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Pell-Lucas collocation method to solve high-order linear Fredholm-Volterra integro-differential equations and residual correction

Şuayip YÜZBAŞI[1,*], **Gamze YILDIRIM**[1,2]
[1]Department of Mathematics, Faculty of Science, Akdeniz University, Antalya, Turkey
[2]Department of Mathematics, Faculty of Science, Akdeniz University, Antalya, Turkey

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Abstract: In this article, a collocation method based on Pell-Lucas polynomials is studied to numerically solve higher order linear Fredholm-Volterra integro differential equations (FVIDE). The approximate solutions are assumed in form of the truncated Pell-Lucas polynomial series. By using Pell-Lucas polynomials and relations of their derivatives, the solution form and its derivatives are brought to matrix forms. By applying the collocation method based on equally spaced collocation points, the method reduces the problem to a system of linear algebraic equations. Solution of this system determines the coefficients of assumed solution. Error estimation is made and also a method with the help of the obtained approximate solution is developed that finds approximate solution with better results. Then, the applications are made on five examples to show that the method is successful. In addition, the results are supported by tables and graphs and the comparisons are made with other methods available in the literature. All calculations in this study have been made using codes written in Matlab.

Key words: Collocation method, collocation points, Fredholm-Volterra integro-differential equations, Pell-Lucas polynomials

1. Introduction

In applied disciplines, some problems cannot be expressed with a single equation, but instead can be expressed as a whole of integro-differential equations consisting of a linear combination of differential and integral equations containing multiple unknown functions. These types of systems of equations have been appeared in many branches of physics, biology and engineering. For example, integro-differential equation systems have been appeared in areas such as thermoelasticity [19], electromagnetic theory [3], mechanics [34], biology [12], diffraction of waves [4].

Recently, Sezer, Doğan, Akyüz-Daşcıoğlu, and Yaslan have studied the Chebyshev collocation method for linear and nonlinear integral equations and systems of linear FVIDEs [1, 2, 30]. Pour-Mahmoud, Rahimi-Ardabili, Shahmorad, Hosseini have presented the Tau method for linear FVIDEs and systems of FVIDEs [15, 16, 24, 31]. Maleknejad, Mahmoudi, Yalçınbaş and Sezer have studied on the Taylor polynomial approach for linear and nonlinear FVIDEs [21, 33]. Also, various methods [7, 8, 20, 29, 37, 38, 40] such as compact the finite difference method [43], the rationalized Haar functions method [22, 27], the CAS wavelet method [5], the differential transform method [6], the improved homotopy perturbation method [35], the sine-cosine

*Correspondence: syuzbasi@akdeniz.edu.tr
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wavelet method [18, 32], the homotopy perturbation method [11, 18], the hybrid function method [17], the sinc method [42], the Legendre method [23, 25, 26, 28], the Bernstein method [9, 10, 36] and the combined Laplace transform-Adomian decomposition method [41] have been studied for solving linear and nonlinear FVIDEs.

In this study, we will develope the Pell-Lucas collocation method using the matrix representation of the Pell-Lucas polynomials to compute approximate solutions of the m-th order linear FVIDE

$$L[y(x)] = \sum_{k=0}^{m} F_k(x) y^{(k)}(x) + \lambda_f \int_a^b K_f(x,t) y(t) dt + \lambda_v \int_a^x K_v(x,t) y(t) dt = g(x) \tag{1.1}$$

under the initial-boundary conditions

$$\sum_{k=0}^{m-1} (a_{jk} y^{(k)}(a) + b_{jk} y^{(k)}(b)) = \mu_j, \quad j = 0,1,2,...,m-1. \tag{1.2}$$

In Table (1), the expressions in the problem (1.1)-(1.2) are described.

Table 1. Some Parameter in the problem (1.1)-(1.2)

Parameter	**Description**
$a_{jk}, b_{jk}, \mu_j, \lambda_f, \lambda_v$	real or complex constants
$y^{(0)}(x) = y(x)$	unknown function
$F_k(x), g(x), K_f(x,t), K_v(x,t)$	the defined functions on interval $a \leq x, t \leq b$
$K_f(x,t), K_v(x,t)$	functions that can be expanded to the Maclaurin series

In this study, three important goals are analyzed. As a first goal, we obtain approximate solutions depending on the Pell-Lucas polynomials of the problem (1.1)-(1.2) in the form

$$y_N(x) = \sum_{n=0}^{N} a_n Q_n(x) \tag{1.3}$$

where $a_n (n = 0, 1, ...N)$ are the Pell-Lucas coefficients to be found in the method and $Q_n(x)$ are the Pell-Lucas polynomials. Also, N is any positive integer, which is the cutting limit in the method.

As a second goal, an error estimation for error function is made with $e_{N,M}(x)$ are made with the help of the residual function $R_N(x)$

$$R_N(x) = L[y_N(x)] - g(x).$$

As a final goal, the improved approximate solutions $y_{N,M}(x)$ and the improved errors $E_{N,M}(x)$ are obtained with the help of the approximate solution $y_N(x)$ and the estimated error function $e_{N,M}(x)$. Also, M is any positive integer, $M \geq N$.

A brief summary of this article is as follows: In Section (2), first some basic relations related to the Pell-Lucas polynomials are given. Then, the matrix relations of the problem (1.1)-(1.2) are given. In Section (3), a method of solution based on collocation points is described. In Section (4), an error estimation method is presented with the help of residual function that finds approximate solution better than the obtained approximate solution. In Section (5), the presented method in the previous sections is implemented on five examples. The obtained results from these applications are shown in tables and graphs. In Section (6), the results of this study are mentioned.

2. Some Fundamental Relations

In this section, some features of Pell-Lucas polynomials to be used in the method will be given in the Subsection (2.1). Then, in the Subsection (2.2), basic matrix relations that will be used in the method will be given.

2.1. The Pell-Lucas polynomials $Q_n(x)$

Some significant characteristics of Pell-Lucas polynomials are as follows [13, 14]:

- The recurrence relationship of Pell-Lucas polynomials $Q_n(x)$ is

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x), \quad n \geq 2$$

and the first two Pell-Lucas polynomials are $Q_0(x) = 2$ and $Q_1(x) = 2x$.

- The Pell-Lucas polynomials $Q_n(x)$ have generating function in the form

$$W(x,t) = \sum_{n=0}^{\infty} Q_{n+1}(x)t^n = (2x + 2t)[1 - 2xt - x^2]^{-1}.$$

- By using the generator function, it can also be clearly expressed as

$$Q_n(x) = \sum_{k=0}^{[n/2]} 2^{n-2k} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}.$$

- By using standard procedures, it can also be expressed the Binet-type formula as

$$Q_n(x) = \alpha^n(x) + \beta^n(x)$$

where $\alpha(x) = x + \Delta(x), \quad \beta(x) = x - \Delta(x), \quad \Delta(x) = \sqrt{(x^2+1)}$. Here, $\alpha(x)$ and $\beta(x)$ are the roots of

$$\lambda^2 - 2x\lambda - 1 = 0$$

so that

$$\alpha(x) + \beta(x) = 2x, \quad \alpha(x) - \beta(x) = 2\Delta(x), \quad \alpha(x)\beta(x) = -1.$$

- Pell-Lucas polynomials $Q_n(x)$ can also be expressed to as

$$Q_n(x) = P_{n+1}(x) + P_{n-1}(x)$$

where $P_n(x)$, (n=0,1,2,...) are Pell polynomials.

- The relationship of Pell-Lucas polynomials $Q_n(x)$ with Lucas polynomials $L_n(x)$ is in the form

$$Q_n(x/2) = L_n(x).$$

- Pell-Lucas polynomials $Q_n(x)$ can be expanded to

$$Q_{-n}(x) = (-1)^n Q_n(x).$$

  including their negative subscript.

- The recurrence relation for the derivative of Pell-Lucas polynomials is

$$Q_n^{'}(x) = 2xQ_{n-1}^{'}(x) + Q_{n-2}^{'}(x) + 2Q_{n-1}(x), \quad n \geq 2$$

  where $Q_0^{'}(x) = 0$ and $Q_1^{'}(x) = 2$.

- For Pell-Lucas polynomials, see [13, 14] for more features.

2.2. Fundamental Matrix Relations

First, for convenience, let's write the Eq. (1.1) as

$$D(x) + \lambda_f I(x) + \lambda_v V(x) = g(x) \tag{2.1}$$

Here, $D(x)$ is the differential part in the form

$$D(x) = \sum_{k=0}^{m} F_k(x) y^{(k)}(x), \tag{2.2}$$

$I(x)$ is the Fredholm integral part in the form

$$I(x) = \int_a^b K_f(x,t) y(t) dt \tag{2.3}$$

and $V(x)$ is the Volterra integral part in the form

$$V(x) = \int_a^x K_v(x,t) y(t) dt. \tag{2.4}$$

Now, let's find the matrix representation of the solution $y(x)$, its derivatives, the parts $D(x)$, $I(x)$ and $V(x)$ and the mixed conditions (1.2) which are also necessary for the method. We will consider them in the following subsections.

2.2.1. Matrix relation for the differential part $D(x)$

In this subsection, we can first express the solution $y(x)$ of the Eq. (1.1) in matrix form as

$$y_N(x) = \mathbf{Q}(x)\mathbf{A} \tag{2.5}$$

where $\mathbf{Q}(x) = \begin{bmatrix} Q_0(x) & Q_1(x) & \cdots & Q_N(x) \end{bmatrix}$ and $\mathbf{A} = \begin{bmatrix} a_0 & a_1 & \cdots & a_N \end{bmatrix}^T$.

Secondly, with the help of $Q_n^{'}(x) = 2xQ_{n-1}^{'}(x) + Q_{n-2}^{'}(x) + 2Q_{n-1}(x)$, the derivative of the expression (2.5), it is obtained as

$$y_N^{'}(x) = \mathbf{Q}^{'}(x)\mathbf{A} = \mathbf{Q}(x)\mathbf{M}\mathbf{A} \tag{2.6}$$

where if N is odd

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 5 & \cdots & (-1)^{\frac{N-1}{2}}N \\ 0 & 0 & 4 & 0 & -8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 6 & 0 & -10 & \cdots & (-1)^{\frac{N-3}{2}}2N \\ 0 & 0 & 0 & 0 & 8 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & 10 & \cdots & (-1)^{\frac{N-5}{2}}2N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & 2N \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}$$

and if N is even

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & -3 & 0 & 5 & \cdots & 0 \\ 0 & 0 & 4 & 0 & -8 & 0 & \cdots & (-1)^{\frac{N-2}{2}}2N \\ 0 & 0 & 0 & 6 & 0 & -10 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 8 & 0 & \cdots & (-1)^{\frac{N-4}{2}}2N \\ 0 & 0 & 0 & 0 & 0 & 10 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}_{(N+1)\times(N+1)}.$$

Similarly, the $k-th$ order derivative of $y_N(x)$, based on Pell-Lucas polynomials, becomes in the form

$$y_N^{(k)}(x) = \mathbf{Q}^{(k)}(x)\mathbf{A} = \mathbf{Q}(x)\mathbf{M}^k\mathbf{A}. \tag{2.7}$$

Finally, in this subsection, when the expression (2.7) is substituted in the expression (2.2), the following expression is obtained

$$[D(x)] = \sum_{k=0}^{m} F_k(x)\mathbf{Q}(x)\mathbf{M}^k\mathbf{A}. \tag{2.8}$$

2.2.2. Matrix relation for the Fredholm integral part $I(x)$

In this subsection, we will find the matrix form of the integral part $I(x)$. For this purpose, we will use the truncated Maclaurin series of the kernel function $K_f(x,t)$. After the kernel function $K_f(x,t)$ is expressed the truncated Pell-Lucas series, and its relationship with the truncated Maclaurin series will be found.

Firstly, let's express the truncated Maclaurin series of the kernel function $K_f(x,t)$ in the form

$$K_f(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}^{T,f} x^m t^n \tag{2.9}$$

where

$$k_{m,n}^{T,f} = \frac{1}{m!n!}\frac{\partial^{m+n}K_f(0,0)}{\partial x^m \partial t^n}; \quad m,n = 0,1,...,N.$$

Similarly, let's express the truncated Pell-Lucas series of the kernel function $K_f(x,t)$ in the form

$$K_f(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}^{Q,f} Q_m(x) Q_n(t). \tag{2.10}$$

Here $k_{m,n}^{Q,f}$ are the coefficients of the Pell-Lucas serial form. In order to determine these coefficients, let's write the matrix form of expressions (2.9) and (2.10) and find the relationship between them.

The serial form of the expression (2.9) can be represented as

$$K_f(x,t) = \mathbf{X}(x)\mathbf{K}_f^T\mathbf{X}^T(t), \quad \mathbf{K}_f^T = [k_{m,n}^{T,f}] \tag{2.11}$$

where

$$\mathbf{X}(x) = \begin{bmatrix} 1 & x & x^2 & \cdots & x^N \end{bmatrix}$$

and the serial form of the expression (2.10) can be expressed as

$$K_f(x,t) = \mathbf{Q}(x)\mathbf{K}_f^Q\mathbf{Q}^T(t), \quad \mathbf{K}_Q^T = [k_{m,n}^{Q,f}] \tag{2.12}$$

where

$$\mathbf{Q}(x) = \begin{bmatrix} Q_0(x) & Q_1(x) & Q_2(x) & \cdots & Q_N(x) \end{bmatrix}.$$

Secondly, from the expression (2.11) and (2.12), it can be written as follows

$$\mathbf{X}(x)\mathbf{K}_f^T\mathbf{X}^T(t) = \mathbf{Q}(x)\mathbf{K}_f^Q\mathbf{Q}^T(t). \tag{2.13}$$

If the expression (2.13) is organized, it is obtained as

$$\mathbf{K}_f^T = \mathbf{X}^{-1}(x)\mathbf{Q}(x)\mathbf{K}_f^Q\mathbf{Q}^T(t)(\mathbf{X}^T)^{-1}(t) \tag{2.14}$$

or

$$\mathbf{K}_f^Q = \mathbf{Q}^{-1}(x)\mathbf{X}(x)\mathbf{K}_f^T\mathbf{X}^T(t)(\mathbf{Q}^T)^{-1}(t). \tag{2.15}$$

Finally, in this subsection, when the expression (2.5) and the expression (2.12) is written in the expression (2.3), the matrix form of integral part $I(x)$ can be represented as

$$[I(x)] = \int_a^b \mathbf{Q}(x)\mathbf{K}_f^Q\mathbf{Q}^T(t)\mathbf{Q}(t)\mathbf{A}dt. \tag{2.16}$$

If the expression (2.16) is organized, it becomes as follows

$$[I(x)] = \mathbf{Q}(x)\mathbf{K}_f^Q\left\{\int_a^b \mathbf{Q}^T(t)\mathbf{Q}(t)dt\right\}\mathbf{A}. \tag{2.17}$$

When the integral $\int_a^b \mathbf{Q}^T(t)\mathbf{Q}(t)dt$ is denoted by $\mathbf{N}_f$, the expression (2.17) can be written as

$$[I(x)] = \mathbf{Q}(x)\mathbf{K}_f^Q\mathbf{N}_f\mathbf{A}. \tag{2.18}$$

2.2.3. Matrix relation for the Volterra integral part $V(x)$

In this subsection, we will find the matrix form of the integral part $V(x)$. For this purpose, we will use the truncated Maclaurin series of the kernel function $K_v(x,t)$. And then the kernel function $K_v(x,t)$ will be written in the truncated Pell-Lucas series, and the matrix relation between it and the truncated Maclaurin series will be constructed.

Now, let's express the truncated Maclaurin series of the kernel function $K_v(x,t)$ in the form

$$K_v(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}^{T,v} x^m t^n \tag{2.19}$$

where

$$k_{m,n}^{T,v} = \frac{1}{m!n!}\frac{\partial^{m+n}K_v(0,0)}{\partial x^m \partial t^n}; \quad m,n = 0,1,...,N.$$

Similarly, we can express the truncated Pell-Lucas series of the kernel function $K_v(x,t)$ in the form

$$K_v(x,t) = \sum_{m=0}^{N}\sum_{n=0}^{N} k_{m,n}^{Q,v} Q_m(x) Q_n(t). \tag{2.20}$$

Here $k_{m,n}^{Q,v}$ are the coefficients of the Pell-Lucas serial form. In order to determine these coefficients, let's write the matrix form of expressions (2.19) and (2.20) and find the relationship between them.

The serial form of the expression (2.19) can be denoted as

$$K_v(x,t) = \mathbf{X}(x)\mathbf{K}_v^T\mathbf{X}^T(t), \quad \mathbf{K}_v^T = [k_{m,n}^{T,v}] \tag{2.21}$$

and the serial form of the expression (2.20) can be conceived as

$$K_v(x,t) = \mathbf{Q}(x)\mathbf{K}_v^Q\mathbf{Q}^T(t), \quad \mathbf{K}_Q^T = [k_{m,n}^{Q,v}]. \tag{2.22}$$

Secondly, from the expression (2.21) and (2.22), it can be obtained as

$$\mathbf{X}(x)\mathbf{K}_v^T\mathbf{X}^T(t) = \mathbf{Q}(x)\mathbf{K}_v^Q\mathbf{Q}^T(t). \tag{2.23}$$

After the expression (2.23) is organized, we have

$$\mathbf{K}_v^T = \mathbf{X}^{-1}(x)\mathbf{Q}(x)\mathbf{K}_v^Q\mathbf{Q}^T(t)(\mathbf{X}^T)^{-1}(t) \tag{2.24}$$

or

$$\mathbf{K}_v^Q = \mathbf{Q}^{-1}(x)\mathbf{X}(x)\mathbf{K}_v^T\mathbf{X}^T(t)(\mathbf{Q}^T)^{-1}(t). \tag{2.25}$$

Finally, in this subsection, when the expression (2.5) and the expression (2.22) is written in the expression (2.4) expression, the matrix form of integral part $V(x)$ can be denoted as

$$[V(x)] = \int_a^x \mathbf{Q}(x)\mathbf{K}_v^Q\mathbf{Q}^T(t)\mathbf{Q}(t)\mathbf{A}dt. \tag{2.26}$$

If the expression (2.26) is organized, it becomes as follows

$$[V(x)] = \mathbf{Q}(x)\mathbf{K}_v^Q \left\{ \int_a^x \mathbf{Q}^T(t)\mathbf{Q}(t)dt \right\} \mathbf{A}. \tag{2.27}$$

When the integral $\int_a^x \mathbf{Q}^T(t)\mathbf{Q}(t)dt$ is represented by $\mathbf{N}_v(x)$, the expression (2.27) can be stated as

$$[V(x)] = \mathbf{Q}(x)\mathbf{K}_v^Q\mathbf{N}_v(x)\mathbf{A}. \tag{2.28}$$

2.2.4. Matrix relation for the mixed conditions (1.2)

In this subsection, we will find the matrix form of the mixed conditions (1.2). For this purpose, by substituting a and b instead of x in the expression (2.7), the expression (1.2) can be written as

$$\sum_{k=0}^{m-1} \left(a_{jk}\mathbf{Q}(a)\mathbf{M}^k\mathbf{A} + b_{jk}\mathbf{Q}(b)\mathbf{M}^k\mathbf{A} \right) = \mu_j, \quad j = 0, 1, 2, ..., m-1$$

or

$$\sum_{k=0}^{m-1} \left(a_{jk}\mathbf{Q}(a) + b_{jk}\mathbf{Q}(b) \right)\mathbf{M}^k\mathbf{A} = \mu_j, \quad j = 0, 1, 2, ..., m-1. \tag{2.29}$$

The expression (2.29) can be expressed briefly as

$$\mathbf{UA} = \mu \quad or \quad [\mathbf{U}; \mu]. \tag{2.30}$$

Here,

$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_0 \\ \mathbf{U}_1 \\ \vdots \\ \mathbf{U}_{m-1} \end{bmatrix}, \quad \mathbf{U}_j = \sum_{k=0}^{m-1} \left(a_{jk}\mathbf{Q}(a) + b_{jk}\mathbf{Q}(b) \right)\mathbf{M}^k, \quad \mu = \begin{bmatrix} \mu_0 \\ \mu_1 \\ \vdots \\ \mu_{m-1} \end{bmatrix}. \tag{2.31}$$

3. The Collocation Method

In this section, we will first define the collocation points x_i. Then, we will get a system of the matrix equations by replacing these collocation points in the equation (1.1) where we found matrix representations in Section (2). Finally, we create a new system of the matrix equations by replacing the obtained matrix for conditions with any m rows of this system of the matrix equations. By solving this system, we will find the coefficient matrix A we are looking for.

The collocation points are defined as

$$x_i = a + \frac{b-a}{N}i, \quad i = 0, 1, ..., N. \tag{3.1}$$

Now, by replacing the matrix relations in Section (2) in the expression (2.1), we have

$$\sum_{k=0}^{m} F_k(x)\mathbf{Q}(x)\mathbf{M}^k\mathbf{A} + \lambda_f\mathbf{Q}(x)\mathbf{K}_f^Q\mathbf{N}_f\mathbf{A} + \lambda_v\mathbf{Q}(x)\mathbf{K}_v^Q\mathbf{N}_v(x)\mathbf{A} = g(x). \tag{3.2}$$

If the collocation points in the expression (2.30) are written instead of x in the expression (3.2), it becomes as follows

$$\sum_{k=0}^{m} F_k(x_i)\mathbf{Q}(x_i)\mathbf{M}^k\mathbf{A} + \lambda_f\mathbf{Q}(x_i)\mathbf{K}_f^Q\mathbf{N}_f\mathbf{A} + \lambda_v\mathbf{Q}(x_i)\mathbf{K}_v^Q\mathbf{N}_v(x_i)\mathbf{A} = g(x_i) \tag{3.3}$$

or briefly it can be written in compact form

$$\left\{\sum_{k=0}^{m} \mathbf{F}_k\mathbf{Q}\mathbf{M}^k + \lambda_f\mathbf{Q}\mathbf{K}_f^Q\mathbf{N}_f + \lambda_v\bar{\mathbf{Q}}\bar{\mathbf{K}_v^Q}\bar{\mathbf{N}}_v\right\}\mathbf{A} = \mathbf{G}. \tag{3.4}$$

Here,

$$\mathbf{F}_k = \begin{bmatrix} F_k(x_0) & 0 & \cdots & 0 \\ 0 & F_k(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & F_k(x_N) \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} g(x_0) \\ g(x_1) \\ \vdots \\ g(x_N) \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}(x_0) \\ \mathbf{Q}(x_1) \\ \vdots \\ \mathbf{Q}(x_N) \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_0(x_0) & \mathbf{Q}_1(x_0) & \mathbf{Q}_2(x_0) & \cdots & \mathbf{Q}_N(x_0) \\ \mathbf{Q}_0(x_1) & \mathbf{Q}_1(x_1) & \mathbf{Q}_2(x_1) & \cdots & \mathbf{Q}_N(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{Q}_0(x_N) & \mathbf{Q}_1(x_N) & \mathbf{Q}_2(x_N) & \cdots & \mathbf{Q}_N(x_N) \end{bmatrix}, \quad \bar{\mathbf{N}}_v = \begin{bmatrix} \mathbf{N}_\mathbf{v}(x_0) \\ \mathbf{N}_\mathbf{v}(x_1) \\ \vdots \\ \mathbf{N}_\mathbf{v}(x_N) \end{bmatrix},$$

$$\bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}(x_0) & 0 & \cdots & 0 \\ 0 & \mathbf{Q}(x_1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Q}(x_N) \end{bmatrix}, \quad \bar{\mathbf{K}_v^Q} = \begin{bmatrix} \mathbf{K}_v^Q & 0 & \cdots & 0 \\ 0 & \mathbf{K}_v^Q & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{K}_v^Q \end{bmatrix}.$$

The expression (3.4) can be written briefly as

$$\mathbf{W}\mathbf{A} = \mathbf{G} \quad or \quad [\mathbf{W};\mathbf{G}] \tag{3.5}$$

where

$$\mathbf{W} = \sum_{k=0}^{m} \mathbf{F}_k\mathbf{Q}\mathbf{M}^k + \lambda_f\mathbf{Q}\mathbf{K}_f^Q\mathbf{N}_f + \lambda_v\bar{\mathbf{Q}}\bar{\mathbf{K}_v^Q}\bar{\mathbf{N}}_v.$$

Now, let's note the dimensions of the matrices in the expression (3.4). The dimensions of the $\bar{\mathbf{Q}}, \bar{\mathbf{K}_v^Q}, \bar{\mathbf{N}}_v$ matrices are $(N+1)\times(N+1)^2, (N+1)^2\times(N+1)^2$ and $(N+1)^2\times(N+1)$ respectively.

The number of lines in the matrix $[\mathbf{W};\mathbf{G}]$ is $N+1$. The number of lines of the matrix $[\mathbf{U};\mu]$ is m. Finally, any $N+1-m$ line of the matrix $[\mathbf{W};\mathbf{G}]$ with the m line of the matrix $[\mathbf{U};\mu]$ is written as a single matrix, and this new matrix is called $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$. For convenience in this study, the first $N+1-m$ line of the matrix $[\mathbf{W};\mathbf{G}]$ is based. Hence, the new augmented matrix becomes in the form

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\begin{bmatrix} w_{0,0} & w_{0,1} & \cdots & w_{0,N} & ; & g(x_0) \\ w_{1,0} & w_{1,1} & \cdots & w_{1,N} & ; & g(x_1) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{N-m,0} & w_{N-m,1} & \cdots & w_{N-m,N} & ; & g(x_{N-m}) \\ u_{0,0} & u_{0,1} & \cdots & u_{0,N} & ; & \mu_0 \\ u_{1,0} & u_{1,1} & \cdots & u_{1,N} & ; & \mu_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ u_{m-1,0} & u_{m-1,1} & \cdots & u_{m-1,N} & ; & \mu_{m-1} \end{bmatrix}. \tag{3.6}$$

Here, if the rank of $\widetilde{\mathbf{W}}$ and $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ matrices is equal to $N+1$, then we can find the coefficients matrix $\mathbf{A}$ we are looking for in the method as

$$\mathbf{A}=(\widetilde{\mathbf{W}})^{-1}\widetilde{\mathbf{G}}.$$

Hence, the determined coefficients $a_0, a_1, ...a_N$ in $\mathbf{A}$ are substituted in the expression (1.3) and the approximate solution $y_N(x)$ is computed as depending on the Pell-Lucas polynomials in the form

$$y_N(x)=\sum_{n=0}^{N} a_n Q_n(x). \tag{3.7}$$

Note that, when $|\widetilde{\mathbf{W}}|=0$, if rank $\widetilde{\mathbf{W}}=$ rank $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]<N+1$, then it can be found a particular solution. If rank $\widetilde{\mathbf{W}}\neq$ rank $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]<N+1$, then it is not a solution.

4. Error estimation and Residual Improvement

In this section, we will develop an error estimation method using the obtained approximate solution as a result of the discussed method in Section (3). We will define the residual function for this method. Then, with the help of the obtained approximate solution in Section (3) and this error estimation, we will get new approximate solution that give better results than the obtained approximate solution in Section (3).

First, let's define the residual function as

$$R_N(x)=L[y_N(x)]-g(x) \tag{4.1}$$

and since the obtained approximate solution in Section (3) satisfy the problem (1.1)-(1.2), it can be written

$$\begin{cases} \sum_{k=0}^{m} F_k(x) y_N^{(k)}(x)+\lambda_f\int_a^b K_f(x,t)y_N(t)dt+\lambda_v\int_a^x K_v(x,t)y_N(t)dt=g(x)+R_N(x) \\ \sum_{k=0}^{m-1}(a_{jk}y_N^{(k)}(a)+b_{jk}y_N^{(k)}(b))=\mu_j, \quad j=0,1,2,...,m-1. \end{cases} \tag{4.2}$$

Thus, the error function can be expressed as

$$e_N(x)=y(x)-y_N(x) \tag{4.3}$$

where $y(x)$ is the exact solution and $y_N(x)$ is the approximate solution.

Now, if we substract the problem (4.2) from problem (1.1)-(1.2),respectively, then we find the error problem in the form

$$\begin{cases} \sum_{k=0}^{m} F_k(x)e_N^{(k)}(x) + \lambda_f \int_a^b K_f(x,t)e_N(t)dt + \lambda_v \int_a^x K_v(x,t)e_N(t)dt = -R_N(x) \\ \sum_{k=0}^{m-1}(a_{jk}e_N^{(k)}(a) + b_{jk}e_N^{(k)}(b)) = 0, \quad j = 0,1,2,...,m-1. \end{cases} \tag{4.4}$$

We solve the error problem (4.4) using the method in Section (3) and thus we get the following approach for the error function $e_N(x)$

$$e_{N,M}(x) = \sum_{n=0}^{M} a_n^* Q_n(x). \tag{4.5}$$

As a result, by summing the approximate solution $y_N(x)$ obtained in Section (3) and the estimated error function $e_{N,M}(x)$ in the expression (4.5), we find the improved approximate solution in the form

$$y_{N,M}(x) = y_N(x) + e_{N,M}(x) \tag{4.6}$$

which gives better results.

Thus, the error function for the improved approximate solution is obtained as

$$E_{N,M}(x) = y_N(x) + y_{N,M}(x). \tag{4.7}$$

5. Applications of the Method

In this section, five examples are applied the method presented in Section (3) and the obtained solutions are improved by the technique in Section (4). Numerical results of these examples have been calculated through the programs written in Matlab. Then, the results are displayed in tables and graphs. Moreover, the comparisons are made with other methods available in the literature.

In this section, the exact solution, the approximate solution, the improved approximate solution, the actual absolute error function, the estimated absolute error function and the improved absolute error function are represented by $y(x)$, $y_N(x)$, $y_{N,M}(x)$, $e_N(x)$, $e_{N,M}(x)$ and $E_{N,M}(x)$, respectively.

Example 5.1 First, let's take the $2nd$ order linear Fredholm-Volterra integro-differential equation

$$y^{''}(x) + xy^{'} - xy(x) - \int_0^1 \sin(x)e^{-t}y(t)dt + \frac{1}{2}\int_0^x \cos(x)e^{-t}y(t)dt = e^x - sin(x) + \frac{1}{2}x\cos(x) \tag{5.1}$$

in interval $0 \leq x,t \leq 1$ with the initial conditions

$$y(0) = 1, \quad y^{'}(0) = 1 \tag{5.2}$$

The exact solution to this problem is e^x. Now, let's find the approximate solution for this problem in the form

$$y_N(x) = \sum_{n=0}^{3} a_n Q_n(x) \tag{5.3}$$

depending on the Pell-Lucas polynomials and taking $N = 3$ as described in Section (2). According to the method in Section (3), collocation points are determined as

$$\left\{ x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1 \right\} \tag{5.4}$$

and $m = 2$, $F_0 = -x$, $F_1 = x$, $F_2 = 1$, $g(x) = e^x - sin(x) + \frac{1}{2}x\cos x$, $\lambda_f = -1$, $\lambda_v = \frac{1}{2}$, $K_f(x,t) = \sin xe^{-t}$, $K_v(x,t) = \cos xe^{-t}$ and thus from the expression (3.4), the basic matrix equation is written as

$$\left\{ \mathbf{F}_0\mathbf{Q} + \mathbf{F}_1\mathbf{QM} + \mathbf{F}_2\mathbf{QM}^2 + \lambda_f\mathbf{QK}_f^Q\mathbf{N}_f + \lambda_v\bar{\mathbf{Q}}\bar{\mathbf{K}}_v^Q\bar{\mathbf{N}}_v \right\}\mathbf{A} = \mathbf{G} \tag{5.5}$$

where

$$\mathbf{F}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -1/3 & 0 & 0 \\ 0 & 0 & -2/3 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad \mathbf{F}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{F}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}(0) \\ \mathbf{Q}(1/3) \\ \mathbf{Q}(2/3) \\ \mathbf{Q}(1) \end{bmatrix} = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 2 & 2/3 & 22/9 & 62/27 \\ 2 & 4/3 & 34/9 & 172/27 \\ 2 & 2 & 6 & 14 \end{bmatrix}, \bar{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}(0) & 0 & 0 & 0 \\ 0 & \mathbf{Q}(1/3) & 0 & 0 \\ 0 & 0 & \mathbf{Q}(2/3) & 0 \\ 0 & 0 & 0 & \mathbf{Q}(1) \end{bmatrix},$$

$$\mathbf{K}_f^Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 27/128 & -63/256 & 9/128 & -3/256 \\ 0 & 0 & 0 & 0 \\ -1/128 & 7/768 & -1/384 & 1/2304 \end{bmatrix}, \mathbf{K}_v^Q = \begin{bmatrix} 15/64 & -35/128 & 5/64 & -5/384 \\ 0 & 0 & 0 & 0 \\ -3/64 & 7/128 & -1/64 & 1/384 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{\mathbf{K}}_v^Q = \begin{bmatrix} \mathbf{K}_v^Q & 0 & 0 & 0 \\ 0 & \mathbf{K}_v^Q & 0 & 0 \\ 0 & 0 & \mathbf{K}_v^Q & 0 \\ 0 & 0 & 0 & \mathbf{K}_v^Q \end{bmatrix}, \mathbf{N}_f = \begin{bmatrix} 4 & 2 & 20/3 & 10 \\ 2 & 4/3 & 4 & 36/5 \\ 20/3 & 4 & 188/15 & 64/3 \\ 10 & 36/5 & 64/3 & 1412/35 \end{bmatrix},$$

$$\mathbf{N}_v(x) = \begin{bmatrix} 4x & 2x^2 & \frac{8x^3}{3} + 4x & 4x^4 + 6x^2 \\ 2x^2 & \frac{4x^3}{3} & 2x^2(x^2+1) & \frac{4x^3(4x^2+5)}{5} \\ \frac{8x^3}{3} + 4x & 2x^2(x^2+1) & \frac{4x(12x^4+20x^2+15)}{15} & \frac{2x^2(8x^4+15x^2+9)}{3} \\ 4x^4 + 6x^2 & \frac{4x^3(4x^2+5)}{5} & \frac{2x^2(8x^4+15x^2+9)}{3} & \frac{4x^3(80x^4+168x^2+105)}{35} \end{bmatrix},$$

$$\mathbf{M} = \begin{bmatrix} 0 & 1 & 0 & -3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \bar{\mathbf{N}}_v = \begin{bmatrix} \mathbf{N}_v(0) \\ \mathbf{N}_v(1/3) \\ \mathbf{N}_v(2/3) \\ \mathbf{N}_v(1) \end{bmatrix}, \mathbf{G} = \begin{bmatrix} g(0) \\ g(1/3) \\ g(2/3) \\ g(1) \end{bmatrix} = \begin{bmatrix} 1 \\ 1145/934 \\ 1211/761 \\ 2615/1218 \end{bmatrix}.$$

Thus, the augmented matrix $\left[\mathbf{W};\mathbf{G}\right]$ from (3.5) is computed as

$$\left[\mathbf{W};\mathbf{G}\right]=\left[\begin{array}{ccccccc} 0 & 0 & 8 & 0 & ; & 1 \\ -28271/34992 & 33335/104976 & 7319659/944784 & 115504939/6613488 & ; & 1145/934 \end{array}\right].$$

On the other hand, the matrix form $\left[\mathbf{U};\mu\right]$ for conditions from (3.1) is obtained as

$$\left[\mathbf{U};\mu\right]=\left[\begin{array}{cccccc} 2 & 0 & 2 & 0 & ; & 1 \\ 0 & 2 & 0 & 6 & ; & 1 \end{array}\right]$$

and according to the collocation method in Section (3), the new augmented matrix form $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ from (3.6) is calculated as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\left[\begin{array}{cccccc} 0 & 0 & 8 & 0 & ; & 1 \\ -28271/34992 & 33335/104976 & 7319659/944784 & 115504939/6613488 & ; & 1145/934 \\ 2 & 0 & 2 & 0 & ; & 1 \\ 0 & 2 & 0 & 6 & ; & 1 \end{array}\right].$$

Hence, this system $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ is solved and we get the coefficient matrix $\mathbf{A}$ we are looking for as

$$\mathbf{A}=\left[\begin{array}{cccc} 3/8 & 512/1199 & 1/8 & 129/5303 \end{array}\right]^T.$$

Finally, by substituting the obtained coefficient matrix $\mathbf{A}$ in the expression (2.5), we get the approximate solution as

$$y_3(x)=1+x+0.5x^2+0.194606977755395x^3.$$

Now, let's apply the method in Section (4) to get a better approximate solution than this approximate solution we get. Thus, we can express the error problem from (4.4) as

$$\left\{\begin{array}{l} e^{''}(x)+xe^{'}-xe(x)-\int_0^1\sin(x)e^{-t}e(t)dt+\frac{1}{2}\int_0^x\cos(x)e^{-t}e(t)dt=-R_N(x), \\ e(0)=0, \quad e^{'}(0)=0, \quad 0\leq x,t\leq 1 \end{array}\right. \tag{5.6}$$

This error problem is solved for $M=5$ by applying the method in Section (3) and the coefficient matrix $\mathbf{A}^*$ is obtained as

$$\mathbf{A}^*=\left[\begin{array}{cccccc} 79/10594 & 79/5704 & -158/15891 & -55/10567 & 102/41035 & 19/53833 \end{array}\right]^T.$$

The obtained coefficient matrix $\mathbf{A}^*$ is substituted in the expression (4.5), and the approach $e_{N,M}$ for $(N,M)=(3,5)$ is calculated as

$$e_{3,5}(x)=0.01129418899x^5+0.039770932201x^4-0.027521295140x^3-1.0164395367e-20x-2.7105054312e-20$$

and thus, the improved approximate solution for $(N,M)=(3,5)$ from (4.6) is computed as

$$y_{3,5}(x)=0.011294188999x^5+0.0397709322012x^4+0.167085682615x^3+0.5x^2+x+1.$$

Consequently, the error function for the improved approximate solution for $(N, M) = (3, 5)$ from (4.7) is written as

$$E_{3,5}(x) = e^x - 1 - x - 0.5x^2 - 0.167085682615x^3 - 0.0397709322012x^4 - 0.011294188999x^5.$$

In Table (2) and Figure (1), the exact solution, the obtained approximate solution by the method in Section (3) and the obtained improved approximate solution by the method in Section (4) of the problem (5.1)-(5.2) are compared for various N and M values. Table (3) shows comparison of the actual errors, estimated errors, and improved errors of the problem (5.1)-(5.2) for various N and M values. In Figure (2)-(a), the actual errors of the problem (5.1)-(5.2), in Figure (2)-(b), the actual errors and the improved errors of the problem (5.1)-(5.2), in the Figure (3)-(a), the actual errors and the estimated errors of the problem (5.1)-(5.2), in Figure (3)-(b), the actual, estimated and improved errors of the problem (5.1)-(5.2) are compared. According to all these comparisons, it can be observed that the errors decrease as the N value increases, the estimated errors are very close to the actual errors, and the improved errors have fewer errors than the actual errors. According to these results, it can be said that the method in Section (3) and Section (4) is quite effective, and the calculations in Matlab show that it is reliable.

Table 2. Numerical results of the exact, the approximate and the improved approximate solution for $(N, M) = (3, 5), (3, 8), (6, 7), (6, 9)$ of the prob. (5.1)-(5.2)

	Exact solution	Approximate solution	Improved approximate solution	
x_i	$y(x_i) = e^{x_i}$	$y_3(x_i)$	$y_{3,5}(x_i)$	$y_{3,8}(x_i)$
0	1	1	1	1
0.2	1.2214027581601	1.2215568558220	1.2214039330929	1.2214027580318
0.4	1.4918246976412	1.4924548465763	1.4918272720470	1.4918246973963
0.6	1.8221188003905	1.8220351071951	1.8221230563946	1.8221188000834
0.8	2.2255409284924	2.2196387726107	2.2255389231796	2.2255409282309
1	2.7182818284590	2.6946069777553	2.7181508038152	2.7182818094869
x_i	$y(x_i) = e^{x_i}$	$y_6(x_i)$	$y_{6,7}(x_i)$	$y_{6,9}(x_i)$
0	1	1	1	1
0.2	1.2214027581601	1.2214027621562	1.2214027613469	1.2214027581651
0.4	1.4918246976412	1.4918251276157	1.4918247042228	1.4918246976516
0.6	1.8221188003905	1.8221204604815	1.8221188100876	1.8221188004060
0.8	2.2255409284924	2.2255450607203	2.2255409408505	2.2255409285141
1	2.7182818284590	2.7182832021744	2.7182814283235	2.7182818277307

Example 5.2 Secondly, let's take the $5th$ order linear Fredholm-Volterra integro-differential equation

$$y^{(5)}(x) - xy^{''} + xy(x) - \frac{1}{2}\int_0^1 e^{2x+t}y(t)dt - \int_0^x xe^t y(t)dt = -e^{-x} - \frac{1}{2}e^{2x} - x^2, \quad 0 \leq x, t \leq 1 \tag{5.7}$$

together with the initial conditions

$$y(0) = 1, \quad y^{'}(0) = -1, \quad y^{''}(0) = 1, \quad y^{(3)}(0) = -1, \quad y^{(4)}(0) = 1. \tag{5.8}$$

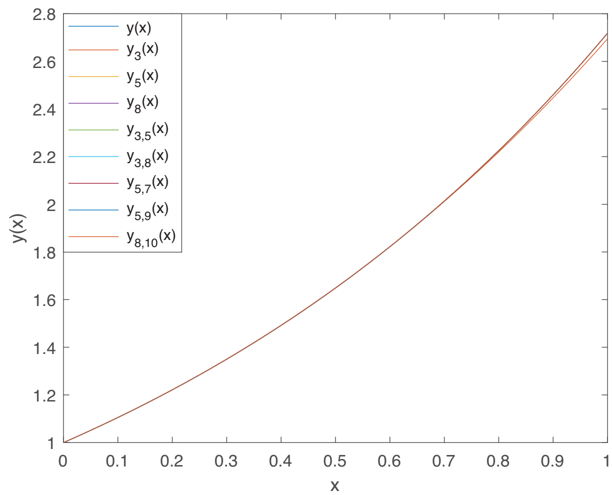


Figure 1. Comparison of the exact, the approximate and the improved approximate solution at some points of the prob. (5.1)-(5.2)

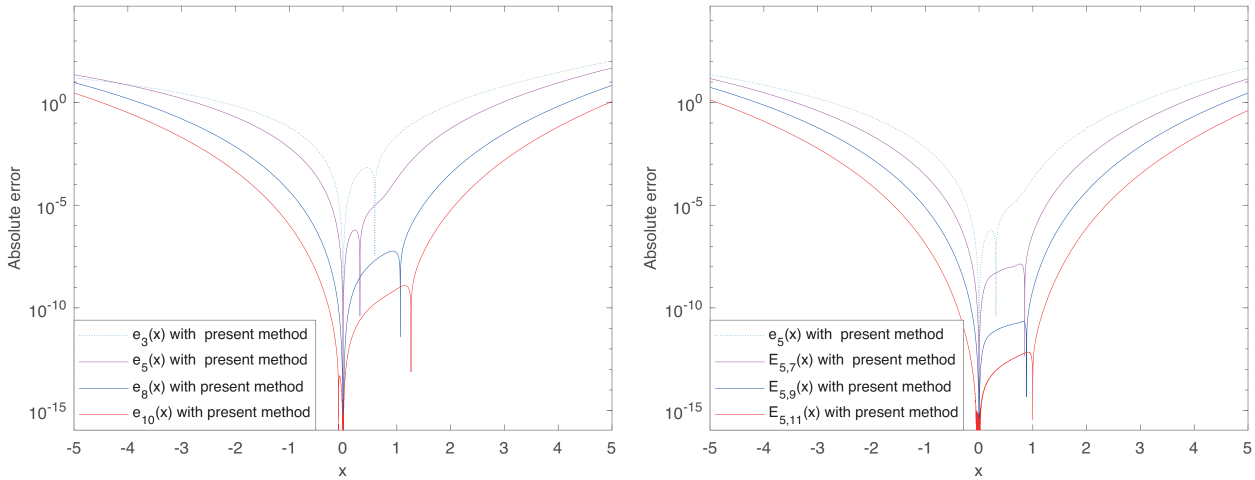
Table 3. Comparison of the absolute errors for $(N, M) = (3, 5), (3, 8), (6, 7), (6, 9)$ of the prob. (5.1)-(5.2)

	Absolute errors for the approximate solution	Estimated errors for the approximate solution		Absolute errors for the improved approximate solution	
x_i	$\lvert e_3(x_i)\rvert = \lvert y(x_i) - y_3(x_i)\rvert$	$\lvert e_{3,5}(x_i)\rvert$	$\lvert e_{3,8}(x_i)\rvert$	$\lvert E_{3,5}(x_i)\rvert$	$\lvert E_{3,8}(x_i)\rvert$
0	0	0	0	0	0
0.2	1.5410e-04	1.5292e-04	1.5410e-04	1.1749e-06	1.2831e-10
0.4	6.3015e-04	6.2757e-04	6.3015e-04	2.5744e-06	2.4488e-10
0.6	8.3693e-05	8.7949e-05	8.3693e-05	4.2560e-06	3.0710e-10
0.8	5.9022e-03	5.9002e-03	5.9022e-03	2.0053e-06	2.6154e-10
1	2.3675e-02	2.3544e-02	2.3675e-02	1.3102e-04	1.8972e-08
x_i	$\lvert e_6(x_i)\rvert = \lvert y(x_i) - y_6(x_i)\rvert$	$\lvert e_{6,7}(x_i)\rvert$	$\lvert e_{6,9}(x_i)\rvert$	$\lvert E_{6,7}(x_i)\rvert$	$\lvert E_{6,9}(x_i)\rvert$
0	0	0	0	0	0
0.2	3.9961e-09	8.0935e-10	3.9912e-09	3.1868e-09	4.9530e-12
0.4	4.2997e-07	4.2339e-07	4.2996e-07	6.5816e-09	1.0410e-11
0.6	1.6601e-06	1.6504e-06	1.6601e-06	9.6971e-09	1.5580e-11
0.8	4.1322e-06	4.1199e-06	4.1322e-06	1.2358e-08	2.1670e-11
1	1.3737e-06	1.7739e-06	1.3744e-06	4.0014e-07	7.2830e-10

The exact solution to this problem is e^{-x}. Now, let's find the approximate solution of this problem in the form

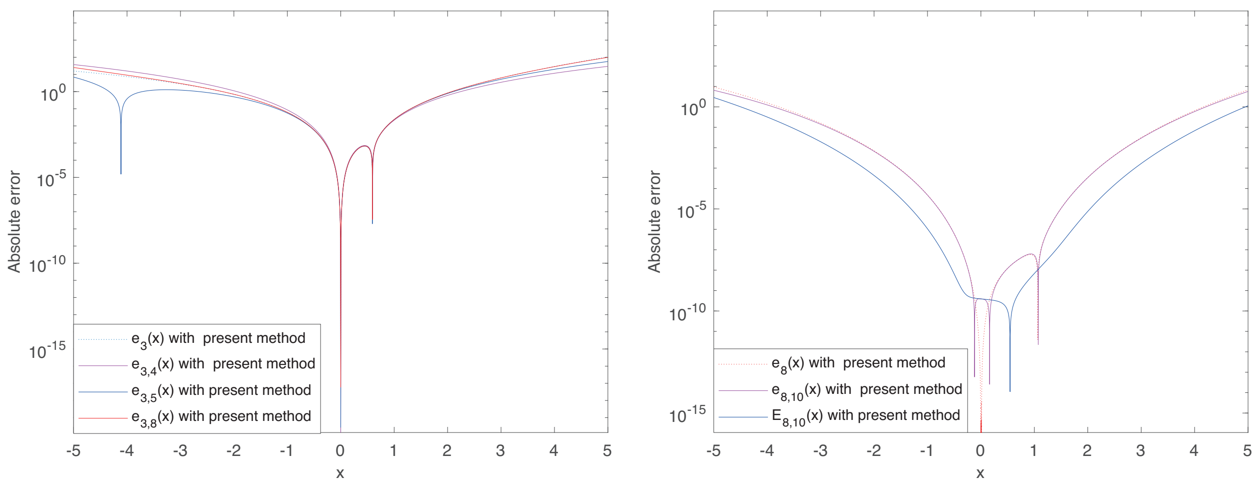
$$y_N(x) = \sum_{n=0}^{6} a_n Q_n(x) \tag{5.9}$$

depending on the Pell-Lucas polynomials and taking $N = 6$ as described in Section (2). According to the method in Section (3), collocation points are determined as $\{x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{1}{3}, \quad x_3 = \frac{1}{2}, \quad x_4 =$



(a) $e_N(x)$ (b) $e_N(x)$ and $E_{N,M}(x)$

Figure 2. Comparison of the actual and the improved errors at some points of the prob. (5.1)-(5.2)



(a) $e_N(x)$ and $e_{N,M}(x)$ (b) $e_N(x)$, $e_{N,M}(x)$ and $E_{N,M}(x)$

Figure 3. Comparison of the actual, the estimated and the improved errors at some points of the prob. (5.1)-(5.2)

$\frac{2}{3}, \quad x_5 = \frac{5}{6}, \quad x_6 = 1\big\}$ and $m = 5,\ F_0 = x,\ F_2 = -x,\ F_5 = 1,\ g(x) = -e^x - \frac{1}{2}e^{2x} - x^2,\ \lambda_f = -\frac{1}{2},\ \lambda_v = -1,$ $K_f(x,t) = e^{2x+t},\ K_v(x,t) = xe^t$ and from the expression (3.4), the basic matrix equation becomes as follows

$$\left\{\mathbf{F}_0\mathbf{Q} + \mathbf{F}_2\mathbf{Q}\mathbf{M}^2 + \mathbf{F}_5\mathbf{Q}\mathbf{M}^5 + \lambda_f\mathbf{Q}\mathbf{K}_f^Q\mathbf{N}_f + \lambda_v\bar{\mathbf{Q}}\mathbf{K}_v^{\bar{Q}}\bar{\mathbf{N}}_v\right\}\mathbf{A} = \mathbf{G}. \tag{5.10}$$

Thus, the augmented matrix $\left[\mathbf{W};\mathbf{G}\right]$ from (3.5) can be written and thus, according to the collocation method in Section (3), the new augmented matrix form $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ from (3.6) is calculated as

$$[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]=\left[\begin{array}{ccccccccc} -433/252 & -40319/40320 & -3404/1079 & -1098/209 & -3779/338 & 19087/5 & -7148/149 & ; & -3/2 \\ -1953/919 & -1247/927 & -403/74 & -1540/181 & -9903/461 & 45619/12 & 83555/11 & ; & -1058/673 \\ 2 & 0 & 2 & 0 & 2 & 0 & 2 & ; & 1 \\ 0 & 2 & 0 & 6 & 0 & 10 & 0 & ; & -1 \\ 0 & 0 & 8 & 0 & 32 & 0 & 72 & ; & 1 \\ 0 & 0 & 0 & 48 & 0 & 240 & 0 & ; & -1 \\ 0 & 0 & 0 & 0 & 384 & 0 & 2304 & ; & 1 \end{array}\right].$$

Hence, this system $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ is solved and we get the coefficient matrix $\mathbf{A}$

$$\mathbf{A}=\left[\begin{array}{ccccccc} 867/2266 & -169/384 & 467/4065 & -5/256 & 102/41059 & -100/384001 & 3/150079 \end{array}\right]^T.$$

Finally, by substituting the obtained coefficient matrix $\mathbf{A}$ in the expression (2.5), we have the approximate solution

$$y_6(x)=0.00127932503853x^6-0.00833331173867x^5+0.0416666666667x^4-0.166666666667x^3+0.5x^2-x+1.$$

Thus, it is applied the method in Section (4) for $M=7$ and it can be obtained $e_{6,7}(x)$, $y_{6,7}(x)$ and $E_{6,7}(x)$.

In Table (4) and Figure (4), the exact solution, the obtained approximate solution by the method in Section (3) and the obtained improved approximate solution by the method in Section (4) of the problem (5.7)-(5.8) have compared for various N and M values. Table (5) compares the actual errors, estimated errors, and improved errors of the problem (5.7)-(5.8) for various N and M values. In Figure (5)-(a), the actual errors of the problem (5.7)-(5.8), in Figure (5)-(b), the actual errors and the improved errors of the problem (5.7)-(5.8), in the Figure (6)-(a), the actual errors and the estimated errors of the problem (5.7)-(5.8), in Figure (6)-(b), the actual, estimated and improved errors of the problem (5.7)-(5.8) are compared.

Table 4. Numerical results of the exact, the approximate and the improved approximate solution for $(N,M)=(6,7),(6,10)$ of the prob. (5.7)-(5.8)

	Exact solution	Approximate solution	Improved approximate solution	
x_i	$y(x_i)=e^{-x_i}$	$y_6(x_i)$	$y_{6,7}(x_i)$	$y_{6,10}(x_i)$
0	1	1	1	1
0.2	0.818730753077982	0.818730748550379	0.818730752804608	0.818730753077949
0.4	0.670320046035639	0.670319907003154	0.670320038400105	0.670320046034900
0.6	0.548811636094026	0.548811689868198	0.548811579877258	0.548811636089891
0.8	0.449328964117222	0.449338041125705	0.449328392845722	0.449328964110598
1	0.367879441171442	0.367946013299856	0.367874755869615	0.367879441549126

Example 5.3 As the third, let's take the first order linear Fredholm integro-differential equation

$$y^{'}-\int_0^1 xy(t)dt=xe^x+e^x-x,\quad 0\leq x,t\leq 1 \tag{5.11}$$

with initial the condition

$$y(0)=0. \tag{5.12}$$

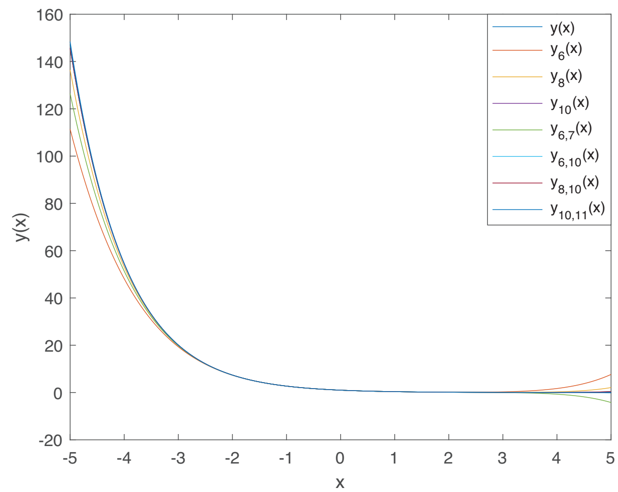


Figure 4. Comparison of the exact, the approximate and the improved approximate solution at some points of the prob. (5.7)-(5.8)

Table 5. Comparison of the absolute errors for $(N,M)=(6,7),(6,10)$ of the prob. (5.7)-(5.8)

	Absolute errors for the approximate solution	Estimated errors for the approximate solution		Absolute errors for the improved approximate solution	
x_i	$\lvert e_6(x_i)\rvert = \lvert y(x_i) - y_6(x_i)\rvert$	$\lvert e_{6,7}(x_i)\rvert$	$\lvert e_{6,10}(x_i)\rvert$	$\lvert E_{6,7}(x_i)\rvert$	$\lvert E_{6,10}(x_i)\rvert$
0	0	0	0	0	0
0.2	4.5276e-09	4.2542e-09	4.5276e-09	2.7337e-10	3.2514e-14
0.4	1.3903e-07	1.3140e-07	1.3903e-07	7.6355e-09	7.3934e-13
0.6	5.3774e-08	1.0999e-07	5.3778e-08	5.6217e-08	4.1349e-12
0.8	9.0770e-06	9.6483e-06	9.0770e-06	5.7127e-07	6.6239e-12
1	6.6572e-05	7.1257e-05	6.6572e-05	4.6853e-06	3.7768e-10

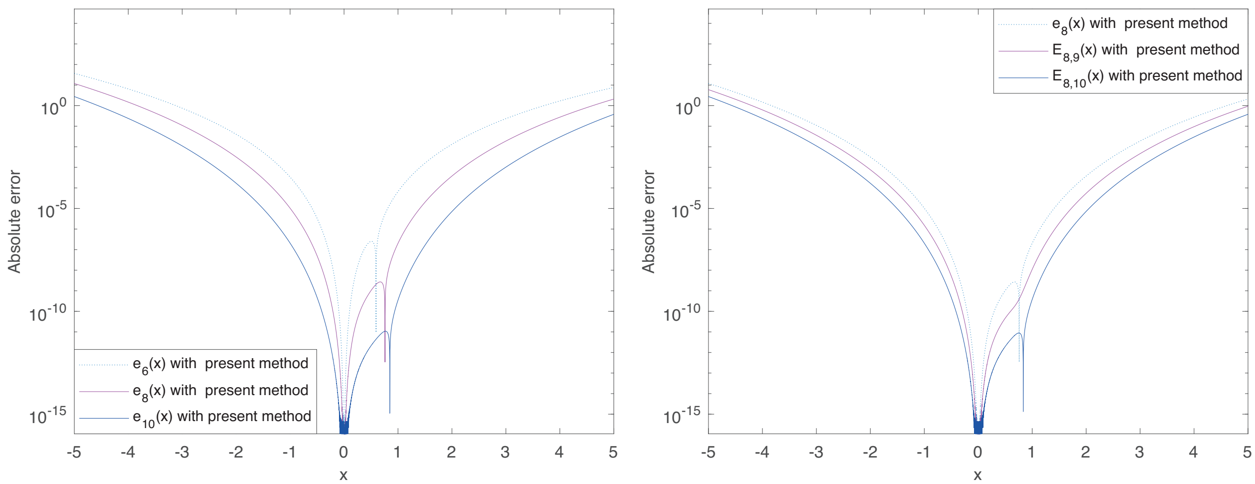
The exact solution to this problem is xe^x. Now, our aim is to find the approximate solution of this problem in the form

$$y_N(x) = \sum_{n=0}^{5} a_n Q_n(x) \tag{5.13}$$

depending on the Pell-Lucas polynomials and taking $N=5$ as described in Section (2). According to the method in Section (3), collocation points are determined as $\left\{x_0=0,\quad x_1=\frac{1}{5},\quad x_2=\frac{2}{5},\quad x_3=\frac{3}{5},\quad x_4=\frac{4}{5},\quad x_5=1\right\}$ and $m=1$, $F_0=0$, $F_1=1$, $g(x)=xe^x+e^x-x$, $\lambda_f=-1$, $\lambda_v=0$, $K_f(x,t)=x$, $K_v(x,t)=0$ and from the expression (3.4), the basic matrix equation is written as

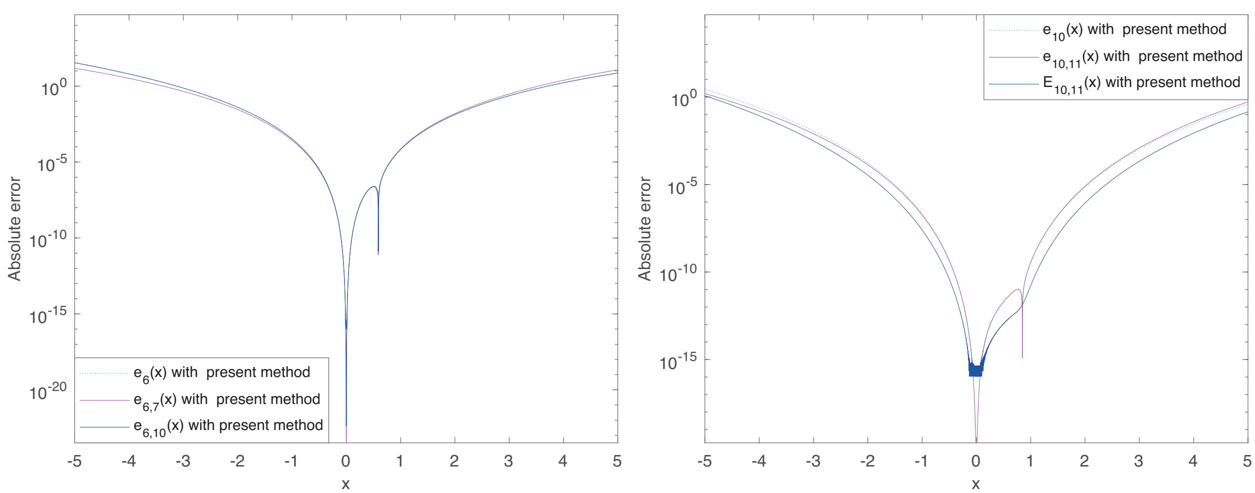
$$\left\{\mathbf{F}_1\mathbf{QM}+\lambda_f\mathbf{QK}_f^Q\mathbf{N}_f\right\}\mathbf{A}=\mathbf{G}. \tag{5.14}$$

Thus, the augmented matrix $\left[\mathbf{W};\mathbf{G}\right]$ from (3.5) can be written and thus, according to the collocation method in Section (3), the new augmented matrix form $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ from (3.6) can be calculated. Hence, this system $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ is solved and thus the coefficient matrix $\mathbf{A}$ is calculated as



(a) $e_N(x)$ (b) $e_N(x)$ and $E_{N,M}(x)$

Figure 5. Comparison of the actual and the improved errors at some points of the prob. (5.7)-(5.8)



(a) $e_N(x)$ and $e_{N,M}(x)$ (b) $e_N(x)$, $e_{N,M}(x)$ and $E_{N,M}(x)$

Figure 6. Comparison of the actual, the estimated and the improved errors at some points of the prob. (5.7)-(5.8)

$$\mathbf{A} = \begin{bmatrix} -250/1121 & 299/906 & 733/3423 & 437/8225 & 61/6873 & 47/22200 \end{bmatrix}^T .$$

Finally, by substituting the obtained coefficient matrix $\mathbf{A}$ in the expression (2.5), the approximate solution is obtained as

$$y_5(x) = 0.0677477594012x^5 + 0.142004818753x^4 + 0.50973034525x^3 + 0.998563502239x^2 + x + 4.33680868994e-19.$$

Thus, it is applied the method in Section (4) for $M = 7$ and it can be obtained $e_{5,7}(x)$, $y_{5,7}(x)$ and $E_{5,7}(x)$.

In Figure (7), we compare the exact solution, the obtained approximate solution by the method in Section

(3) and the obtained improved approximate solution by the method in Section (4) of the problem (5.11)-(5.12) for various N and M values. In Table (7) and Figure (8), the actual errors, estimated errors, and improved errors of the problem (5.11)-(5.12) are compared for various N and M values. In Table (6), the actual absolute errors of the problem (5.11)-(5.12) are compared with CASW[5], IHP[35], BC[39], EM[37] and DT[6] methods. According to these comparisons, the presented method gives better results than CASW[5], EM[37] and DT[6] methods, while it gives close results with IHP[35] and BC[39] methods. Matrix operations in the presented method are less and for higher order problems, it can be calculated in a short time with the help of Matlab. These CPU times (in seconds) are shown in Table (7). According to these results, it can be said that the method in Section (3) and Section (4) is quite effective, and the calculations in Matlab show that it is reliable.

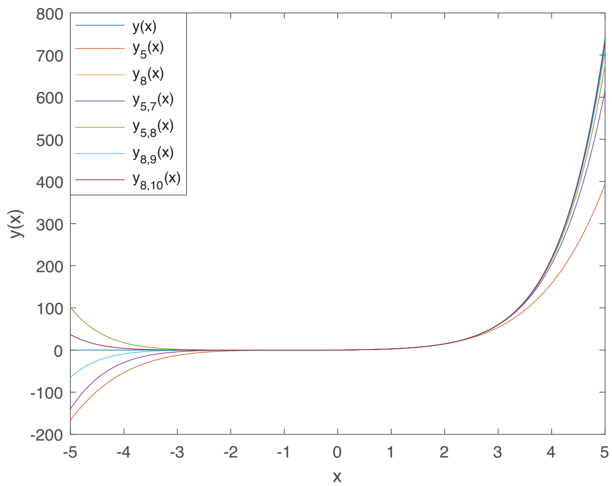


Figure 7. Comparison of the exact, the approximate and the improved approximate solutions at some points of the prob. (5.11)-(5.12)

Table 6. Numerical results of the absolute errors of the problem (5.11)-(5.12)

x_i	CASWM[5]	DTM[6]	IHPM[35]	BCM[39] $e_5(x_i)$	EM[37] $E_{8,8}(x_i)$	PM $e_5(x_i)$
0.1	1.34917637e-03	1.00118319e-02	2.314814815e-06	6.8485e-06	6.3466e-004	6.8485e-06
0.2	1.15960044e-03	2.78651355e-02	9.259259259e-06	1.1282e-05	6.0386e-004	1.1282e-05
0.4	5.93105645e-02	7.55356316e-02	3.703703704e-05	7.9162e-06	6.4560e-004	7.9162e-06
0.6	4.39287720e-02	1.09551714e-01	8.333333333e-05	1.4775e-05	7.0993e-004	1.4775e-05
0.8	1.34514117e-02	6.94512700e-02	1.481481481e-04	5.4050e-06	7.9955e-004	5.4050e-06
0.9	1.32045209e-02	1.00034260e-02	1.875000000e-04	3.9206e-05	8.7135e-004	3.9206e-05

Example 5.4 Fourthly, let's take the first order linear Volterra integro-differential equation

$$y^{'} + \int_0^x y(t)dt = 1, \quad 0 \le x, t \le 1 \tag{5.15}$$

with the initial condition

$$y(0) = 0. \tag{5.16}$$

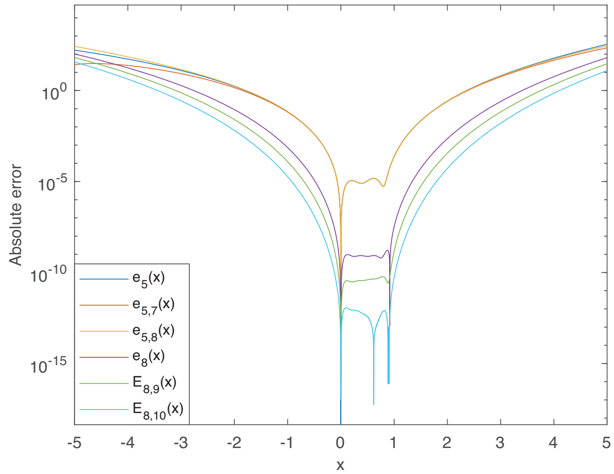


Figure 8. Comparison of the actual, the estimated and the improved errors at some points of the prob. (5.11)-(5.12)

Table 7. Comparison of the absolute errors for $(N, M) = (5,7), (5,8), (8,9), (8,10)$ of the prob. (5.11)-(5.12)

	Absolute errors for the approximate solution	Estimated errors for the for approximate solution		Absolute errors for the corrected approximate solution	
x_i	$\lvert e_5(x_i)\rvert = \lvert y(x_i) - y_5(x_i)\rvert$	$\lvert e_{5,7}(x_i)\rvert$	$\lvert e_{5,8}(x_i)\rvert$	$\lvert E_{5,7}(x_i)\rvert$	$\lvert E_{5,8}(x_i)\rvert$
0	4.3368e-19	0	1.1189e-16	4.3368e-19	1.1232e-16
0.2	1.1282e-05	1.1259e-05	1.1283e-05	2.2665e-08	8.1718e-10
0.4	7.9162e-06	7.8910e-06	7.9170e-06	2.5198e-08	8.4300e-10
0.6	1.4775e-05	1.4747e-05	1.4775e-05	2.7325e-08	8.5072e-10
0.8	5.4050e-06	5.3778e-06	5.4059e-06	2.7236e-08	8.3637e-10
1	2.3540e-04	2.3464e-04	2.3537e-04	7.6359e-07	3.2821e-08
CPU time(s)	0.2344	0.2500	0.2656	0.2969	0.3125
x_i	$\lvert e_8(x_i)\rvert = \lvert y(x_i) - y_8(x_i)\rvert$	$\lvert e_{8,9}(x_i)\rvert$	$\lvert e_{8,10}(x_i)\rvert$	$\lvert E_{8,9}(x_i)\rvert$	$\lvert E_{8,10}(x_i)\rvert$
0	2.0470e-16	2.3766e-16	3.2752e-15	3.2960e-17	3.0705e-15
0.2	8.1718e-10	8.4788e-10	8.1620e-10	3.0698e-11	9.7666e-13
0.4	8.4300e-10	8.7922e-10	8.4200e-10	3.6224e-11	1.0002e-12
0.6	8.5072e-10	8.9475e-10	8.4973e-10	4.4035e-11	9.8660e-13
0.8	8.3637e-10	8.9446e-10	8.3552e-10	5.8092e-11	8.4573e-13
1	3.2821e-08	3.1358e-08	3.2770e-08	1.4628e-09	5.1118e-11
CPU time(s)	0.3281	0.3594	0.4688	0.5156	0.5938

The exact solution to this problem is $\sin x$. Now, let's find the approximate solutions for this problem in the form

$$y_N(x) = \sum_{n=0}^{3} a_n Q_n(x) \tag{5.17}$$

by taking $N = 3$ in Section (2). According to the method in Section (3), collocation points are computed as $\left\{x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1\right\}$ and $m = 1$, $F_0 = 0$, $F_1 = 1$, $g(x) = 1$, $\lambda_f = 0$, $\lambda_v = 1$, $K_f(x,t) = 0$,

$K_v(x,t) = 1$ and from the expression (3.4), we write the basic matrix equation

$$\left\{ \mathbf{F}_1\mathbf{Q}\mathbf{M} + \lambda_v \bar{\mathbf{Q}}\bar{\mathbf{K}}_v^Q \bar{\mathbf{N}}_v \right\}\mathbf{A} = \mathbf{G}. \tag{5.18}$$

Thus, the augmented matrix $\left[\mathbf{W};\mathbf{G}\right]$ from (3.5) can be written and thus according to the collocation method in Section (3), the new augmented matrix form $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ from (3.6) can be calculated. We solve this system $[\widetilde{\mathbf{W}};\widetilde{\mathbf{G}}]$ and so we get the coefficient matrix

$$\mathbf{A} = \left[\begin{array}{cccc} 27/23992 & 1675/2999 & -27/23992 & -117/5998 \end{array}\right]^T.$$

Lastly, after this coefficient matrix $\mathbf{A}$ in the expression (2.5), we gain the approximate solution

$$y_3(x) = -0.156052017339x^3 - 0.00450150050017x^2 + x.$$

Applying the method in Section (4) for $M = 7$ and we have $e_{3,5}(x)$, $y_{3,5}(x)$ and $E_{3,3}(x)$.

The approximate solutions $y_N(x)$, $y_{N,M}(x)$ and the exact solution of the problem (5.15)-(5.16) are compared for various values of N and M in Figure (9).

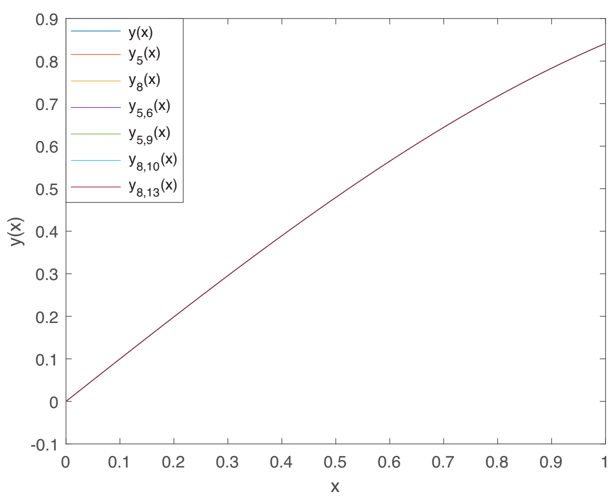


Figure 9. Comparison of the exact, the approximate and the improved approximate solutions at some points of the prob. (5.15)-(5.16)

Table (9) and Figure (10) show comparison of the actual errors, estimated errors and improved errors of the problem (5.15)-(5.16) for various N and M values. In Table (8), the actual absolute errors of the problem (5.15)-(5.16) are compared with TC[33], Tau[31] and BC[39] methods. According to these comparisons, in TC[33] and Tau[31] method, it gives good results at near zero points, but not as close to 1. Thus, it may not be very useful considering a wider range. But according to BC[39] and the presented method, it gives consistent results in the given range. The fact that there are fewer matrix operations in the presented method, added positive results to the method.

Table 8. Numerical results of the actual errors of the problem (5.15)-(5.16)

x_i	TCM[33] $e_5(x_i)$	TM[31]	BCM[39] $e_5(x_i)$	PM $e_5(x_i)$
0	0	0	0	0
0.2	2.60e-09	0.25e-8	4.0240e-007	4.0240e-07
0.4	3.24e-07	0.3244e-6	2.0574e-007	2.0574e-07
0.6	5.53e-06	0.55266e-5	3.7576e-007	3.7576e-07
0.8	4.12e-05	0.412424e-4	1.8172e-007	1.8172e-07
1	1.96e-04	0.1956819e-3	9.6665e-006	9.6665e-06

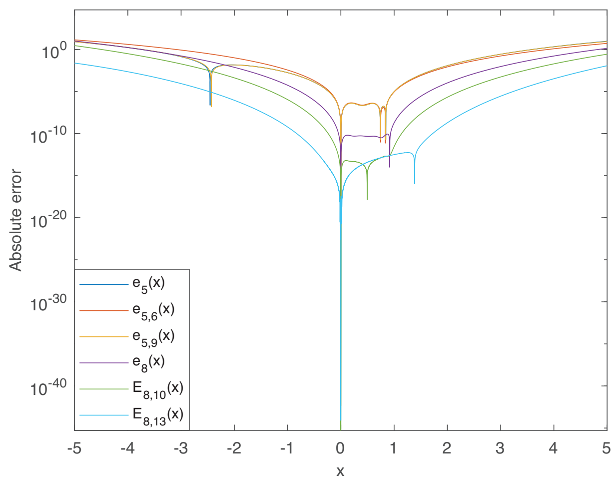


Figure 10. Comparison of the actual, the estimated and the improved errors at some points of the prob. (5.15)-(5.16)

Example 5.5 Finally, let's take the first order linear Fredholm integro-differential equation

$$y^{'} - \int_0^1 xty(t)dt = 1 - \frac{1}{3}x, \quad 0 \le x, t \le 1 \tag{5.19}$$

together with initial the condition

$$y(0) = 0. \tag{5.20}$$

The exact solution to this problem is x. Now, let's find the approximate solutions of this problem in the form

$$y_N(x) = \sum_{n=0}^{3} a_n Q_n(x) \tag{5.21}$$

by selecting $N = 3$ as described in Section (2). The collocation points becomes as follows $\left\{x_0 = 0, \quad x_1 = \frac{1}{3}, \quad x_2 = \frac{2}{3}, \quad x_3 = 1\right\}$ and $m = 1$, $F_0 = 0$, $F_1 = 1$, $g(x) = 1 - \frac{1}{3}x$, $\lambda_f = -1$, $\lambda_v = 0$, $K_f(x,t) = xt$, $K_v(x,t) = 0$ and from the expression (3.4), the basic matrix equation is obtained as

$$\left\{\mathbf{F}_1\mathbf{Q}\mathbf{M} + \lambda_f \mathbf{Q}\mathbf{K}_f^Q \mathbf{N}_f\right\}\mathbf{A} = \mathbf{G}. \tag{5.22}$$

Table 9. Comparison of the absolute errors for $(N,M)=(5,6),(5,9),(8,10),(8,13)$ of the prob. (5.15)-(5.16)

	Absolute errors for the approximate solution	Estimated errors for the for approximate solution		Absolute errors for the corrected approximate solution	
x_i	$\lvert e_5(x_i)\rvert = \lvert y(x_i)-y_5(x_i)\rvert$	$\lvert e_{5,6}(x_i)\rvert$	$\lvert e_{5,9}(x_i)\rvert$	$\lvert E_{5,6}(x_i)\rvert$	$\lvert E_{5,9}(x_i)\rvert$
0	0	0	4.4842e-43	0	4.4842e-43
0.2	4.0240e-07	4.4736e-07	4.0239e-07	4.4970e-08	7.1752e-13
0.4	2.0574e-07	2.3950e-07	2.0574e-07	3.3755e-08	7.0356e-13
0.6	3.7576e-07	4.0431e-07	3.7576e-07	2.8543e-08	6.4362e-13
0.8	1.8172e-07	1.2607e-07	1.8173e-07	5.5658e-08	6.4605e-13
1	9.6665e-06	8.7373e-06	9.6664e-06	9.2920e-07	3.8633e-11
x_i	$\lvert e_8(x_i)\rvert = \lvert y(x_i)-y_8(x_i)\rvert$	$\lvert e_{8,10}(x_i)\rvert$	$\lvert e_{8,13}(x_i)\rvert$	$\lvert E_{8,10}(x_i)\rvert$	$\lvert E_{8,13}(x_i)\rvert$
0	0	5.2549e-46	8.0575e-45	5.2549e-46	8.0575e-45
0.2	5.3200e-11	5.3251e-11	5.3200e-11	5.1078e-14	4.4013e-19
0.4	5.2356e-11	5.2405e-11	5.2356e-11	4.9697e-14	4.1374e-19
0.6	4.8571e-11	4.8615e-11	4.8571e-11	4.4475e-14	3.7321e-19
0.8	4.1821e-11	4.1852e-11	4.1821e-11	3.1134e-14	3.2667e-19
1	1.8402e-09	1.8426e-09	1.8402e-09	2.3700e-12	3.1767e-17

By following the other steps of the method, we get the coefficient matrix $\mathbf{A}$

$$\mathbf{A}=\begin{bmatrix} 0 & 1/2 & 0 & 0 \end{bmatrix}^T.$$

We substitute the obtained coefficient matrix $\mathbf{A}$ in the expression (2.5) and thus we get the approximate solution

$$y_3(x)=x$$

which is the exact solution. If the exact solution of the given equation is of the polynomial type, the obtained approximate solution with this technique give us the exact solution. This is one of the advantages of the method.

6. Conclusions

In this article, a collocation method based on Pell-Lucas polynomials is presented to solve linear FVIEs. In addition, by presenting the error estimation technique, the errors can be estimated in cases where there is no exact solution to the problem. Moreover, the obtained approximate solutions are improved with the help of the residual correction method. For all these methods, five examples are given and the results of these examples are shown in tables and graphs, and their implementation is clearly made. According to these results, it can be said that the errors decrease as the N value increases, the estimated errors are very close to the actual errors and the improved errors give better results than the actual errors. An advantage of the method is that if the exact solution of the given equation is of the polynomial type, the approximate solution obtained by the presented method gives us the exact solution. This result is seen from Example (5.5). Additionally, the comparisons with other methods available in the literature are also made in Table (6) and Table (8). It can be observed that the presented method according to these data gives good results compared to most methods. All calculations in this study can be calculated in a very short time with the help of Matlab, so the method is both reliable and useful. Additionally, the presented method can be developed for integro-differential equations or integral equations.

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