that the above Lemma fails for n-cubes with $n > 2$.

Example 5.5 *Consider the digital image $(X, 4)$ shown in Figure 6, where in parts (a) and (b), the subsets A and B of X, respectively, are shown in darker shades of blue. Elements of interiors $Int_4(A)$ and $Int_4(B)$ in $(X, 4)$ are shown in part (c) of Figure 6 as gray-shaded pixels and with double-line borders, respectively. Clearly, $X = Int_4(A) \cup Int_4(B)$. In these figures, we have labeled some elements of X as a, b, c and d. Define a digital 3-cube T as follows: $T(0,0,0) = a$, $T(1,0,0) = T(0,1,0) = T(0,0,1) = b$, $T(1,1,0) = T(0,1,1) = T(1,0,1) = c$, $T(1,1,1) = d$. It is clear that neither $Im(T) \subset A$ nor $Im(T) \subset B$.*



(a) (b) (c)

Figure 6. (a) Digital image $(X, 4)$ with its subset A, (b) subset $B \subset X$ (c) interiors $Int_4(A)$ and $Int_4(B)$ in $(X, 4)$. Parts (a) and (b) show subsets A and B of X in darker shades of blue, respectively, while part (c) shows the interior $Int_4(A)$ in gray and the interior $Int_4(B)$ with double-line borders.

The following theorem is similar to Excision axiom of homology theory except that it holds only for n less than 2.

Theorem 5.6 *Let (X, κ) be a digital image.*

- *For subsets $A, W \subset X$ such that $Cl_\kappa(W) \subset Int_\kappa(A)$, the inclusion $(X - W, A - W) \to (X, A)$ induces isomorphisms $dH_n(X - W, A - W) \to dH_n(X, A)$, for $n < 2$.*

Equivalently,

- *For subsets $A, B \subset X$ such that $X = Int_\kappa(A) \cup Int_\kappa(B)$, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms $dH_n(B, A \cap B) \to dH_n(X, A)$, for $n < 2$.*

Proof The equivalence of the two statements follows from Proposition 5.2 (iv) by taking $B = X - W$, which implies $W = X - B$ and $A - W = A \cap B$.
One can verify that for all n, $dC_n(A) \cap dC_n(B) = dC_n(A \cap B)$ and for $n \leq 2$, $dC_n(X) = dC_n(A) + dC_n(B)$ using Lemma 5.4. Furthermore, the map $\frac{dC_n(B)}{dC_n(A) \cap dC_n(B)} \to \frac{dC_n(A)+dC_n(B)}{dC_n(A)}$ induced by inclusion is an isomorphism by second isomorphism theorem of groups. Therefore, we get:

$$dC_n(B, A \cap B) = \frac{dC_n(B)}{dC_n(A \cap B)} \approx \frac{dC_n(A) + dC_n(B)}{dC_n(A)} = \frac{dC_n(X)}{dC_n(A)} = dC_n(X, A),$$

where only the second last equality is restricted to $n \leq 2$. It follows that $dH_n(X, A) \approx dH_n(B, A \cap B)$, for integers $n < 2$.

□

Theorem 5.6 is restricted to $n < 2$.

The first of the two versions of Theorem 5.6 states that there is no change in the n^{th}-relative homology groups of the digital image pair (X, A), when $n < 2$, if we excise out a subset W, which is contained 'well-inside' A. In order to extend this idea to higher homology groups ($n \geq 2$), we need the subset W to be contained deeper inside A. This can be done by iterative applications of interior and closure operators. This gives rise to the following definitions and results similar to those in Proposition 5.2.

Definition 5.7 *Let (X, κ) be a digital image and $A \subset X$. We define the operators $Int^i_\kappa : \mathcal{P}(X) \to \mathcal{P}(X)$ and $Cl^i_\kappa : \mathcal{P}(X) \to \mathcal{P}(X)$, for nonnegative integers i, recursively, as follows:*

$$Int^0_\kappa(A) = A, \quad Int^i_\kappa(A) = Int_\kappa(Int^{i-1}_\kappa(A)), \ \textit{for positive integer } i,$$

$$Cl^0_\kappa(A) = A, \quad Cl^i_\kappa(A) = Cl_\kappa(Cl^{i-1}_\kappa(A)), \ \textit{for positive integer } i.$$

Proposition 5.8 *Let (X, κ) be a digital image, $A, B \subset X$ and $x, y \in X$. Then:*

(i) $Cl^i_\kappa(A) \subset Cl^{i+1}_\kappa(A)$, $Int^{i+1}_\kappa(A) \subset Int^i_\kappa(A)$

(ii) $Int^i_\kappa(X - A) = X - Cl^i_\kappa(A)$, $X - Int^i_\kappa(A) = Cl^i_\kappa(X - A)$

(iii) $X = Int^i_\kappa(A) \cup Int^i_\kappa(B) \Leftrightarrow Cl^i_\kappa(X - B) \subset Int^i_\kappa(A)$

Proof The proofs are simple and follow easily from Definitions 5.1 and 5.7, and Proposition 5.2.

□

We give a generalization of Lemma 5.4, using Definitions 5.7 and Proposition 5.8.

Lemma 5.9 *Let (X, κ) be a digital image, with subsets A and B such that there is a positive integer i with $X = Int^i_\kappa(A) \cup Int^i_\kappa(B)$. Then for $n \leq i + 1$ and for every digital n-cube T, $Im(T) \subset A$ or $Im(T) \subset B$.*

Proof Consider a digital n-cube $T : I^n \to X$, $n \in \{0, 1, \ldots, i + 1\}$. The set $Im(T) \subset X$ can be partitioned into sets S_j for $j = 0, 1, \ldots, n$ defined as follows:

$$S_j = \{T(x_1, x_2, \ldots, x_n) \mid \Sigma^n_{i=1} x_i = j\}$$

Note that for $j \in \{1, 2, \ldots, n-1\}$, elements of S_j are κ-neighbors of elements of S_{j+1} and S_{j-1} and that S_0 and S_n are singletons.

Case: $n = 0$ In this case, the partition of $Im(T)$ consists of single set $S_0 \subset X$. Thus, $S_0 \subset Int^i_\kappa(A)$ or $S_0 \subset Int^i_\kappa(B)$, implying $Im(T) \subset A$ or $Im(T) \subset B$.

Case: $0 < n < i+1$ We can assume without loss of generality that the singleton $S_0 \subset Int^i_\kappa(A)$. Then by definition of Int^i_κ operator, $S_j \subset Int^{i-j}_\kappa(A)$, for $j = 1, 2, \ldots, n$. Thus, for all j, $S_j \subset A$, since $Int^i_\kappa(A) \subset A$ from Proposition 5.8 (i). Therefore, $Im(T) \subset A$.

Case: $n = i+1$ Again, we can assume without loss of generality that the set $S_0 \subset Int^i_\kappa(A)$. From the definition of Int^i_κ, for $j = 1, 2, \ldots, n-1$, $S_j \subset Int^{i-j}_\kappa(A)$. Now $Im(T) - S_n \subset A$ and S_n may or may not lie in A. If $S_n \subset A$, then $Im(T) \subset A$, which completes the proof.

However, if $S_n \subset X - A$, then we claim that $Im(T) \subset B$, which also completes the proof. Our claim follows from the following argument: From the definition of Cl_κ operator, $S_n \subset X - A$ implies S_{n-1} is contained in $Cl_\kappa(X - A)$. Using Proposition 5.8, we get the following:

$$X - A \subset Cl_\kappa(X - A) \subset Cl^i_\kappa(X - A) \subset Int^i_\kappa(B),$$

$$\Rightarrow S_{n-1} \subset Int^i_\kappa(B) \quad \Rightarrow S_{n-j} \subset Int^{i-j+1}_\kappa(B), \text{ for } j = 2, 3, \ldots, n \quad \Rightarrow Im(T) \subset B.$$

□

Theorem 5.10 [Excision-like property]
Let (X, κ) be a digital image.

- *For subsets $A, W \subset X$ such that there is a positive integer i, with $Cl^i_\kappa(W) \subset Int^i_\kappa(A)$, the inclusion $(X-W, A-W) \to (X, A)$ induces isomorphisms $dH_n(X-W, A-W) \to dH_n(X, A)$, for integers $n < i+1$.*

Equivalently,

- *For subsets $A, B \subset X$ such that there is a positive integer i, with $X = Int^i_\kappa(A) \cup Int^i_\kappa(B)$, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms $dH_n(B, A \cap B) \to dH_n(X, A)$, for integers $n < i+1$.*

Proof The equivalence of the two statements follows from Proposition 5.8 (iii) as in the proof of Theorem 5.6. Rest of the proof is also similar to the proof of Theorem 5.6 except that the equality $(dC_n(A) + dC_n(B))/dC_n(A) = dC_n(X)/dC_n(A)$ holds for $n \leq i+1$ from Lemma 5.9. □

The following result states the condition under which Excision-like property for n^{th}-digital cubical-singular homology holds for all n.

Corollary 5.11 *Let (X, κ) be a digital image.*

- *For subsets $A, W \subset X$ such that $W \subset A$, $Cl_\kappa(W) = W$ and $Int_\kappa(A) = A$, the inclusion $(X - W, A - W) \to (X, A)$ induces isomorphisms $dH_n(X - W, A - W) \to dH_n(X, A)$, for all n.*

Equivalently,

- *For subsets $A, B \subset X$ such that $X = A \cup B$, $Int_\kappa(A) = A$ and $Int_\kappa(B) = B$, the inclusion $(B, A \cap B) \to (X, A)$ induces isomorphisms $dH_n(B, A \cap B) \to dH_n(X, A)$, for all n.*

Proof The equivalence of the statements can be shown in a similar way as in the proof of Theorem 5.6. Using the hypothesis of first statement, one can show that for all integers i, $Cl^i_\kappa(W) = W$ and $Int^i_\kappa(A) = A$; therefore, $Cl^i_\kappa(W) \subset Int^i_\kappa(A)$ also holds for all integers i. Rest follows from Theorem 5.10. □

6. Digital homology theory

In this section, we show that digital cubical singular homology satisfies digital analogs of Eilenberg–Steenrod axioms of homology theory. Other homologies for digital images have not been proven to exhibit this coherence with homology theory of topological spaces.

We define category of digital-image pairs Dig^2 with digital-image pairs as objects and (κ, λ)-continuous maps of pairs as morphisms. It can be shown that $dH_n(-,-)$ is a functor from Dig^2 to Ab in a similar way as in Proposition 3.4.

Definition 6.1 *We say that (κ, λ)-continuous maps of pairs $f, g : (X, A) \to (Y, B)$ are (κ, λ)-homotopic as maps of pairs, if $H : [0, m]_{\mathbb{Z}} \times X \to Y$ is (κ, λ)-homotopy from $f : X \to Y$ to $g : X \to Y$ and $H(t, A) \subset B$, $\forall t \in [0, m]_{\mathbb{Z}}$.*

Definition 6.2 *Digital homology theory consists of functors $dH_n(-,-)$ from the category of digital image pairs Dig^2 to the category of Abelian groups Ab along with natural transformations $\partial_* : dH_n(X, A) \to dH_{n-1}(A)$, (where $dH_{n-1}(A, \emptyset)$ is denoted as $dH_{n-1}(A)$) satisfying following axioms:*

[*Homotopy axiom*]

If $f, g : (X, A) \to (Y, B)$ are homotopically equivalent, then $f_*, g_* : dH_n(X, A) \to dH_n(Y, B)$ are equal maps.

[*Exactness axiom*]

For each digital image pair (X, A), and inclusion maps $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, there is a long-exact sequence:

$$\cdots \xrightarrow{\partial_*} dH_n(A) \xrightarrow{i_*} dH_n(X) \xrightarrow{j_*} dH_n(X, A) \xrightarrow{\partial_*} dH_{n-1}(A) \xrightarrow{i_*} \cdots$$

[*Excision axiom*]

For a digital image pair (X, A) and a subset $W \subset A$ such that there is a positive integer i with $Cl^i_\kappa(W) \subset Int^i_\kappa(A)$, the inclusion $(X - W, A - W) \to (X, A)$ induces isomorphism $dH_n(X - W, A - W) \to dH_n(X, A)$ for $0 \le n \le i + 1$.

[*Dimension axiom*]

If $X = \{x_0\}$ is a one-point digital image, $dH_n(X) = 0$, for all $n > 0$.

[*Additivity axiom*]

If $\{(X_\alpha, \kappa) \mid \alpha \in \Lambda\}$ is a collection of mutually κ-disconnected digital images with $X_\alpha \subset \mathbb{Z}^d$ and (X, κ) is the digital image $X = \bigcup_\alpha X_\alpha$, then $dH_n(X) \approx \bigoplus_\alpha dH_n(X_\alpha)$.

Theorem 6.3 *The relative cubical singular homology groups $dH_n(-,-)$ form a digital homology theory.*

Proof We prove the axioms of digital homology theory one-by-one:

[*Homotopy axiom*] It can be shown using Theorem 3.7 that if $f, g : (X, A) \to (Y, B)$ are homotopically equivalent, then f and g induce the same map $f_* = g_*$ from $dH_n(X, A)$ to $dH_n(Y, B)$.

[*Exactness axiom*] For a digital image pair (X, A), we have chain complexes $(dC_\bullet(A), \partial)$, $(dC_\bullet(X), \partial)$ and $(dC_\bullet(X, A), \partial)$. We also have chain maps $i_* : dC_n(A) \to dC_n(X)$ and $j_* : dC_n(X) \to dC_n(X, A)$, induced by inclusions $i : A \hookrightarrow X$ and $j : (X, \emptyset) \hookrightarrow (X, A)$. This gives the following short exact sequence of chain-complexes.

$$0 \longrightarrow dC_\bullet(A) \xrightarrow{i_*} dC_\bullet(X) \xrightarrow{j_*} dC_\bullet(X, A) \longrightarrow 0$$

The above short-exact sequence induces the following long-exact sequence of homology groups:

$$\cdots \xrightarrow{\partial_*} dH_n(A) \xrightarrow{i_*} dH_n(X) \xrightarrow{j_*} dH_n(X, A) \xrightarrow{\partial_*} dH_{n-1}(A) \xrightarrow{i_*} \cdots$$

by zig-zag lemma ([17], Lemma 24.1). The zig-zag lemma also asserts the existence and uniqueness of the homomorphism $\partial_* : dC_n(X, A) \to dC_{n-1}(A)$.

[*Excision axiom*] , see Theorem 5.10.

[*Dimension axiom*] can be easily proved using Proposition 3.6.

[*Additivity axiom*] , see Proposition 3.2.

□

7. Conclusion

In digital topology, researchers are interested in exploring topological properties of digital images. Some researchers have attempted to develop a theory for digital images which parallels with general topology, thereby defining digital continuity, digital connectedness, digital homotopy equivalence, digital fundamental group, and homology groups for digital images using various approaches (simplicial, cubical, and singular). This work extends the research in this direction by introducing concepts and proving results for digital images that are in line with algebraic topology and homology. Although some researchers have already defined homology for digital images (simplicial, cubical, and singular), there were two important gaps that are filled in by this piece of work. Firstly, cubical singular homology for digital images have not been defined previously. Secondly, the already-available approaches to homology (simplicial, cubical, and singular) had failed to produce results that may be considered digital analogs of Hurewicz theorem, homotopy invariance and excision property of homology groups that are found in algebraic topology, while digital Hurewicz theorem (Theorem 4.4), Proposition 3.7, and excision-like theorem (Theorem 5.10) fill in this gap using digital cubical singular homology.

Computability of digital cubical singular homology groups of various digital images is still a challenge not accomplished in this work. It is well-known that singular homology for topological spaces is in general difficult to compute and similar difficulty carries over to the case of cubical singular homology for digital images. More theoretical study is required to make computations possible to some extent. Furthermore, it can be explored whether it is possible to develop an algorithm to compute these groups for a given digital image. Such an algorithm can be used to further explore the applicability of digital homology in the fields such as digital image processing, image analysis and computer vision.

The work presented and proposed in this document can be extended in various directions. Cohomology theory for digital images can be developed. Our work is restricted to black-and-white digital images, one might extend this work to develop homology theory for gray-scale and colored digital images and for unbounded digital images.

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