

A Study on Horadam Hybrid Numbers

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Abstract: In this paper, we study Horadam hybrid numbers. For these numbers, we give the exponential generating function, Poisson generating function, generating matrix, Vajda's, Catalan's, Cassini's, and d'Ocagne's identities. In addition, we offer Honsberger formula, general bilinear formula, and some summation formulas for these numbers.

Key words: Hybrid number, Horadam hybrid number, Horadam sequence

1. Introduction

The hybrid numbers, defined by Özdemir [9], are a mixture of real numbers, complex numbers, dual numbers, and hyperbolic numbers. First of all, we review these numbers briefly.

The foundation of complex numbers was laid by Rene Descartes in 1637 and they have been studied for many years. The well-known set of these numbers is

$$C = \{z = x + iy \mid i^2 = -1, x, y \in \mathbb{R}\}.$$

Dual numbers were first proposed by William Kingdon Clifford in [3] to solve some algebraic problems and were developed over time. A dual number is in the form of $a + \varepsilon a^*$ where a and $a^* \in \mathbb{R}$. These numbers are similar to a complex number at first glance, but the most striking difference is that $\varepsilon^2 = 0$. Dual numbers has interesting features and the set of dual numbers is shown as

$$D = \{z = x + \varepsilon y \mid \varepsilon^2 = 0, x, y \in \mathbb{R}\}$$

and forms a zero-divisional ring. The readers can find more information on these numbers in [3]. The last type of numbers is hyperbolic numbers. The set including the number h which is not a real number but its square is equal to 1, is called a set of hyperbolic numbers and defined as

$$P = \{z = x + hy \mid h^2 = 1, x, y \in \mathbb{R}\}.$$

Some works on these sets of numbers are listed in [1, 10].

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Özdemir [9], with the help of these three sets of numbers, defined a new set called Hybrid numbers:

$$K = \{a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \mid a, b, c, d \in \mathbb{R}, \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{i}\mathbf{h} = -\mathbf{ih} = \varepsilon + \mathbf{i}\}. \tag{1.1}$$

This set is a new set of numbers that are completely different from the three sets of numbers mentioned above, but with similar characteristics in some respects. The author obtained the following multiplication table:

Table 1. Multiplication rules for 1, i , ε , and h .

\cdot	1	i	ε	h
1	1	i	ε	h
i	i	-1	$1 - h$	$\varepsilon + i$
ε	ε	$h + 1$	0	$-\varepsilon$
h	h	$-\varepsilon - i$	ε	1

The conjugation of hybrid number $Z = a + b\mathbf{i} + c\varepsilon + d\mathbf{h}$ is shown as follows:

$$\bar{Z} = a - b\mathbf{i} - c\varepsilon - d\mathbf{h}.$$

The character of hybrid number Z is as

$$\mathcal{C}(Z) = Z\bar{Z} = a^2 + (b - c)^2 - c^2 - d^2$$

and the norm of Z is calculated as

$$N(Z) = \sqrt{|\mathcal{C}(Z)|} = \sqrt{|a^2 + (b - c)^2 - c^2 - d^2|}.$$

In addition, the hybrid number Z can be divided into scalar and vector parts

$$S_Z = a, \quad V_Z = b\mathbf{i} + c\varepsilon + d\mathbf{h}$$

and we can show hybrid number Z as

$$Z = S_Z + V_Z.$$

The readers can find detailed information about the character and norm of a hybrid number in [9].

Horadam first defined the generalized Fibonacci numbers in [4] and then the Horadam numbers in [6]. Also Horadam gave the formula for negative index terms of Horadam numbers in [6]. Other studies on the Horadam numbers can be found in [5, 7, 8]. These numbers are defined as follows:

$$w_n = pw_{n-1} - qw_{n-2}, \quad n \geq 2 \tag{1.2}$$

with initial conditions $w_0 = a$ and $w_1 = b$. Here, a, b, p , and q are integers and $p^2 - 4q \geq 0$. The Binet's formula for Horadam numbers is

$$w_n = \frac{A\alpha^n - B\beta^n}{\Delta}, \tag{1.3}$$

where $A = b - a\beta$ and $B = b - a\alpha$. Also α and β are

$$\alpha = \frac{p + \sqrt{p^2 - 4q}}{2} \text{ and } \beta = \frac{p - \sqrt{p^2 - 4q}}{2}. \tag{1.4}$$

Clearly, α and β are the roots of the characteristic equation

$$t^2 - pt + q = 0 \tag{1.5}$$

and also satisfy

$$\alpha + \beta = p, \quad \alpha - \beta = \Delta, \quad \alpha\beta = q, \quad \alpha^2 = p\alpha - q, \quad \beta^2 = p\beta - q, \tag{1.6}$$

where $\Delta = \sqrt{p^2 - 4q}$. In addition, A and B have the following properties

$$A - B = a\Delta, \quad A + B = 2b - ap, \quad AB = b^2 - pab + qa^2. \tag{1.7}$$

The generating function for the sequences $\{w_n(a, b; p, q)\}_{n=0}^\infty$ is

$$g(t) = \frac{a + (b - pa)t}{1 - pt + qt^2} \tag{1.8}$$

and the exponential generating function for the sequences $\{w_n(a, b; p, q)\}_{n=0}^\infty$ is

$$w_k(t) = \sum_{n=0}^\infty w_n^k t^n. \tag{1.9}$$

The number sequences $\{u_n\}$ and $\{v_n\}$ are special cases of the sequence $\{w_n\}$ and defined as

$$\mathbf{u}_n(p, q) = w_n(1, p; p, q) \tag{1.10}$$

and

$$v_n(p, q) = w_n(2, p; p, q), \tag{1.11}$$

respectively.

In Eq. (1.3), if we write $a = 1$ and $b = p$, we obtain the following Binet's formula for the sequence $\{\mathbf{u}_n\}$:

$$\mathbf{u}_n = \frac{\alpha^{n+1} - \beta^{n+1}}{\Delta}.$$

However, in this article, we use the following representation of Cerda-Morales [2]:

$$u_n = \frac{\alpha^n - \beta^n}{\Delta}. \tag{1.12}$$

If we take $a = 2$ and $b = p$, we get the following Binet's formula

$$v_n = \alpha^n + \beta^n. \tag{1.13}$$

In [2], Cerda-Morales gave the 2×2 matrix

$$U(p, q) = \begin{bmatrix} p & -q \\ 1 & 0 \end{bmatrix} \tag{1.14}$$

and showed that

$$U^n(p, q) = \begin{bmatrix} u_{n+1} & -qu_n \\ u_n & -qu_{n-1} \end{bmatrix}. \tag{1.15}$$

For abbreviation, we will omit (p, q) , denoting it by U .

Recently, Szynal-Liana [11, 12] mentioned an attractive definition, namely Horadam hybrid numbers. The author defined these numbers and gave Binet’s formula, generating function and character for Horadam hybrid numbers. According to the study, for any natural number n , n th Horadam hybrid number is

$$\mathcal{H}_n = w_n + \mathbf{i}w_{n+1} + \varepsilon w_{n+2} + \mathbf{h}w_{n+3}.$$

The Binet’s formula for the Horadam Hybrid numbers is

$$\mathcal{H}_n = \frac{A\underline{\alpha}^n - B\underline{\beta}^n}{\Delta} \tag{1.16}$$

where $\underline{\alpha} = 1 + \mathbf{i}\alpha + \varepsilon\alpha^2 + \mathbf{h}\alpha^3$ and $\underline{\beta} = 1 + \mathbf{i}\beta + \varepsilon\beta^2 + \mathbf{h}\beta^3$. The generating function for Horadam hybrid numbers is

$$G(t) = \frac{\mathcal{H}_0 + (\mathcal{H}_1 - p\mathcal{H}_0)t}{1 - pt + qt^2}.$$

The character of Horadam hybrid numbers is given by

$$\begin{aligned} \mathcal{C}(\mathcal{H}_n) = & w_n^2(1 - p^2q^2) + w_nw_{n+1}(2q + 2p^3q - 2pq^2) \\ & + w_{n+2}^2(1 - 2p - p^4 + 2p^2q - q^2). \end{aligned} \tag{1.17}$$

Throughout the paper, we prefer the lowercase letters for integer sequences and the uppercase letters for hybrid number sequence to cancel any complexity except for the usual representations.

2. Main results

We give our results in this section. Note that the definition is reduced to certain special cases depending on the choice of the parameters a, b, p , and q as shown in Table 2.

Table 2. Special hybrid numbers associated with choices of a, b, p , and q .

Horadam hybrid numbers	a	b	p	q	Symbol
Generalized Fibonacci	0	1	p	q	\mathcal{U}_n
Generalized Lucas	2	p	p	q	\mathcal{V}_n
Fibonacci	0	1	1	-1	\mathcal{Q}_n
Lucas	2	1	1	-1	\mathcal{K}_n
Pell	0	1	2	-1	\mathcal{P}_n
Pell-Lucas	2	2	2	-1	\mathcal{R}_n
Modified Pell	1	1	2	-1	\mathcal{MP}_n
Jacobsthal	0	1	1	-2	\mathcal{J}_n
Jacobsthal-Lucas	2	1	1	-2	j_n
Mersenne	0	1	3	2	\mathcal{M}_n
Fermat	1	3	3	-2	\mathcal{F}_n

We start with the following theorem.

Theorem 2.1 *Let n be a positive integer. Then the Horadam hybrid numbers satisfy the recurrence relation*

$$\mathcal{H}_{n+1} = p\mathcal{H}_n - q\mathcal{H}_{n-1} \tag{2.1}$$

with initial conditions

$$\mathcal{H}_0 = a + ib + \varepsilon(pb - qa) + h(p^2b - q(a + b))$$

and

$$\mathcal{H}_1 = b + i(pb - qa) + \varepsilon(p^2b - q(a + b)) + h(p^3 + q^2a - pq(a + 2b) - b).$$

Proof By using Eq. (1.2), the proof can be completed readily. □

According to Eq.(1.17), we have the norm of Horadam hybrid numbers as follows

$$\begin{aligned} \mathcal{N}(\mathcal{H}_n) &= \sqrt{|\mathcal{C}(\mathcal{H}_n)|} \\ &= \sqrt{\left| \begin{matrix} w_n^2(1 - p^2q^2) + w_nw_{n+1}(2q + 2p^3q - 2pq^2) \\ + w_{n+2}^2(1 - 2p - p^4 + 2p^2q - q^2) \end{matrix} \right|}. \end{aligned} \tag{2.2}$$

More simplification gives

$$\mathcal{N}(\mathcal{H}_n) = \sqrt{|w_n^2 + (w_{n+1} - w_{n+2})^2 - w_{n+2}^2 - w_{n+3}^2|}. \tag{2.3}$$

Note that Özdemir [9] explained how to find the character and norm of a hybrid number by calculating the determinant of a 2×2 matrix W . Inspired by this approach, we can define the following matrix to compute the character and norm of the Horadam hybrid numbers

$$W_n = \begin{bmatrix} w_n + w_{n+2} & w_{n+1} - w_{n+2} + w_{n+3} \\ w_n + 2 - w_{n+1} + w_{n+3} & w_n - w_{n+2} \end{bmatrix}.$$

In E_2^4 , \mathcal{H}_n is a spacelike, timelike, and lightlike if $\mathcal{C}(\mathcal{H}_n) < 0$, $\mathcal{C}(\mathcal{H}_n) > 0$ and $\mathcal{C}(\mathcal{H}_n) = 0$, respectively. Therefore, we can say that while it is possible to mention these three cases for the Horadam hybrid numbers and the first two sequences in Table 2, others in Table 2 such as Pell or Mersenne hybrid numbers are only spacelike. In addition, the hybrid number sequences whose entries consist of the components of a strictly monotonic increasing sequence are also spacelike. In order to compute our next result, we will use Table 3. Here, $\theta = 1 + q - pq - q^3$ and $\eta = -\mathbf{i}u_2 + \varepsilon(qu_1 - u_2) + \mathbf{h}u_1$. To convince the reader, we prove one of them here. From the definitions of $\underline{\alpha}$ and $\underline{\beta}$, we can write

$$\begin{aligned} \underline{\alpha}\underline{\beta} &= (1 + \mathbf{i}\alpha + \varepsilon\alpha^2 + \mathbf{h}\alpha^3)(1 + \mathbf{i}\beta + \varepsilon\beta^2 + \mathbf{h}\beta^3) \\ &= 1 + \mathbf{i}\beta + \varepsilon\beta^2 + \mathbf{h}\beta^3 \\ &\quad + \mathbf{i}\alpha - \alpha\beta + (1 - \mathbf{h})\alpha\beta^2 + (\varepsilon + \mathbf{i})\alpha\beta^3 \\ &\quad + \varepsilon\alpha^2 + (h + 1)\alpha^2\beta - \varepsilon\alpha^2\beta^3 \\ &\quad + h\alpha^3 - (\varepsilon + \mathbf{i})\alpha^3\beta - \varepsilon\alpha^3\beta^2 + (\alpha\beta)^3 \\ &= 2\underline{\beta} - \theta + \Delta \begin{pmatrix} \mathbf{i}u_1 + \varepsilon u_2 + \mathbf{h}u_3 \\ +q(-\mathbf{i}u_2 + \varepsilon(qu_1 - u_2) + \mathbf{h}u_1) \end{pmatrix} \\ &= 2\underline{\beta} - \theta + \Delta(\mathcal{U}_0 + q\eta). \end{aligned}$$

Table 3. Relationships between $\underline{\alpha}$ and $\underline{\beta}$.

\cdot	$\underline{\alpha}$	$\underline{\beta}$
$\underline{\alpha}$	$2\underline{\alpha} - \mathcal{C}(\underline{\alpha})$	$2\underline{\beta} - \theta + \Delta (\mathcal{U}_0 + q\eta)$
$\underline{\beta}$	$2\underline{\alpha} - \theta - \Delta (\mathcal{U}_0 + q\eta)$	$2\underline{\beta} - \mathcal{C}(\underline{\beta})$

We now give the following useful result, which we will use to obtain a number of special identities.

Theorem 2.2 (Vajda’s identity) For any integers n, r , and s , we have

$$\mathcal{H}_{n+r}\mathcal{H}_{n+s} - \mathcal{H}_n\mathcal{H}_{n+r+s} = ABq^n u_r [2\mathcal{U}_s - u_s\theta - v_s (\mathcal{U}_0 + q\eta)]. \tag{2.4}$$

Proof From the Binet’s formula for the Horadam hybrid numbers, we get

$$\begin{aligned} \mathcal{H}_{n+r}\mathcal{H}_{n+s} - \mathcal{H}_n\mathcal{H}_{n+r+s} &= \frac{1}{\Delta^2} [(A\underline{\alpha}\alpha^{n+r} - B\underline{\beta}\beta^{n+r})(A\underline{\alpha}\alpha^{n+s} - B\underline{\beta}\beta^{n+s}) \\ &\quad - (A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n)(A\underline{\alpha}\alpha^{n+r+s} - B\underline{\beta}\beta^{n+r+s})] \\ &= \frac{AB}{\Delta^2} (-\underline{\alpha}\underline{\beta}\alpha^{n+r}\beta^{n+s} - \underline{\beta}\underline{\alpha}\alpha^{n+s}\beta^{n+r} + \underline{\alpha}\underline{\beta}\alpha^n\beta^{n+r+s} + \underline{\beta}\underline{\alpha}\alpha^{n+r+s}\beta^n) \\ &= \frac{AB}{\Delta^2} q^n (\alpha^r - \beta^r) (\underline{\beta}\underline{\alpha}\alpha^s - \underline{\alpha}\underline{\beta}\beta^s) \\ &= \frac{AB}{\Delta} q^n u_r [(2\underline{\alpha} - \theta - \Delta (\mathcal{U}_0 + q\eta)) \alpha^s \\ &\quad - (2\underline{\beta} - \theta + \Delta (\mathcal{U}_0 + q\eta)) \beta^s] \\ &= \frac{AB}{\Delta} q^n u_r [2 (\underline{\alpha}\alpha^s - \underline{\beta}\beta^s) - \theta (\alpha^s - \beta^s) - \Delta (\mathcal{U}_0 + q\eta) (\alpha^s + \beta^s)] \\ &= ABq^n u_r [2\mathcal{U}_s - u_s\theta - v_s (\mathcal{U}_0 + q\eta)]. \end{aligned}$$

□

We have the following particular cases from Vajda’s identity:

- For $r = -s$, we reduce to Catalan’s identity

$$\mathcal{H}_{n-s}\mathcal{H}_{n+s} - [\mathcal{H}_n]^2 = -ABq^{n-s} u_s [2\mathcal{U}_s - u_s\theta - v_s (\mathcal{U}_s + q\eta)], \tag{2.5}$$

- For $s = -r = 1$, we find Cassini’s identity

$$\mathcal{H}_{n-1}\mathcal{H}_{n+1} - [\mathcal{H}_n]^2 = -ABq^{n-1} [2\mathcal{U}_1 - \theta - p (\mathcal{U}_0 + q\eta)], \tag{2.6}$$

- For $s = m - n$ and $r = 1$, we obtain d’Ocagne’s identity

$$\mathcal{H}_{n+1}\mathcal{H}_m - \mathcal{H}_n\mathcal{H}_{m+1} = ABq^n [2\mathcal{U}_{m-n} - u_{m-n}\theta - v_{m-n} (\mathcal{U}_0 + q\eta)]. \tag{2.7}$$

Now we give some numerical examples for the above well-known generalized identities.

Example 2.3 If we take $(a, b; p, q) = (0, 1; 1, -1)$, the sequence $\{\mathcal{H}\}_{n=0}^\infty$ reduces to the sequence of the Fibonacci hybrid numbers $\{\mathcal{Q}\}_{n=0}^\infty$ according to Table 2. Then we obtain $A = 1$ and $B = 1$. From Eqs. (1.12) and (1.13), the sequences $\{u_n\}_{n=0}^\infty$ and $\{v_n\}_{n=0}^\infty$ give the sequences $\{F_n\}_{n=0}^\infty$ and $\{L_n\}_{n=0}^\infty$, respectively, where F_n is the n th Fibonacci number and L_n is the n th Lucas number. Also, we have $\theta = 2$ and $\eta = -\mathbf{i} - 2\varepsilon + \mathbf{h}$. Thus, we obtain Vajda's, Catalan's, and Cassini's identities for the sequence the Fibonacci hybrid numbers as

$$\mathcal{Q}_{n+r}\mathcal{Q}_{n+s} - \mathcal{Q}_n\mathcal{Q}_{n+r+s} = (-1)^n F_r [2(\mathbf{i}F_{s+1} + \varepsilon F_{s+2} + \mathbf{h}F_{s+3}) - L_s(2\mathbf{i} + 3\varepsilon + \mathbf{h})],$$

$$\mathcal{Q}_{n-s}\mathcal{Q}_{n+s} - [\mathcal{Q}_n]^2 = (-1)^{n+s+1} F_s [2(\mathbf{i}F_{s+1} + \varepsilon F_{s+2} + \mathbf{h}F_{s+3}) - L_s(2\mathbf{i} + 3\varepsilon + \mathbf{h})]$$

and

$$\mathcal{Q}_{n-1}\mathcal{Q}_{n+1} - [\mathcal{Q}_n]^2 = (-1)^n(\varepsilon + 5\mathbf{h})$$

respectively. If we substitute $n = 10$, $r = 4$ and $s = 6$ into the these three identities, we obtain

$$\mathcal{Q}_{14}\mathcal{Q}_{16} - \mathcal{Q}_{10}\mathcal{Q}_{20} = -30\mathbf{i} - 36\varepsilon + 150\mathbf{h},$$

$$\mathcal{Q}_4\mathcal{Q}_{16} - [\mathcal{Q}_{10}]^2 = 80\mathbf{i} + 96\varepsilon - 400\mathbf{h}$$

and

$$\mathcal{Q}_9\mathcal{Q}_{11} - [\mathcal{Q}_{10}]^2 = \varepsilon + 5\mathbf{h}$$

respectively.

Here we give the exponential and Poisson generating function for the Horadam hybrid numbers. To do this, we first define the following series

$$G_e(t) = \sum_{n=0}^\infty \mathcal{H}_n \frac{t^n}{n!}. \tag{2.8}$$

Theorem 2.4 The exponential generating function for the Horadam hybrid numbers is

$$G_e(t) = \frac{A\underline{\alpha}e^{\alpha t} - B\underline{\beta}e^{\beta t}}{\Delta}. \tag{2.9}$$

Proof Considering the MacLaurin expansion for the exponential function, we write

$$\begin{aligned} G_e(t) &= \sum_{n=0}^\infty \mathcal{H}_n \frac{t^n}{n!} = \sum_{n=0}^\infty \frac{A\underline{\alpha}\alpha^n - B\underline{\beta}\beta^n}{\alpha - \beta} \\ &= \frac{1}{\alpha - \beta} \left(A\underline{\alpha} \sum_{n=0}^\infty \frac{(\alpha t)^n}{n!} - B\underline{\beta} \sum_{n=0}^\infty \frac{(\beta t)^n}{n!} \right) \\ &= \frac{1}{\Delta} (A\underline{\alpha}e^{\alpha t} - B\underline{\beta}e^{\beta t}). \end{aligned}$$

□

Theorem 2.5 The Poisson generating function for Horadam hybrid numbers is

$$G_P(t) = \frac{A\underline{\alpha}e^{\alpha t} - B\underline{\beta}e^{\beta t}}{\Delta e^t}. \tag{2.10}$$

Proof Since $G_p(t) = e^{-t}G_e(t)$, we have the result by Theorem 2.4. □

Substituting $G_P(t) = e^{-t}G_e(t)$ into Eq. (2.9), the proof can be completed directly.

In next theorem, we give some summation formulas for the Horadam hybrid numbers.

Theorem 2.6 For every positive integer n , we have

$$\sum_{t=1}^n \mathcal{H}_t = \frac{1}{p-q-1} [\mathcal{H}_{n+1} - q \mathcal{H}_n - \mathcal{H}_1 + q\mathcal{H}_0], \tag{2.11}$$

$$\sum_{t=1}^n \mathcal{H}_{2t-1} = \frac{1}{p^2 - (1+q)^2} [\mathcal{H}_{2n+1} - q^2\mathcal{H}_{2n-1} - (q+1)\mathcal{H}_1 + p\mathcal{H}_0], \tag{2.12}$$

$$\sum_{t=1}^n \mathcal{H}_{2t} = \frac{1}{p^2 - (1+q)^2} [\mathcal{H}_{2n+2} + q^2\mathcal{H}_{2n} - p\mathcal{H}_1 + q(q+1)\mathcal{H}_0] \tag{2.13}$$

and

$$\sum_{t=1}^n (-1)^t \mathcal{H}_t = \frac{1}{p-q+1} [(-1)^{n+1} (\mathcal{H}_{n+1} + q\mathcal{H}_n) + \mathcal{H}_1 + q\mathcal{H}_0]. \tag{2.14}$$

Proof Here, we only show the validity of Eq.(2.11) using the induction method. The others can be demonstrated similarly. For $n = 1$, it is clear that Eq.(2.11) is true. Based on the assumption such that it is true for $n = k$, we can write

$$\begin{aligned} \sum_{t=1}^{k+1} \mathcal{H}_t &= \mathcal{H}_{k+1} + \sum_{t=1}^k \mathcal{H}_t = \mathcal{H}_{k+1} + \frac{1}{p-q-1} [\mathcal{H}_{n+1} - q\mathcal{H}_n - \mathcal{H}_1 + q\mathcal{H}_0] \\ &= \frac{p\mathcal{H}_{k+1} - q\mathcal{H}_k - q\mathcal{H}_{k+1} - \mathcal{H}_{k+1} + \mathcal{H}_{k+1} - \mathcal{H}_1 + q\mathcal{H}_0}{p-q-1} \\ &= \frac{\mathcal{H}_{k+2} - q\mathcal{H}_{k+1} - \mathcal{H}_1 + q\mathcal{H}_0}{p-q-1}, \end{aligned}$$

which is the desired result. □

In order to give some important properties, we define the following matrices

$$Q_n = \begin{bmatrix} \mathcal{H}_{n+1} & \mathcal{H}_n \\ \mathcal{H}_n & \mathcal{H}_{n-1} \end{bmatrix} \text{ and } R_n = \begin{bmatrix} w_{n+1} & w_n \\ w_n & w_{n-1} \end{bmatrix}. \tag{2.15}$$

Using the recurrence relations in (1.2) and (2.1), we obtain

$$Q_n = UQ_{n-1} \text{ and } R_n = UR_{n-1}$$

and

$$Q_n = U^n Q_0 \text{ and } R_n = U^n R_0. \tag{2.16}$$

We give the Honsberger formula for Horadam hybrid numbers in the following theorem.

Theorem 2.7 (Honsberger formula) *Let n and m be any integers. Then,*

$$\mathcal{H}_{n+m} = \frac{1}{AB} [T_{n+1}\mathcal{H}_m - qT_n\mathcal{H}_{m-1}],$$

where $T_n = bw_n - aw_{n+1}$.

Proof Using the matrices (2.15), we can write

$$Q_{n+m} = U^{n+m}Q_0 = U^nU^mQ_0 = R_nR_0^{-1}Q_m.$$

Considering the entries the term 21th of matrix in the left-hand side is equal to the term 21th of the product matrix in the right-hand side, which gives the result. □

Substituting $(m, n) = (n - t, m + t)$ into the Honsberger formula we obtain the following result.

Corollary 2.8 *For any integers m, n , and t , we have*

$$\mathcal{H}_{n+m} = \frac{1}{AB} (T_{n-t+1}\mathcal{H}_{m+t} - qT_{n-t}\mathcal{H}_{m+t-1}).$$

Theorem 2.9 (General bilinear formula) *Let k, l, m , and n be any integers satisfying that $k + l = m + n$ and r be arbitrary integer. We have*

$$\mathcal{H}_k\mathcal{H}_l - \mathcal{H}_m\mathcal{H}_n = (-q)^r (\mathcal{H}_{k-r}\mathcal{H}_{l-r} - \mathcal{H}_{m-r}\mathcal{H}_{n-r}). \tag{2.17}$$

Proof Employing the matrix equations in (2.15) and (2.16), we obtain $U^kQ_l = U^mQ_n$. Considering the entry (2,1) of the last equation, we write

$$\mathcal{H}_k\mathcal{H}_l - \mathcal{H}_m\mathcal{H}_n = (-q)(\mathcal{H}_{k-1}\mathcal{H}_{l-1} - \mathcal{H}_{m-1}\mathcal{H}_{n-1}).$$

Repeating the same operations of r times yields Eq. (2.17). □

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