Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1222 – 1237
© TÜBİTAK
doi:10.3906/mat-1909-56

A special cone construction and its connections to structured tensors and their spectra

Vehbi Emrah PAKSOY*
Department of Mathematics, Halmos College of Natural Sciences and Oceanography,
Nova Southeastern University, Fort Lauderdale-Davie, USA

Received: 17.09.2019 • **Accepted/Published Online:** 04.05.2020 • **Final Version:** 08.07.2020

Abstract: In this work we construct a cone comprised of a group of tensors (hypermatrices) satisfying a special condition, and we study its relations to structured tensors such as M-tensors and H-tensors. We also investigate its applications to spectra of certain Z-tensors. We obtain an inequality for the spectral radius of certain tensors when the order m is odd.

Key words: H-tensor, M-tensor, Hadamard product, sum-of-squares tensor

1. Introduction

Tensors (hypermatrices) are higher order generalizations of matrices. Due to their frequent appearance in the applications in several different areas, such as computer engineering, medical imaging, and quantum entanglement, they have become popular and effective in research and exploration.

Various useful cone structures of tensors were introduced and studied by several authors ([1–3, 9, 17, 25, 26]). In this paper, we study certain types of tensors and some related constructions that can help explain their intrinsic structures such as decomposition and spectra. More explicitly, we construct a cone of tensors and examine its interaction with structured tensors. Our cone construction contains certain tensors with strong connections to hypergraph theory and its applications. For instance the Laplacian tensor $\mathcal{L}$ and signless Laplacian tensor $\mathcal{D}$ ([8, 21]) are in the cone that we construct in this work. Furthermore, we include some results that connects this work to positive (semi) definite tensors. These tensors are shown to be useful in quantum field theories ([12]).

The structure of this paper is as follows; after giving fundamental definitions, notations, and basic results, we study a certain condition and show that it gives rise to a convex cone. We present several results about the types of tensors contained in this cone. Then, with a specific construction, we concentrate on its effects on Hadamard products. Finally, we prove our main results for several structured tensors including H-, M-, and SOS tensors.

*Correspondence: vp80@nova.edu
2010 *AMS Mathematics Subject Classification:* 15A69

2. Basic definitions and some background

We closely follow the notations presented in the book by L. Qi and Z. Luo ([24]). A tensor (hypermatrix) $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ is a multidimensional array of entries $a_{i_1 i_2 \cdots, i_m}$ in a field K. $i_1 \in \{1, 2, \ldots n_1\}$, $i_2 \in \{1, 2, \ldots n_2\}, \ldots, i_m \in \{1, 2, \ldots, n_m\}$, where $m \geq 2$, and $n_1, \ldots n_m$ are positive integers. In this paper let $n_1 = \cdots = n_m = n$ and consider the tensors with real entries ($K = \mathbb{R}$). The set of all such tensors is denoted by $T_{m,n}$. Here, m is the order and n is the dimension of $\mathcal{A} \in T_{m,n}$. If all entries of $\mathcal{A}$ are nonnegative, we call it a nonnegative tensor. A tensor $\mathcal{A}$ is symmetric if its entries are unchanged under any permutation of indices. $\mathcal{I}$ denotes the identity tensor. Its off-diagonal entries are 0 and diagonal entries are 1. In other words, we have $\mathcal{I}_{i_1 i_2 \cdots i_m} = \delta_{i_1 i_2 \cdots i_m}$, where δ is the generalized Kronecker symbol with m indices. Namely,

Given two tensors $\mathcal{A}$, $\mathcal{B}$, we write $\mathcal{A} \leq \mathcal{B}$ if $a_{i_1 i_2 \cdots i_m} \leq b_{i_1 i_2 \cdots i_m}$ for all indicies $(i_1, i_2 \ldots, i_m)$. For any $\mathcal{A}$ and a vector $\mathbf{x} = (x_1, \ldots, x_n)^T$, we have a homogeneous polynomial

$$\mathcal{A}\mathbf{x}^m = \sum_{i_1, i_2, \ldots, i_m = 1}^{n} a_{i_1 i_2 \cdots i_m} x_{i_1} x_{i_2} \cdots x_{i_m}.$$

When m is even, $\mathcal{A}$ is called positive semidefinite (PSD) if for any nonzero real vector $\mathbf{x}$, $\mathcal{A}\mathbf{x}^m \geq 0$. If the inequality is strict, then $\mathcal{A}$ is called positive definite(PD). If $\mathcal{A}$ is symmetric and we can write

$$\mathcal{A}\mathbf{x}^m = \sum_{j=1}^{p} f_j(\mathbf{x})^2$$

where $f_j(\mathbf{x})(j = 1, \ldots, p)$ are homogeneous polynomials of degree $m/2$, then we call $\mathcal{A}$ a sum-of-squares (SOS) tensor.

We say that a constant λ is an eigenvalue of $\mathcal{A}$ with an eigenvector $\mathbf{x}$, if for all $i \in [n] = \{1, 2, \ldots, n\}$,

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2, \ldots, i_m = 1}^{n} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \lambda x_i^{m-1}. \tag{2.1}$$

We can also write this vector equation as $\mathcal{A}\mathbf{x}^{m-1} = \lambda \mathbf{x}^{[m-1]}$. Set of all eigenvalues of $\mathcal{A}$ is called the spectrum of $\mathcal{A}$. The maximum modulus of the eigenvalues of $\mathcal{A}$ is called the spectral radius, and denoted by $\rho(\mathcal{A})$. An eigenvalue λ is called an H-eigenvalue of $\mathcal{A}$, if it has a real eigenvector $\mathbf{x}$. Consequently, any H-eigenvalue is a real number. The eigenvalues and H-eigenvalues are introduced independently by L. Qi and L.H. Lim ([14, 15, 22]).

Definition 2.1 *Let* $\mathcal{A} = (a_{i_1 \cdots i_m}) \in T_{m,n}$.

(1) $\mathcal{A}$ *is called diagonally dominated if for any* $i \in [n]$,

$$a_{ii \cdots i} \geq \sum_{\substack{i_2, \ldots, i_m \\ \delta_{i i_2 \cdots i_m} = 0}} |a_{i i_2 \ldots i_m}|.$$

If the strict inequality holds, then we call $\mathcal{A}$ *strictly diagonally dominated.*

(2) $\mathcal{A}$ *is called weakly irreducible if the matrix with entries*

$$m_{ij} = \sum_{j\in\{i_2,\ldots,i_m\}} |a_{ii_2\cdots i_m}|$$

is irreducible.

If a nonnegative tensor is weakly irreducible, then its spectral radius is a positive H-eigenvalue with a unique (up to multiplication by a scalar) positive eigenvector ([7]).

Definition 2.2 *Let* $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ *and* $\mathcal{B} = (b_{i_1i_2\cdots i_m})$ *be two tensors. Their Hadamard product* $\mathcal{A}\circ\mathcal{B}$ *is a tensor whose entries are given by*

$$(\mathcal{A}\circ\mathcal{B})_{i_1i_2\cdots i_m} = a_{i_1i_2\cdots i_m} b_{i_1i_2\cdots i_m}.$$

Definition 2.3 *Let* $\mathcal{A}^{(1)},\ldots,\mathcal{A}^{(m)} \in T_{m,n}$*. The BM-product ([18, 19]) of these tensors, denoted by* $Prod(\mathcal{A}^{(1)},\ldots,\mathcal{A}^{(m)})$*, is given by*

$$Prod(\mathcal{A}^{(1)},\ldots,\mathcal{A}^{(m)})_{i_1\cdots i_m} = \sum_{t=1}^{n} a^{(1)}_{i_1t\cdots i_m} a^{(2)}_{i_1i_2t\cdots i_m}\cdots a^{(m)}_{ti_2\cdots i_m}.$$

Note that BM-product is the regular matrix multiplication if $m = 2$. We can only compute the BM-product of m tensors in $T_{m,n}$. Consider the following condition for the tensors in $T_{m,n}$.

$$|a_{ii\cdots i}|^{m-1} \geq \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij\cdots j}|^{m-1}, \qquad \forall i \in [n]. \tag{2.2}$$

We define C=$\{\mathcal{A} \in T_{m,n} : \mathcal{A}$ satisfies (2.2) and $a_{ii\cdots i} \geq 0\}$. We write $\mathcal{A} \in$ C$^\circ$ if the diagonal entries of $\mathcal{A}$ is positive and the inequality in (2.2) is strict.

3. Main results

Proposition 3.1 *C is a convex cone in* $T_{m,n}$

Proof Let $\mathcal{A}\in$ C. For any $t > 0$ and for any $i \in [n]$ we have

$$(t\mathcal{A})^{m-1}_{ii\cdots i} = t^{m-1}a^{m-1}_{ii\cdots i} \geq \sum_{\substack{j=1\\ j\neq i}}^{n} |ta_{ij\cdots j}|^{m-1} = \sum_{\substack{j=1\\ j\neq i}}^{n} |(t\mathcal{A})_{ij\cdots j}|^{m-1}.$$

Therefore, $t\mathcal{A}\in$ C. This shows that C is a cone. Next, suppose that $\mathcal{A}$, $\mathcal{B}\in$ C. For any $t \in [0,1]$ and $i \in [n]$, let $\Lambda = \Big(\sum_{\substack{j=1\\ j\neq i}}^{n} |ta_{ij\cdots j} + (1-t)b_{ij\cdots j}|^{m-1}\Big)^{1/(m-1)}$. Then, using Minkowski inequality

$$\begin{aligned}
\Lambda &\leq \Big(\sum_{\substack{j=1\\ j\neq i}}^{n} |ta_{ij\cdots j}|^{m-1}\Big)^{1/(m-1)} + \Big(\sum_{\substack{j=1\\ j\neq i}}^{n} |(1-t)b_{ij\cdots j}|^{m-1}\Big)^{1/(m-1)} \\
&\leq (t^{m-1}a^{m-1}_{ii\cdots i})^{1/(m-1)} + ((1-t)^{m-1}b^{m-1}_{ii\cdots i})^{1/(m-1)} \\
&= ta_{ii\cdots i} + (1-t)b_{ii\cdots i}.
\end{aligned}$$

Therefore, we obtain

$$\Lambda^{m-1} = \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} |ta_{ij\cdots j} + (1-t)b_{ij\cdots j}|^{m-1}\Big) \leq (ta_{ii\cdots i} + (1-t)b_{ii\cdots i})^{m-1},$$

which implies that $t\mathcal{A} + (1-t)\mathcal{B} \in$ C for all $t \in [0,1]$. □

Theorem 3.2 *Let $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(p)} \in C$. Then, the Hadamard product $\mathcal{A}^{(1)} \circ \cdots \circ \mathcal{A}^{(p)}$ is also in C.*

Proof Let $\mathcal{B} = \mathcal{A}^{(1)} \circ \cdots \circ \mathcal{A}^{(p)}$. Then, the entries of the tensor $\mathcal{B}$ are $b_{i_1 i_2 \cdots i_m} = \prod_{k=1}^{p} a^{(k)}_{i_1 i_2 \cdots i_m}$. In particular, we observe that $b_{ii\cdots i} = \prod_{k=1}^{p} a^{(k)}_{ii\cdots i}$, and $b_{ij\cdots j} = \prod_{k=1}^{p} a^{(k)}_{ij\cdots j}$. We compute that

$$\begin{aligned} |b_{ii\cdots i}|^{m-1} &= |a^{(1)}_{ii\cdots i}|^{m-1} \cdots |a^{(p)}_{ii\cdots i}|^{m-1} \\ &\geq \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a^{(1)}_{ij\cdots j}|^{m-1}\Big) \cdots \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a^{(p)}_{ij\cdots j}|^{m-1}\Big) \\ &\geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a^{(1)}_{ij\cdots j}|^{m-1} \cdots |a^{(p)}_{ij\cdots j}|^{m-1} = \sum_{\substack{j=1 \\ j\neq i}}^{n} |b_{ij\cdots j}|^{m-1}. \end{aligned}$$

Therefore, $\mathcal{B} \in$C □

Proposition 3.3 *Let $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in C$. Then, we have the stronger inequality*

$$\prod_{k=1}^{m-1} |a^{(k)}_{ii\cdots i}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} \prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}|. \tag{3.1}$$

Proof In order to prove the propositionosition, we need the following result by Kwon and Bae ([13]).

Theorem 3.4 *([13]), Theorem 1.1) Let $a_{ji} > 0,\ p_j > 0 (i = 1, 2, \ldots, n; j = 1, 2, \ldots, m),\ 1 - e_i + e_j \geq 0 (i, j = 1, 2, \ldots, n),\ \Lambda := \sum_{j=1}^{m} \frac{1}{p_j}$ and $\{p_{j_1}, p_{j_2}\} \subset \{p_1, p_2, \ldots, p_m\}$. Then*

$$\sum_{i=1}^{n} \prod_{j=1}^{m} a_{ji} \leq n^{1-\min\{\Lambda,1\}} \Bigg\{1 - \Bigg(\frac{\sum_{i=1}^{n} a^{p_{j_1}}_{j_1 i} e_i}{\sum_{i=1}^{n} a^{p_{j_1}}_{j_1 i}} - \frac{\sum_{i=1}^{n} a^{p_{j_2}}_{j_2 i} e_i}{\sum_{i=1}^{n} a^{p_{j_2}}_{j_2 i}}\Bigg)^2\Bigg\}^{\frac{1}{2\max\{p_{j_1}, p_{j_2}\}}} \prod_{j=1}^{m} \Bigg(\sum_{i=1}^{n} a^{p_j}_{ji}\Bigg)^{\frac{1}{p_j}}.$$

To use this theorem, we need to make some changes in its notation. First, we rename i and j in Theorem 3.4 as j and k respectively. We also change the limits n and m to $n-1$ and $m-1$. We choose all e_r equal, and $p_k = m-1$ for all $k \in [m-1]$. With this set up we obtain $\Lambda = \sum_{k=1}^{m-1} \frac{1}{p_k} = 1$. Finally, we let $a_{kj} = |a^{(k)}_{ij\cdots j}|$. With these modifications and rearranging the values of index j when necessary, Theorem 3.4 becomes

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} \prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}| \leq \prod_{k=1}^{m-1} \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a^{(k)}_{ij\cdots j}|^{m-1}\Big)^{1/(m-1)}. \tag{3.2}$$

On the other hand, since all $\mathcal{A}^{(k)} \in$ C, we have

$$\prod_{k=1}^{m-1} |a_{ii\cdots i}^{(k)}| \geq \prod_{k=1}^{m-1} \Big(\sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij\cdots j}^{(k)}|^{m-1} \Big)^{1/(m-1)}. \tag{3.3}$$

(4) and (5) gives the desired result. □

It is worth mentioning that the condition (2.2) is weaker than the diagonal dominance.

Example 3.5 *Consider the tensor* $\mathcal{A} \in T_{3,2}$ *given by entries* $a_{111} = a_{222} = 0.1$, $a_{122} = a_{211} = 0$ *and all other* $a_{ijk} = -3$. *This tensor satisfies (2.2) but it is not diagonally dominated.*

Let $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in T_{m,n}$. We construct a special BM-product tensor by $\mathcal{P} = \mathrm{Prod}(\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)}, \mathcal{I})$, where $\mathcal{I}$ is the identity tensor whose diagonal entries are 1 and all off-diagonal entries are 0. We have

$$\mathcal{P}_{i_1 i_2 \cdots i_m} = \sum_{t=1}^{n} a_{i_1 t i_3 \cdots i_m}^{(1)} a_{i_1 i_2 t \cdots i_m}^{(2)} \cdots a_{i_1 i_2 \cdots t}^{(m-1)} \delta_{t i_2 \cdots i_m}.$$

It is easy to see that the entries of tensor $\mathcal{P}$ are given by

$$\mathcal{P}_{i_1 i_2 \cdots i_m} = \begin{cases} a_{ii\cdots i}^{(1)} a_{ii\cdots i}^{(2)} \cdots a_{ii\cdots i}^{(m-1)} & \text{if } i_1 = i_2 = \cdots = i_m = i, \\ a_{ij\cdots j}^{(1)} a_{ij\cdots j}^{(2)} \cdots a_{ij\cdots j}^{(m-1)} & \text{if } i_2 = \ldots = i_m = j \neq i. \end{cases}$$

and $\mathcal{P}_{i_1 i_2 \cdots i_m} = 0$ otherwise. For the rest of the paper we use $\mathcal{P}$ only for this special construction. Recall that a tensor is called a Z-tensor if its off-diagonal entries are nonpositive. $\mathcal{P}$ produces a Z-tensor as long as certain requirements are satisfied.

Proposition 3.6 *Suppose* $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in$ *C. Then,* $\mathcal{P} \in$ *C.*

Proof We observe that taking $p = m-1$ in Theorem 3.2 gives the results since $\mathcal{P}_{ii\cdots i} = (\mathcal{A}^{(1)} \circ \cdots \circ \mathcal{A}^{(m-1)})_{ii\cdots i}$ and $\mathcal{P}_{ij\cdots j} = (\mathcal{A}^{(1)} \circ \cdots \circ \mathcal{A}^{(m-1)})_{ij\cdots j}$ □

Proposition 3.7 *Suppose that* m *is even and* $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in T_{m,n}$ *are Z-tensors. Then,* $\mathcal{P}$ *is also a Z-tensor.*

Proof Since the off-diagonal entries of $\mathcal{P}$ are of the form $a_{ij\cdots j}^{(1)} a_{ij\cdots j}^{(2)} \cdots a_{ij\cdots j}^{(m-1)}$, and each term in the product is nonpositive, we conclude that off-diagonal entries of $\mathcal{P}$ are also nonpositive. □

Let $\mathcal{A} \in T_{m,n}$. We define the comparison tensor $M(\mathcal{A})$ of $\mathcal{A}$ by

$$M(\mathcal{A})_{i_1 i_2 \cdots i_m} = \begin{cases} |a_{ii\cdots i}| & \text{if } i_1 = i_2 = \ldots = i_m = i, \\ -|a_{i_1 i_2 \cdots i_m}| & \text{if } (i_2, \ldots i_m) \neq (i_1, i_1 \ldots i_1). \end{cases}$$

Clearly $M(\mathcal{A})$ is a Z-tensor. If we compute the comparison tensor $M(\mathcal{P})$, we obtain $M(\mathcal{P})_{ii\cdots i} = \prod_{k=1}^{m-1} |a_{ii\cdots i}^{(k)}|$, and $M(\mathcal{P})_{ij\cdots j} = -\prod_{k=1}^{m-1} |a_{ij\cdots j}^{(k)}|$.

Definition 3.8 *A tensor $\mathcal{A} \in T_{m,n}$ is called an M-tensor if $\mathcal{A} = s\mathcal{I} - \mathcal{B}$, where $\mathcal{B}$ is a nonnegative tensor and $s \geq \rho(\mathcal{B})$. If $s > \rho(\mathcal{B})$, then $\mathcal{A}$ is a strong M-tensor. We call $\mathcal{A} \in T_{m,n}$ an H-tensor (strong H-tensor) if $M(\mathcal{A})$ is an M-tensor (strong M-tensor).*

For a given Z-tensor $\mathcal{A}$, there are several equivalent conditions for $\mathcal{A}$ to be a strong M-tensor (See [5, 24, 28]). We use the following.

Theorem 3.9 *([24]) Z-tensor $\mathcal{A}$ is a strong M-tensor if and only if there exists some nonnegative vector $\mathbf{x} \in \mathbb{R}^n$ such that $\mathcal{A}\mathbf{x}^{m-1} > \mathbf{0}$.*

The following theorem is our first main result in this paper.

Theorem 3.10 *Let $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in T_{m,n}$ such that each $\mathcal{A}^{(k)}$ satisfies the strict inequality in (2.2). Then, $\mathcal{P}$ is a strong H-tensor. In particular, whenever $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in C^{\circ}$, then $\mathcal{P}$ is strong H-tensor.*

Proof Let $\mathbf{x} \in \mathbb{R}^n$ be any vector. For any i we have

$$\begin{aligned}(M(\mathcal{P})\mathbf{x}^{m-1})_i &= \sum_{i_2,\ldots,i_m} M(\mathcal{P})_{ii_2\cdots i_m} x_{i_2} \cdots x_{i_m} \\ &= M(\mathcal{P})_{ii\cdots i} x_i^{m-1} + \sum_{\substack{j=1 \\ i\neq j}}^{n} M(\mathcal{P})_{ij\cdots j} x_j^{m-1} \\ &= \Big(\prod_{k=1}^{m-1} |a_{ii\cdots i}^{(k)}|\Big) x_i^{m-1} - \sum_{\substack{j=1 \\ i\neq j}}^{n} \Big(\prod_{k=1}^{m-1} |a_{ij\cdots j}^{(k)}|\Big) x_j^{m-1}. \end{aligned} \tag{3.4}$$

Consider the positive vector $\mathbf{x} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$. With this $\mathbf{x}$ and for any i we have

$$(M(\mathcal{P})\mathbf{x}^{m-1})_i = \Big(\prod_{k=1}^{m-1} |a_{ii\cdots i}^{(k)}|\Big) - \sum_{\substack{j=1 \\ i\neq j}}^{n} \Big(\prod_{k=1}^{m-1} |a_{ij\cdots j}^{(k)}|\Big) > 0.$$

The last inequality follows from Proposition 2. This means that $M(\mathcal{P})$ is a strong M-tensor; thus, $\mathcal{P}$ is a strong H-tensor. □

In ([5]) Ding, Qi and Wei proved that a tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is a strong H- tensor if and only if there exists a positive vector $\mathbf{y} = (y_1, \cdots, y_n) \in \mathbb{R}^n$ such that

$$|a_{i\cdots i}| y_i^{m-1} > \sum_{\substack{(i_2,\ldots,i_m), \\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}| y_{i_2} \cdots y_{i_m}, \quad \forall i \in [n]. \tag{3.5}$$

We note that a tensor $\mathcal{A}$ satisfying the inequality (2.2) does not automatically satisfy inequality (3.5). The tensor of Example 3.5 can be used as a counterexample.

The product tensor $\mathcal{P}$ is not symmetric in general. However, we consider the unique symmetric tensor $\mathrm{Sym}(\mathcal{P})$ having the property that $\mathrm{Sym}(\mathcal{P})\mathbf{x}^m = \mathcal{P}\mathbf{x}^m$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proposition 3.11 *The symmetrization $Sym(\mathcal{P})$ is given by*

$$Sym(\mathcal{P})_{i_1 i_2 \cdots i_m} = \begin{cases} \mathcal{P}_{ii\cdots i} & \text{if } \delta_{i_1 i_2 \cdots i_m} = 1, \\ \frac{1}{m}\mathcal{P}_{ij\cdots j} & \text{if } (i_1, \ldots, i_m) = \sigma(i, j, \ldots, j), \\ 0 & \text{otherwise}, \end{cases}$$

where $\sigma(i, j, \ldots, j)$ is a permutation of indices $i, j, \ldots, j$.

Proof The symmetrization of $\mathcal{P}$ does not change the diagonal entries. We also have

$$\mathrm{Sym}(\mathcal{P})_{ij\cdots j} = \mathrm{Sym}(\mathcal{P})_{ji\cdots j} = \cdots = \mathrm{Sym}(\mathcal{P})_{jj\cdots i}.$$

We have m different choices for the location of the index i. Therefore, we must have $\mathrm{Sym}(\mathcal{P})_{ij\cdots j} = \frac{1}{m}\mathcal{P}_{ij\cdots j}$. This way, $\mathcal{P}\mathbf{x}^m$ and $\mathrm{Sym}(\mathcal{P})\mathbf{x}^m$ will be identical for any $\mathbf{x} \in \mathbb{R}^n$ □.

By Theorem 3.10, we know that if $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in \mathrm{C}^{\circ}$, then $\mathcal{P}$ is a strong H-tensor. However, this is not necessarily true if we consider $\mathrm{Sym}(\mathcal{P})$ due to the existence of additional potentially nonzero terms. Nevertheless, we have the following.

Lemma 3.12 *Let $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(m-1)} \in C^{\circ}$. Suppose that for all $i, j \in [n]$,*

$$\prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}| = \prod_{k=1}^{m-1} |a^{(k)}_{ji\cdots i}|.$$

Then, $Sym(\mathcal{P})$ is a strong H-tensor.

Proof Let $\mathcal{M} = M(Sym(\mathcal{P}))$. For positive vector $\mathbf{x} = (1, 1, \ldots, 1)^T \in \mathbb{R}^n$, we have

$$\begin{aligned} (\mathcal{M}\mathbf{x}^{m-1})_i &= \sum_{i_2, \ldots, i_m} M(Sym(\mathcal{P}))_{ii_2\cdots i_m} \\ &= M(Sym(\mathcal{P}))_{ii\cdots i} + \sum_{\substack{i_2, \ldots, i_m \\ \delta_{ii_2\cdots i_m}=0}} M(Sym(\mathcal{P}))_{ii_2\cdots i_m} \\ &= |Sym(\mathcal{P})_{ii\cdots i}| - \sum_{\substack{i_2, \ldots, i_m \\ \delta_{ii_2\cdots i_m}=0}} |Sym(\mathcal{P})_{ii_2\cdots i_m}| \end{aligned}$$

For a fixed i, the potentially nonzero entries of the form $\mathrm{Sym}(\mathcal{P})_{ii_2\cdots i_m}$ are,

1) $Sym(\mathcal{P})_{ii\cdots i} = \mathcal{P}_{ii\cdots i}$,

2) $Sym(\mathcal{P})_{ij\cdots j} = \frac{1}{m}\mathcal{P}_{ij\cdots j}$, for $i \neq j$,

3) $Sym(\mathcal{P})_{iji\cdots i} = \mathrm{Sym}(\mathcal{P})_{iij\cdots i} = \ldots = Sym(\mathcal{P})_{ii\cdots ij} = \frac{1}{m}\mathcal{P}_{jii\cdots i}$, for $j \neq i$ since $Sym(\mathcal{P})_{ii\cdots ij} = Sym(\mathcal{P})_{ji\cdots ii} = \frac{1}{m}\mathcal{P}_{jii\cdots i}$. There are exactly $m-1$ such potentially nonzero entries obtained this way.

Therefore, using $\mathbf{x} = (1, 1, \dots, 1)^T$ we obtain

$$\begin{aligned}(\mathcal{M}\mathbf{x}^{m-1})_i &= |\mathcal{P}_{ii\cdots i}| - \frac{1}{m}\sum_{\substack{j=1\\ i\neq j}}^{n} |\mathcal{P}_{ij\cdots j}| - \frac{m-1}{m}\sum_{\substack{j=1\\ i\neq j}}^{n} |\mathcal{P}_{ji\cdots i}| \\ &= \prod_{k=1}^{m-1} |a^{(k)}_{ii\cdots i}| - \frac{1}{m}\sum_{\substack{j=1\\ i\neq j}}^{n}\prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}| - \frac{m-1}{m}\sum_{\substack{j=1\\ i\neq j}}^{n}\prod_{k=1}^{m-1} |a^{(k)}_{ji\cdots i}| \\ &= \prod_{k=1}^{m-1} |a^{(k)}_{ii\cdots i}| - \sum_{\substack{j=1\\ i\neq j}}^{n}\prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}| > 0\end{aligned}$$

by the assumptions and Proposition 3.3. Therefore, $M(Sym(\mathcal{P}))$ is a strong M-tensor. As a result, $Sym(\mathcal{P})$ is a strong H-tensor. □

Theorem 3.13 *Suppose that m is even and $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(m-1)} \in C^{\circ}$. Suppose further that for all $i, j \in [n]$,*

$$\prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}| = \prod_{k=1}^{m-1} |a^{(k)}_{ji\cdots i}|.$$

Then, $\mathcal{P}$ is a positive definite tensor.

Proof By Lemma 3.12, $Sym(\mathcal{P})$ is a symmetric strong H-tensor with positive diagonal entries. Then, by Theorem 5.36 of [24] it is a positive definite tensor. Therefore, $\mathcal{P}\mathbf{x}^m = Sym(\mathcal{P})\mathbf{x}^m > 0$ for all $\mathbf{x} \in \mathbb{R}^n$ □

It is known that when m is even, a symmetric strong H-tensor with nonnegative diagonal entries is a SOS tensor ([24], Theorem 5.52). We have

Corollary 3.14 *Let m be even. Suppose that $\mathcal{A}^{(1)}, \dots, \mathcal{A}^{(m-1)} \in C^{\circ}$, and for all $i, j \in [n]$,*

$$\prod_{k=1}^{m-1} |a^{(k)}_{ij\cdots j}| = \prod_{k=1}^{m-1} |a^{(k)}_{ji\cdots i}|.$$

Then, $Sym(\mathcal{P})$ is a SOS tensor.

Proof With the conditions of the corollary, $Sym(\mathcal{P})$ becomes a symmetric strong H-tensor with nonnegative diagonal entries by Theorem 3.13. Then, the result follows. □

Suppose that $\mathcal{A}^{(1)} = \cdots = \mathcal{A}^{(m-1)} = \mathcal{A}$. We set

$$\mathcal{P}_{\mathcal{A}} = \text{ Prod } (\mathcal{A}, \dots, \mathcal{A}, \mathcal{I}).$$

It is easy to see that $(\mathcal{P}_{\mathcal{A}})_{ii\cdots i} = a^{m-1}_{ii\cdots i}$, $(\mathcal{P}_{\mathcal{A}})_{ij\cdots j} = a^{m-1}_{ij\cdots j}$ for $i \neq j$, and all other entries are 0.

Corollary 3.15 *Suppose that m is even and $\mathcal{A} \in C^{\circ}$ is a Z-tensor. Then, $\mathcal{P}_{\mathcal{A}}$ is a strong M-tensor.*

Proof Since $\mathcal{A}$ is a Z-tensor with positive diagonal entries and $m-1$ is odd, $\mathcal{P}_{\mathcal{A}}$ is also a Z-tensor. Then, $\mathcal{P}_{\mathcal{A}} = M(\mathcal{P}_{\mathcal{A}})$. The result follows from Theorem 3.10. □

When we set all $\mathcal{A}^{(k)}$ equal, the condition

$$\prod_{k=1}^{m-1} |a_{ij\cdots j}^{(k)}| = \prod_{k=1}^{m-1} |a_{ji\cdots i}^{(k)}|.$$

becomes the equality of $|a_{ij\cdots j}|$ and $|a_{ji\cdots i}|$ for all $i,j \in [n]$. Therefore, we have

Lemma 3.16 *Let m be even. Assume that $\mathcal{A} \in C^{\circ}$, and $|a_{ij\cdots j}| = |a_{ji\cdots i}|$ for all $i,j \in [n]$. Then, $Sym(\mathcal{P}_{\mathcal{A}})$ is a SOS tensor.*

Proof With the requirements of the lemma, $Sym(\mathcal{P}_{\mathcal{A}})$ becomes a symmetric strong H-tensor. Since m is even, and diagonal entries of $Sym(\mathcal{P}_{\mathcal{A}})$ are nonnegative, the corollary follows from ([24], Theorem 5.52). □

Based on these results, when m is even, and $|a_{ij\cdots j}| = |a_{ji\cdots i}|$ for all $i,j \in [n]$ we define a map

$$\pi : \mathrm{C}^{\circ} \longrightarrow \mathrm{SOS}_{m,n}$$

by $\pi(\mathcal{A}) = Sym(\mathcal{P}_{\mathcal{A}})$. Here $\mathrm{SOS}_{m,n}$ is the convex cone of SOS tensors ([9]). This gives a nontrivial and algorithmic way of producing a SOS (and positive definite) tensor from a given $\mathcal{A}$ satisfying some mild conditions.

Suppose that m is even and $\mathcal{A}$ is a Z-tensor. In this case, $Sym(\mathcal{P}_{\mathcal{A}})$ is also a Z-tensor whose entries are given by

$$Sym(\mathcal{P}_{\mathcal{A}})_{i_1 i_2 \cdots i_m} = \begin{cases} a_{ii\cdots i}^{m-1} & \text{if } \delta_{i_1 i_2 \cdots i_m} = 1, \\ \frac{1}{m} a_{ij\cdots j}^{m-1} & \text{if } (i_1, \ldots, i_m) = \sigma(i,j,\ldots,j) \\ 0 & \text{otherwise.} \end{cases}$$

where $\sigma(i,j,\ldots,j)$ is a permutation of indices $i,j,\ldots,j$.

Observe that $M(\mathrm{Sym}(\mathcal{P}_{\mathcal{A}})) = Sym(\mathcal{P}_{\mathcal{A}})$, since all off-diagonal entries of $\mathcal{A}$ are nonpositive. We have

Corollary 3.17 *Let m be even and n be any positive integer. Consider the polynomial*

$$f(x_1,\ldots,x_n) = \sum_{i=1}^{n} \big(a_i^{m-1} x_i^m + \sum_{\substack{j=1 \\ i \neq j}}^{n} a_{ij}^{m-1} x_i x_j^{m-1}\big),$$

where for all i, $a_i > 0$, and for all $j \neq i$, $a_{ij} \leq 0$, $a_i^{m-1} > \sum_{j\neq i} |a_{ij}|^{m-1}$, and $a_{ij} = a_{ji}$. Then, $f(x_1,\ldots,x_n)$ is a SOS polynomial.

Proof We construct a Z-tensor $\mathcal{A}$ as $a_{ii\cdots i} = a_i$, $a_{ij\cdots j} = a_{ij}$, and $a_{i_1 i_2 \cdots i_m} = 0$ otherwise. Conditions of the corollary imply that $\mathcal{A} \in \mathrm{C}^{\circ}$. Then, we obtain $f(x_1,\ldots,x_n) = Sym(\mathcal{P}_{\mathcal{A}})\mathbf{x}^m = \mathcal{P}_{\mathcal{A}}\mathbf{x}^m$ with $\mathbf{x} = (x_1,\ldots,x_n)^T$. The result follows from Lemma 3.16. □

Now, we use $\mathcal{P}_{\mathcal{A}}$ to determine certain spectral properties of some Z-tensors. First, we need the following lemma.

Lemma 3.18 *Let* $\mathcal{A} = (a_{i_1\cdots i_m}) \in T_{m,n}$. *Suppose that for any* $i, j \in [n]$ *the entries* $a_{ij\cdots j} \neq 0$. *Then,* $\mathcal{A}$ *is weakly irreducible.*

Proof We compute the relevant matrix given in Definition 2.1. For any i and $j \neq i$,

$$m_{ij} = \sum_{j\in\{i_2,\ldots i_m\}} |a_{ii_2\cdots i_m}| = |a_{ij\cdots j}| + \sum |a_{ii_2\cdots i_m}| \neq 0.$$

That is, off diagonal entries are all nonzero. Therefore, the matrix is irreducible and the tensor is weakly irreducible as required. □

Corollary 3.19 *Let* $\mathcal{A}$ *be a nonnegative tensor such that for any* $i, j \in [n]$ *the entries* $a_{ij\cdots j} \neq 0$. *Then,* $\mathcal{A}$ *has a positive eigenvector associated to* $\rho(\mathcal{A})$. *Moreover,* $a_{ii\cdots i} \leq \rho(\mathcal{A})$ *for all* $i \in [n]$.

Proof The existence of positive eigenvalue is due to weakly irreducibility of $\mathcal{A}$ as established in Lemma 3.18. Let $\mathbf{x}$ be the positive eigenvector corresponding to $\rho(\mathcal{A})$. For any $i \in [n]$,

$$(\mathcal{A}\mathbf{x}^{m-1})_i = a_{ii\cdots i}x_i^{m-1} + \sum_{\substack{i_2,\ldots,i_m \\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} = \rho(\mathcal{A})x_i^{m-1},$$

then,

$$(a_{ii\cdots i} - \rho(\mathcal{A}))x_i^{m-1} + \sum_{\substack{i_2,\ldots,i_m \\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} = 0.$$

Since $x_i > 0$, we must have $a_{ii\cdots i} - \rho(\mathcal{A}) \leq 0$ as required. □

Theorem 3.20 *Let* m *be even and* $\mathcal{A} \in C^\circ$ *be a Z-tensor. Suppose that for any* $i, j \in [n]$ *the entries* $a_{ij\cdots j} \neq 0$. *Then, the Hadamard product* $\mathcal{A}^{\circ(m-1)} = \underbrace{\mathcal{A} \circ \cdots \circ \mathcal{A}}_{(m-1)-copies}$ *has a H-eigenvalue* λ *with a positive eigenvector* $\mathbf{x}$. *Moreover, if* $\lambda > 0$, *then* $\mathcal{A}^{\circ(m-1)}$ *becomes a strong M-tensor.*

Proof Since $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is a Z-tensor and m is even, $\mathcal{A}^{\circ(m-1)}$ is also a Z-tensor. $\mathcal{A} \in \mathrm{C}^\circ$ implies that $\mathcal{P}_\mathcal{A}$ is a strong M-tensor by Corollary 3.15. Therefore, we write

$$\mathcal{P}_\mathcal{A} = s\mathcal{I} - \mathcal{B}, \tag{3.6}$$

where $\mathcal{B}$ is a nonnegative tensor and $s > \rho(\mathcal{B})$. Note that for $i \neq j$, $b_{ij\cdots j} = -a_{ij\cdots j}^{m-1}$. We define another nonnegative tensor $\tilde{\mathcal{A}}$ by

$$(\tilde{\mathcal{A}})_{i_1i_2\cdots i_m} = \begin{cases} |a_{i_1i_2\cdots i_m}|^{m-1} = -a_{i_1i_2\cdots i_m}^{m-1}, & \text{if } \delta_{i_2\cdots i_m} = 0 \\ 0 & \text{otherwise.} \end{cases}$$

$\tilde{\mathcal{A}}$ has zero diagonal entries and we can write

$$\mathcal{A}^{\circ(m-1)} = \mathcal{P}_\mathcal{A} - \tilde{\mathcal{A}}. \tag{3.7}$$

Combining equations (3.6) and (3.7) we obtain

$$\mathcal{A}^{\circ(m-1)} = s\mathcal{I} - (\mathcal{B} + \tilde{\mathcal{A}}), \tag{3.8}$$

where $\mathcal{B} + \tilde{\mathcal{A}}$ is a nonnegative tensor. Examining closely we see that for any $i \in [n]$ and $j \neq i$,

$$(\mathcal{B} + \tilde{\mathcal{A}})_{ij\cdots j} = b_{ij\cdots j} + (\tilde{\mathcal{A}})_{ij\cdots j} = -a_{ij\cdots j}^{m-1} \neq 0.$$

Lemma 3.18 and Corollary 3.19 imply that $(\mathcal{B} + \tilde{\mathcal{A}})$ is a weakly irreducible nonnegative tensor; hence, it has a positive eigenvector $\mathbf{x}$ corresponding to the positive H-eigenvalue $\rho(\mathcal{B} + \tilde{\mathcal{A}})$. Then, for any $i \in [n]$ we have

$$(\mathcal{A}^{\circ(m-1)}\mathbf{x}^{m-1})_i = (s\mathcal{I}\mathbf{x}^{m-1})_i - ((\mathcal{B} + \tilde{\mathcal{A}})\mathbf{x}^{m-1})_i = (s - \rho(\mathcal{B} + \tilde{\mathcal{A}}))x_i^{m-1}.$$

Therefore, $\mathbf{x}$ is a positive eigenvector for the tensor $\mathcal{A}^{\circ(m-1)}$ corresponding to H-eigenvalue $\lambda = s - \rho(\mathcal{B} + \tilde{\mathcal{A}})$. Note that if $\lambda > 0$, then $s > \rho(\mathcal{B} + \tilde{\mathcal{A}})$. Together with equation (3.8) this implies $\mathcal{A}^{\circ(m-1)}$ is a strong M-tensor. □

We can relax the condition of being in C° as follows.

Proposition 3.21 *Let m be even and $\mathcal{A}$ be a Z-tensor with identical nonnegative diagonal entries. Suppose also that for any $i, j \in [n]$, $a_{ij\cdots j} \neq 0$. Then, the Hadamard product $\mathcal{A}^{\circ(m-1)}$ has a H-eigenvalue λ with a positive eigenvector $\mathbf{x}$. Moreover, if $\lambda > 0$, then $\mathcal{A}^{\circ(m-1)}$ becomes a strong M-tensor.*

Proof The statement follows immediately if $\mathcal{A} \in \mathrm{C}^\circ$. Assume that $\mathcal{A} \notin \mathrm{C}^\circ$ and let $a_{ii\cdots i} = a > 0$ for all $i \in [n]$. Define

$$\alpha = \max_{i \in [n]} \Big\{ \Big(\sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij\cdots j}|^{m-1} \Big)^{1/(m-1)} \Big\} > 0,$$

and a tensor $\mathcal{A}_\alpha = \mathcal{A} + \alpha\mathcal{I}$. Since $\mathcal{A}$ is a Z-tensor, so is $\mathcal{A}_\alpha$. Moreover, for any $i \in [n]$

$$\begin{aligned} (\mathcal{A}_\alpha)_{ii\cdots i}^{m-1} &= (\mathcal{A} + \alpha\mathcal{I})_{ii\cdots i}^{m-1} = (a_{ii\cdots i} + \alpha)^{m-1} \\ &> a_{ii\cdots i}^{m-1} + \alpha^{m-1} \\ &> \alpha^{m-1} \\ &> \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij\cdots j}|^{m-1}, \end{aligned}$$

since $a_{ii\cdots i}$ and α are positive. We obtain

$$(\mathcal{A}_\alpha)_{ii\cdots i}^{m-1} > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij\cdots j}|^{m-1} = \sum_{\substack{j=1 \\ i \neq j}}^{n} |(\mathcal{A}_\alpha)_{ij\cdots j}|^{m-1}.$$

for any $i \in [n]$. Therefore, $\mathcal{A}_\alpha \in \mathrm{C}^\circ$, and by the proof of Theorem 3.20 we write

$$\mathcal{A}_\alpha^{\circ(m-1)} = s_\alpha\mathcal{I} - (\mathcal{B}_\alpha + \tilde{\mathcal{A}}_\alpha), \tag{3.9}$$

where $\mathcal{B}_\alpha$ and $\tilde{\mathcal{A}}_\alpha$ are nonnegative tensors such that $s_\alpha > \rho(\mathcal{B}_\alpha)$. Note also that $(\mathcal{A}_\alpha^{\circ(m-1)})_{i_1 i_2 \cdots i_m} = (\mathcal{A}^{\circ(m-1)})_{i_1 i_2 \cdots i_m}$ for $\delta_{i_1 i_2 \cdots i_m} = 0$, and for any $i \in [n]$

$$(\mathcal{A}_\alpha^{\circ(m-1)})_{ii\cdots i} = (a_{ii\cdots i} + \alpha)^{m-1} = (a+\alpha)^{m-1} = (\mathcal{A}^{\circ(m-1)})_{ii\cdots i} + \gamma(\alpha),$$

where $\gamma(\alpha)$ is a positive polynomial in α of degree $m-1$. Therefore, we have

$$\mathcal{A}_\alpha^{\circ(m-1)} = \mathcal{A}^{\circ(m-1)} + \gamma(\alpha)\mathcal{I}. \tag{3.10}$$

Furthermore, for any $i \neq j$, $(\mathcal{A}_\alpha)_{ij\cdots j} = a_{ij\cdots j} \neq 0$. Since $\mathcal{A}_\alpha \in \mathrm{C}^\circ$ and it satisfies the conditions of Theorem 3.20, $\mathcal{A}_\alpha^{\circ(m-1)}$ has a H-eigenvalue $\lambda_\alpha = s_\alpha - \rho(\mathcal{B}_\alpha + \tilde{\mathcal{A}}_\alpha)$ with a positive eigenvector $\mathbf{y}$. This gives,

$$\begin{aligned}
(\mathcal{A}_\alpha^{\circ(m-1)})\mathbf{y}^{m-1} &= \lambda_\alpha \mathbf{y}^{[m-1]} \\
(\mathcal{A}^{\circ(m-1)} + \gamma(\alpha)\mathcal{I})\mathbf{y}^{m-1} &= \lambda_\alpha \mathbf{y}^{[m-1]} \\
\mathcal{A}^{\circ(m-1)}\mathbf{y}^{m-1} &= (\lambda_\alpha - \gamma(\alpha))\mathbf{y}^{[m-1]}.
\end{aligned}$$

Therefore, $\lambda = \lambda_\alpha - \gamma(\alpha)$ is a H-eigenvalue of $\mathcal{A}^{\circ(m-1)}$ with the positive eigenvector $\mathbf{y}$. Note also that combining equations (3.9) and (3.10), we can write

$$\mathcal{A}^{\circ(m-1)} = (s_\alpha - \gamma(\alpha))\mathcal{I} - (\mathcal{B}_\alpha + \tilde{\mathcal{A}}_\alpha).$$

We know that $\mathcal{B}_\alpha + \tilde{\mathcal{A}}_\alpha$ is a nonnegative tensor. If $\lambda = \lambda_\alpha - \gamma(\alpha) > 0$, we have $s_\alpha - \rho(\mathcal{B}_\alpha + \tilde{\mathcal{A}}_\alpha) - \gamma(\alpha) > 0$. This implies that $s_\alpha - \gamma(\alpha) > \rho(\mathcal{B}_\alpha + \tilde{\mathcal{A}}_\alpha)$. Therefore, $\mathcal{A}^{\circ(m-1)}$ becomes a strong M-tensor as required. □

For any tensor $\mathcal{A}$, it is relatively easy to work with the tensor $\mathcal{P}_\mathcal{A}$, since it has only n^2 entries which may be nonzero. This gives the tensor a computational edge when it comes to compute the eigenvalues. Moreover, if $\mathcal{A} \in \mathrm{C}^\circ$, $\mathcal{P}_\mathcal{A}$ becomes a strong H-tensor. The proof of the Theorem 3.20 presents some ideas about constructing non trivial strong M-tensors when m is even. More explicitly, one can choose n^2 entries $a_{ii\cdots i}$, $a_{ij\cdots j}$ in a way that they satisfy the requirements of the Theorem 3.20. Then, construct $\tilde{\mathcal{A}}$ such that the relevant H-eigenvalue of the Hadamard product becomes positive. Thus, obtain a strong M-tensor. We also mention that there are several efficient algoritheorems to determine the largest eigenvalues of nonnegative tensors ([16, 20, 24]).

Corollary 3.22 *Let m be even and $\mathcal{A} \in C^\circ$ be a symmetric Z-tensor satisfying requirements of Theorem 3.20. Assume further that the H-eigenvalue of $\mathcal{A}^{\circ(m-1)}$ associated to the positive eigenvector is also positive. Then, $\mathcal{A}^{\circ(m-1)}$ is a SOS tensor.*

Proof In this case, Theorem 3.20 implies that $\mathcal{A}^{\circ(m-1)}$ is a strong M-tensor. Any symmetric strong M-tensor (for even m) is a SOS tensor ([4, 10]). □

Corollary 3.23 *Let m be even and $\mathcal{B}$ be any Z-tensor with identical positive diagonal entries. Then, $\mathcal{B}$ can be written as the difference of a (not necessarily diagonal) strong M-tensor and a nonnegative tensor.*

Proof For any Z-tensor $\mathcal{A}$, we can construct $\mathcal{A}_\alpha \in \mathrm{C}^\circ$ as in the proof of Theorem 3.20. Using (3.7) with $\mathcal{A}_\alpha$, and using (3.10) we obtain

$$\begin{aligned} \mathcal{A}_\alpha^{\circ(m-1)} &= \mathcal{P}_{\mathcal{A}_\alpha} - \tilde{\mathcal{A}}_\alpha \\ \mathcal{A}^{\circ(m-1)} + \gamma(\alpha)\mathcal{I} &= \mathcal{P}_{\mathcal{A}_\alpha} - \tilde{\mathcal{A}}_\alpha \\ \mathcal{A}^{\circ(m-1)} &= \mathcal{P}_{\mathcal{A}_\alpha} - (\gamma(\alpha)\mathcal{I} + \tilde{\mathcal{A}}_\alpha), \end{aligned}$$

where $\mathcal{P}_{\mathcal{A}_\alpha}$ is a strong M-tensor and $\gamma(\alpha)\mathcal{I} + \tilde{\mathcal{A}}_\alpha$ is a nonnegative tensor. Observe that for even m, the map $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = x^{m-1}$ is a bijection. Therefore, for a given Z-tensor $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ we obtain a unique tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ by setting $a_{i_1 i_2 \cdots i_m} = (b_{i_1 i_2 \cdots i_m})^{1/(m-1)}$. Then, clearly $\mathcal{A}$ is a Z-tensor and $\mathcal{B} = \mathcal{A}^{\circ(m-1)}$. By the first part of this proof, the corollary follows. □

4. The Case of odd m

If m is odd, then certain structures of the tensor $\mathcal{A}$ may be lost when we obtain $\mathcal{P}_{\mathcal{A}}$. For instance, since $m-1$ is even, $\mathcal{P}_{\mathcal{A}}$ will not be a Z-tensor even if $\mathcal{A}$ is a Z-tensor. Moreover, concept of positive definiteness and dependently, SOS tensor is not defined for odd m. Nevertheless, we can still obtain the following properties.

Let $\mathcal{A}$ be any tensor such that $\mathcal{A}^{\circ(m-1)}$ is diagonally dominated. Note that $\mathcal{A}^{\circ(m-1)}$ is a nonnegative tensor. Let $\mathcal{P}_{\mathcal{A}}$ be given as before. Clearly, it is also a nonnegative tensor. For any $\mathcal{A} \in T_{m,n}$, let $\Delta_{min}(\mathcal{A})$ and $\Delta_{max}(\mathcal{A})$ denote the minimum and maximum diagonal entries of $\mathcal{A}$ respectively. We also define the ith row sum of $\mathcal{A}$ as

$$r_i(\mathcal{A}) = \sum_{i_2,\ldots,i_m} a_{ii_2\cdots i_m}.$$

It is shown that ([23]) for a nonnegative tensor $\mathcal{A}$,

$$\min_{i\in[n]} r_i(\mathcal{A}) \leq \rho(\mathcal{A}) \leq \max_{i\in[n]} r_i(\mathcal{A}). \tag{4.1}$$

Proposition 4.1 *Let m be odd and $\mathcal{A} \in T_{m,n}$ such that $\mathcal{A}^{\circ(m-1)}$ is diagonally dominated. Then,*

$$\Delta_{min}(\mathcal{P}_{\mathcal{A}}) \leq \rho(\mathcal{P}_{\mathcal{A}}) \leq \rho(\mathcal{A}^{\circ(m-1)}) \leq 2\Delta_{max}(\mathcal{P}_{\mathcal{A}}).$$

Proof Since m is odd, both $\mathcal{P}_{\mathcal{A}}$ and $\mathcal{A}^{\circ(m-1)}$ are nonnegative. For any $i \in [n]$, $r_i(\mathcal{P}_{\mathcal{A}}) \geq \Delta_{min}(\mathcal{P}_{\mathcal{A}})$. By (4.1), $\rho(\mathcal{P}_{\mathcal{A}}) \geq \Delta_{min}(\mathcal{P}_{\mathcal{A}})$.

For any index i, the ith row sum of $\mathcal{A}^{\circ(m-1)}$ is

$$\begin{aligned} r_i(\mathcal{A}^{\circ(m-1)}) &= \sum_{i_2,\cdots,i_m} a_{ii_2\cdots i_m}^{m-1} = a_{ii\cdots i}^{m-1} + \sum_{\substack{i_2,\cdots,i_m \\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m}^{m-1} \\ &\leq 2a_{ii\cdots i}^{m-1} \leq 2\Delta_{max}(\mathcal{P}_{\mathcal{A}}) \end{aligned}$$

Therefore,

$$\max_{i\in[n]} r_i((\mathcal{A}^{\circ(m-1)})) \leq 2\Delta_{max}(\mathcal{P}_{\mathcal{A}}).$$

Again by (4.1), $\rho(\mathcal{A}^{\circ(m-1)}) \leq 2\Delta_{max}(\mathcal{P}_{\mathcal{A}})$. Note also that $\mathcal{P}_{\mathcal{A}} \leq \mathcal{A}^{\circ(m-1)}$. Then, by ([24, 27]) we obtain $\rho(\mathcal{P}_{\mathcal{A}}) \leq \rho(\mathcal{A}^{\circ(m-1)})$. □

The requirement that $\mathcal{A}^{\circ(m-1)}$ is diagonally dominated does not imply diagonal dominance of $\mathcal{A}$. For example, If we let $a_{111} = a_{222} = 1$, $a_{122} = a_{211} = 2/3$, and $a_{121} = a_{221} = -2/3$, we obtain a tensor $\mathcal{A} \in$ C$^{\circ}$, $\mathcal{A} \circ \mathcal{A}$ is diagonally dominated, but $\mathcal{A}$ is not diagonally dominated. However, when $\mathcal{A}^{\circ(m-1)}$ is diagonally dominated, then $\mathcal{A}$ is essentially in C as long as it has nonnegative diagonal entries.

Given a tensor $\mathcal{A} \in T_{m,n}$ for any m and n, the determinant of $\mathcal{A}$, denoted by det $\mathcal{A}$, can be defined as the resultant of the homogeneous polynomial system $\mathcal{A}\mathbf{x}^{m-1} = \mathbf{0}$. We also have

$$\det \mathcal{A} = \prod \lambda_{\mathcal{A}},$$

where $\lambda_{\mathcal{A}}$ is an eigenvalue of $\mathcal{A}$. It is known that $\mathcal{A} \in T_{m,n}$ has $n(m-1)^{n-1}$ eigenvalues ([11, 22, 24]).

Corollary 4.2 *Let m be odd and $\mathcal{A} \in T_{m,n}$ such that $\mathcal{A}^{\circ(m-1)}$ is diagonally dominated. Then,*

$$|det\ \mathcal{A}^{\circ(m-1)}| \leq 2^d(\Delta_{max}(\mathcal{P}_{\mathcal{A}}))^d, \tag{4.2}$$

where $d = n(m-1)^{n-1}$.

Proof Corollary follows directly from Proposition 4.1, and the modulus inequality $|\lambda_{\mathcal{A}^{\circ(m-1)}}| \leq \rho(\mathcal{A}^{\circ(m-1)})$ for the eigenvalues of $\mathcal{A}^{\circ(m-1)}$. □

Let $f(x)$ be a real valued function, and $\mathcal{A} \in T_{m,n}$. Define a tensor $f(\mathcal{A}) \in T_{m,n}$ by $f(\mathcal{A})_{i_1 i_2 \cdots i_m} = f(a_{i_1 i_2 \cdots i_m})$. Suppose that m is odd, and $f(x) = x^{1/(m-1)}$. $f(x)$ is injective on nonnegative real numbers. Assume further that $\mathcal{A} \in T_{m,n}$ is a nonnegative, diagonally dominated tensor. Let $\mathcal{B} = f(\mathcal{A})$. Then, $\mathcal{A} = \mathcal{B}^{\circ(m-1)}$ is a diagonally dominated tensor. By Proposition 4.1, we have

$$\Delta_{min}(\mathcal{P}_{\mathcal{B}}) \leq \rho(\mathcal{P}_{\mathcal{B}}) \leq \rho(\mathcal{B}^{\circ(m-1)}) \leq 2\Delta_{max}(\mathcal{P}_{\mathcal{B}}).$$

In other words, we have

$$\Delta_{min}(\mathcal{P}_{f(\mathcal{A})}) \leq \rho(\mathcal{P}_{f(\mathcal{A})}) \leq \rho(\mathcal{A}) \leq 2\Delta_{max}(\mathcal{P}_{f(\mathcal{A})}).$$

Therefore, we obtain

Corollary 4.3 *Let m be odd, and $\mathcal{A}$ be a diagonally dominated, nonnegative tensor. Then,*

$$\Delta_{min}(\mathcal{P}_{f(\mathcal{A})}) \leq \rho(\mathcal{P}_{f(\mathcal{A})}) \leq \rho(\mathcal{A}) \leq 2\Delta_{max}(\mathcal{P}_{f(\mathcal{A})}).$$

Moreover,

$$|det\ \mathcal{A}| \leq 2^d(\Delta_{max}(\mathcal{P}_{f(\mathcal{A})}))^d,$$

where $d = n(m-1)^{n-1}$.

Our final goal is to relax the diagonal dominance of $\mathcal{A}^{\circ(m-1)}$. We define

Definition 4.4 *A tensor $\mathcal{A} \in T_{m,n}$ is called quasidiagonally dominated if for each $i \in [n]$*

$$|a_{ii\cdots i}| \geq \sum_{\substack{i_2,\ldots,i_m \\ \delta_{i_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|$$

Example 4.5 *Let $\mathcal{A}$ be a tensor with $a_{111} = 1/2$, $a_{222} = a_{333} = 1$, $a_{122} = 2/5$, $a_{211} = a_{311} = 2/3$, $a_{133} = a_{233} = a_{322} = 1/4$, $a_{123} = a_{213} = 1/5$, and $a_{ijk} = 0$ otherwise. Then, $\mathcal{A}$ is quasi-diagonally dominated. Neither $\mathcal{A}$, nor $\mathcal{A}^{\circ 2}$ is diagonally dominated. However, $\mathcal{A} \in C^{\circ}$ and $\mathcal{A}^{\circ 2}$ is quasidiagonally dominated.*

Proposition 4.6 *Let m be odd. Suppose that $\mathcal{A} \in C^{\circ}$ is a tensor such that $\mathcal{A}^{\circ(m-1)}$ is quasidiagonally dominated. Then,*

$$\rho(\mathcal{A}^{\circ(m-1)}) \leq 3\Delta_{max}(\mathcal{P}_{\mathcal{A}}).$$

Proof Since $\mathcal{A} \in \mathrm{C}^{\circ}$, for any $i \in [n]$ we have

$$a_{ii\cdots i}^{m-1} > \sum_{\substack{j=1 \\ i\neq j}}^{n} a_{ij\cdots j}^{m-1}.$$

We compute the row sum for any $i \in [n]$ as

$$\begin{aligned} r_i(\mathcal{A}^{\circ(m-1)}) &= \sum_{i_2,\cdots,i_m} a_{ii_2\cdots i_m}^{m-1} = a_{ii\cdots i}^{m-1} + \sum_{\substack{j=1 \\ i\neq j}}^{n} a_{ij\cdots j}^{m-1} + \sum_{\substack{i_2,\ldots,i_m \\ \delta_{i_2\cdots i_m}=0}} a_{ii_2\cdots i_m}^{m-1} \\ &\leq 3a_{ii\cdots i}^{m-1} \leq 3\Delta_{max}(\mathcal{P}_{\mathcal{A}}). \end{aligned}$$

Then, $\rho(\mathcal{A}^{\circ(m-1)}) \leq 3\Delta_{max}(\mathcal{P}_{\mathcal{A}})$ as claimed. □

Acknowledgments

This paper is written during the sabbatical leave of the author. The author would like to thank Department of Mathematics of Shanghai University, in particular, Dr. Qingwen Wang, for their hospitality and support.

References

[1] Chang H, Paksoy VE, Zhang F. Polytopes of stochastic tensors. Annals of Functional Analysis 2016; 7: 386-393.

[2] Che M, Bu C, Qi L, Wei Y. Nonnegative tensors revisited: plane stochastic tensors. Linear and Multilinear Algebra 2018; 67: 1364-1391.

[3] Che M, Wei Y. Theory and Computation of Complex Tensors and its Applications, Springer, 2020

[4] Chen H, Li G, Qi L. SOS tensor decomposition: Theory and applications. Communications in Mathematical Sciences 2016; 14: 2073- 2100.

[5] Ding W, Qi L, Wei Y. M- tensors and nonsingular M-tensors. Linear Algebra and Applications 2013; 439: 3264-3278.

[6] Dow SJ, Gibson PM. Permanents of d-dimensional matrices. Linear Algebra and Applications 1987; 90: 133-145.

[7] Friedlands S, Gaubert S, Han L. Perron Frobenius theorem for nonnegative multilinear forms and extensions. Linear Algebra and Applications 013; 438: 738-749.

[8] Hu S, Qi L, Xie J. The Largest Laplacian and signless H-eigenvalues of a uniform hypergraph. Linear Algebra and Applications 2015; 469: 1-27.

[9] Hu S, Li G, Qi L. A tensor analogy of Yuan's theorem of the alternative and polynomial optimization with sign structure. Journal of Optimization Theory and Applications 2016; 168: 446-474.

[10] Hu S, Li G, Qi L, Song Y. Finding the maximum eigenvalue of essentially nonnegative symmetric tensors via sum of squares programming. Journal of Optimization Theory and Applications 2013; 158: 17-738.

[11] Hu S, Huang Z, Ling C, Qi L. On determinants and eigenvalue theory of tensors. Journal of Symbolic Computation 2013; 50: 508-531.

[12] Kannike K. Vacuum stability of a general scalar potential of a few fields. European Physical Journal C 2016; 76: 324. doi: 10.1140/epjc/s10052-016-4160-3

[13] Kwon EG, Bae JE. On a generalized Holder inequality. Journal of Inequalities and Applications 2015;88. doi: 10.1186/s13660-015-0612-9

[14] Lim LH. Singular values and eigenvalues of tensors: A variational approach. Computational advances in multi-sensor adaptive processing, 1st IEEE International workshop, IEEE, Piscataway, NJ, 2005; 129-132.

[15] Lim LH, Ng MK, Qi L. The spectral theory of tensors and its applications. Numerical Linear Algebra and Applications 2013; 20: 889-890.

[16] Liu Y, Zhou G, Ibrahim NF. An always convergent algorithm for the largest eigenvalue of an irreducible nonnegative tensor. Journal of Computational and Applied Mathematics 2010; 235: 286-292.

[17] Luo Z, Qi L, Ye Y. Linear operators and positive semidefiniteness of symmetric tensor spaces. Science China Mathematics 2015; 58: 197-212.

[18] Mesner DM, Bhattacharya P. Association schemes on triples and a ternary algebra. Journal of Combinatorial Theory Series A 1990; 55: 204-234.

[19] Mesner DM, Bhattacharya P. A ternary algebra arising from associative schemes on triples. Journal of Algebra 1994; 264: 595-613.

[20] Ng M, Qi L, Zhou G. Finding the largest eigenvalue of a nonnegative tensor. SIAM Journal of Matrix Analysis and Applications 2009; 31: 1090-1099.

[21] Qi L. H^+-eigenvalues of Laplacian tensor and signless Laplacians. Communications in Mathematical Sciences 2014; 12: 228-238.

[22] Qi L. Eigenvalues of a real supersymmetric tensor. Journal of Symbolic Computation 2005; 40: 1302-1324.

[23] Qi L. Symmetric nonnegative tensors and copositive tensors. Linear Algebra and Applications 2013; 439: 228-238.

[24] Qi L, Luo Z. Tensor analysis: Spectral theory and special tensors. Philadelphia USA: SIAM 2017

[25] Qi L, Chen H, Chen Y. Tensor Eigenvalues and Their Applications. Springer, 2018.

[26] Wei Y, Ding W. Theory and Computation of Tensors:Multi-Dimensional Arrays. New York, NY, USA: Academic Press 2016.

[27] Yang Y, Yang Q. Further results for Perron-Frobenius theorem for nonnegative tensors. SIAM Journal of Matrix Analysis and Applications 2010; 31: 2517-2530.

[28] Zhang L, Qi L, and Zhou G. M-tensors and some applications. SIAM Journal of Matrix Analysis and Applications 2014; 35: 437-452.