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Abstract Korovkin-type theorems in the filter setting with respect to relative uniform convergence

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Abstract: We prove a Korovkin-type approximation theorem using abstract relative uniform filter convergence of a net of functions with respect to another fixed filter, a particular case of which is that of all neighborhoods of a point, belonging to the domain of the involved functions. We give some examples, in which we show that our results are strict generalizations of the classical ones.

Key words: Korovkin theorem, relative uniform convergence, bivariate Kantorovich operator

1. Introduction

One of the most commonly used theorems in approximation theory, thanks to its simplicity and power, is the Korovkin theorem (see [39]), which deals with uniform approximation of continuous functions defined on a compact space, using sequences or nets of positive linear operators defined on the space of continuous functions.

The classical Bohman-Korovkin theorem yields uniform convergence in the space $C([a,b])$ of all continuous real-valued functions defined on the compact subinterval $[a,b]$ of the real line, with the only hypothesis of convergence on the test functions 1, x, x^2 (see for instance [18, 38, 39]). Since then, the Korovkin theorem has been extended to abstract functional spaces, such as L^p spaces (see e.g., [31, 36, 46]), Orlicz spaces (see e.g., [41, 47]), and general modular spaces (see e.g., [10, 12]).

Korovkin-type theorems hold even when one considers some other types of test functions, for example trigonometric functions. An abstract presentation of test functions is given by the so-called *Korovkin sets* (see e.g., [2, 3]).

The Korovkin theorems are related to several tools, useful to get different applications to various branches of Mathematics. Among them, we quote sampling-type operators, density approximation theorems, Radon measures, Banach lattices, Banach algebras, C^*-algebras, harmonic analysis, (locally) compact topological groups, differential equations, Markov processes, stochastic processes, fuzzy numbers (see e.g., [1–3, 5, 43, 44] and the references therein).

There have been also several studies on Korovkin-type theorems related to convergence associated with summability methods, statistical and filter convergence (see e.g., [5, 8, 9, 29, 32–34, 49]).

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In [17] it is dealt with Korovkin-type theorems for operators acting on modular function spaces, with respect to abstract convergences satisfying suitable axioms (see e.g., [10]).

In [28] some Korovkin-type theorems are proved in the setting of relative uniform convergence with respect to scale functions, investigated in [20–22, 37], and different examples and applications are given.

In this paper we deal with Korovkin theorems in the setting of an abstract convergence, which extends the one investigated in [28], where the set of the neighborhoods of the point at which relative uniform convergence is considered, is replaced with a general filter of the domain of the involved functions, and the "classical" (uniform) convergence of the functions is replaced by the (uniform) filter/ideal convergence. We also suppose that our functions are defined on an abstract topological space, endowed with a uniform structure (see also [10]). We give some examples, showing that our results are strict generalizations of the corresponding classical ones. Filter/ideal convergence, whose statistical convergence is a particular case, is related, in our setting, to convergences of sequences or nets, which are not necessarily topological structures (see also [35, 40]). Observe that it is possible to give an axiomatic definition of filter convergence even without nets (see also [30, 35, 40] and the references therein).

Furthermore, there are recent applications of these subjects to emergent and concrete issues as, for example signal processes, image reconstruction, neural networks, thermography and seismic engineering (see e.g., [4, 6, 7, 11, 13–16, 19, 23–27, 48]).

2. Preliminaries

Let Λ be any nonempty set and $\mathcal{P}(\Lambda)$ be the class of all subsets of Λ. A nonempty family $\mathcal{F} \subset \mathcal{P}(\Lambda)$ is called a *filter of* Λ iff $\emptyset \notin \mathcal{F}$, $A \cap B \in \mathcal{F}$ whenever A, $B \in \mathcal{F}$, and for every $A \in \mathcal{F}$ and $B \supset A$ it is $B \in \mathcal{F}$.

If $\Lambda = (\Lambda, \succeq)$ is a directed set, then for any $\lambda \in \Lambda$ set $M_\lambda = \{l \in \Lambda : l \succeq \lambda\}$. A filter $\mathcal{F}$ of Λ is *free* iff $M_\lambda \in \mathcal{F}$ for all $\lambda \in \Lambda$.

Some classical examples of free filters are the filter $\mathcal{F}_{\text{cofin}}$ of all subsets of $\mathbb{N}$ whose complement is finite and the filter $\mathcal{F}_d$ of all subsets of $\mathbb{N}$ whose asymptotic density is 1. As we know, the asymptotic density of a set $A \subseteq \mathbb{N}$ is defined as

$$\delta(A) = \lim_n \frac{|\{k \le n : k \in A\}|}{n} \tag{2.1}$$

whenever the limit in (2.1) exists, where $|B|$ denotes the cardinality of the set B ([42]). Here, $(\mathbb{N}, \succeq)$ is meant with respect to the usual order $\geq$.

From now on, we always suppose that $\mathcal{F}$ is a free filter of Λ.

A family $(x_\lambda)_{\lambda \in \Lambda}$ of real numbers is said to be $\mathcal{F}$*-convergent to* $x \in \mathbb{R}$ iff for every $\varepsilon > 0$ there is $F \in \mathcal{F}$ with $|x_\lambda - x| \le \varepsilon$ whenever $\lambda \in \mathcal{F}$.

Let G be a locally compact Hausdorff topological space, $\mathcal{P}(G \times G)$ be the class of all subsets of the Cartesian product $G \times G$, and let $\mathcal{Y} \subset \mathcal{P}(G \times G)$ be a uniform structure which generates the topology of G. Let $\mathcal{C}_b(G)$ be the space of all real-valued continuous and bounded functions defined on G, and $\mathcal{C}_c(G)$ be the subspace of $\mathcal{C}_b(G)$ of all functions with compact support on G.

Let a_r, e_r, $r = 0, 1, \ldots, m$, be elements of $\mathcal{C}_b(G)$ with $e_0(t) = 1$ for each $t \in G$. Set

$$P_s(t) := \sum_{r=0}^{m} a_r(s) e_r(t), \quad s, t \in G, \tag{2.2}$$

and assume that

(P1) $P_s(s) = 0$ for all $s \in G$;

(P2) for every $U \in \mathcal{Y}$ there is $\eta > 0$ such that $P_s(t) \geq \eta$ for each s, $t \in G$, $(s,t) \notin U$.

Example 2.1 (see also [10, 12])

(a) Let I be a connected subset of the real line and $G = I^m$ be equipped with the norm $\| \cdot \|_2$. Let $\varphi : I \to \mathbb{R}$ be monotone and such that φ^{-1} is uniformly continuous. Examples of such functions are $\varphi(t) = t$ or $\varphi(t) = e^t$, where $I = [a,b] \subset \mathbb{R}$.

For each $t = (t_1, t_2, \ldots, t_m) \in G$ let $e_r(t) := \varphi(t_r)$, $r = 1, 2, \ldots, m$, and

$$e_{m+1}(t) = \sum_{r=1}^{m} (\varphi(t_r))^2.$$

For every $s = (s_1, \ldots, s_m) \in G$, let

$$a_0(s) = \sum_{r=1}^{m} (\varphi(s_r))^2, \quad a_r(s) = -2\varphi(s_r), \quad r = 1, 2, \ldots, m,$$

and $a_{m+1}(s) = 1$. It is

$$P_s(t) = \sum_{r=0}^{m+1} a_r(s)\, e_r(t) = \sum_{r=1}^{m} (\varphi(s_r) - \varphi(t_r))^2.$$

It is not difficult to check that (P1) and (P2) hold.

(b) Let $G = [a,b]$ be with $0 < a < b < \pi/2$, $m = 2$, $e_1(t) = \cos t$, $e_2(t) = \sin t$, $t \in G$. Set $a_0(s) = 1$, $a_1(s) = -\cos s$, $a_2(s) = -\sin s$, $s \in G$. For any s, $t \in G$ it is

$$P_s(t) = 1 - \cos s \cos t - \sin s \sin t = 1 - \cos(s - t). \tag{2.3}$$

It is not difficult to see that (P1) and (P2) are satisfied.

(c) Let $q \in \mathbb{N}$ be fixed, $m = 2\,q$, $G = \left(\frac{a}{q}, \frac{b}{q}\right)^q$, with $0 < a < b < \pi/2$, $t = (t_1, \ldots, t_q)$, $s = (s_1, \ldots, s_q) \in G$, set $e_{2j-1}(t) = \cos j\, t_j$, $e_{2\,j}(t) = \sin j\, t_j$, $j = 1, \ldots, q$. Put $a_0(s) = q$, $a_{2j-1}(s) = -\cos j\, s_j$, $a_{2\,j}(s) = -\sin j\, s_j$, $j = 1, \ldots, r$. For each s, $t \in G$ it is

$$\begin{aligned} P_s(t) &= \sum_{r=0}^{m} a_r(s) e_r(t) = r - \sum_{j=1}^{r} \cos j s_j \cos j t_j - \sum_{j=1}^{r} \sin j s_j \sin j t_j \\ &= r - \sum_{j=1}^{r} \cos(j(s_j - t_j)). \end{aligned}$$

It is not difficult to check that (P1) and (P2) are fulfilled.

Definition 2.2 Let G be as above, $\Lambda \neq \emptyset$ be a fixed set, $\sigma : G \to \mathbb{R} \setminus \{0\}$ be a function (which in the literature is called *scale function*), $\mathcal{F}$ and $\mathcal{H}$ be free filters of Λ and G respectively, $(f_\lambda)_{\lambda\in\Lambda}$ be a family of real-valued functions, defined on G. We say that $(f_\lambda)_{\lambda\in\Lambda}$ $(\mathcal{F},\mathcal{H})$*-converges with respect to* σ or $(\mathcal{F},\mathcal{H},\sigma)$*-converges to a function* $f : G \to \mathbb{R}$ iff for each $\varepsilon > 0$ there exist $F \in \mathcal{F}$ and $H \in \mathcal{H}$ with $|f_\lambda(t) - f(t)| \le \varepsilon\,|\sigma(t)|$ whenever $\lambda \in F$ and $t \in H$.

If x_0 is any fixed point of G, then an example of filter of G is the set of all neighborhoods of x_0 (see e.g., [28]).

Example 2.3 Let $G = [0,1]$, $\Lambda = \mathbb{N}$, $\mathcal{H} \subset \mathcal{P}(G)$ be the set of all neighborhoods of $x_0 = 0$, $\mathcal{F} = \mathcal{F}_d$, $F \in \mathcal{F}$ and define $g_n : G \to \mathbb{R}$ by

$$g_n(x) = \begin{cases} 2\,n\,x^2, & n \notin F \\ \dfrac{n\,x^2}{3+n\,x^2}, & n \in F \end{cases}. \tag{2.4}$$

We claim that $(g_n)_n$ $(\mathcal{F}_d,\mathcal{H})$-converges to $g(x) = 0$ with respect to the scale function

$$\sigma(x) = \begin{cases} \dfrac{1}{x}, & 0 < x \le 1 \\ 1, & x = 0 \end{cases}. \tag{2.5}$$

Indeed, let $\varepsilon > 0$ be given and choose $\delta = \varepsilon$. Let $n \in F$, $x \in [0,1]$ be with $|x| < \delta$. Then,

$$\left|\frac{g_n(x)}{\sigma(x)}\right| \le \frac{n\,x^3}{3+n\,x^2} < x < \delta = \varepsilon.$$

However, $(g_n)_n$ does not $(\mathcal{F}_d,\mathcal{H})$-converge to $g(x) = 0$; hence, $(g_n)_n$ does not $\mathcal{F}_d$-converge to $g(x) = 0$ on $[0,1]$. Indeed, for $\varepsilon = \frac{1}{5}$, $x = \frac{1}{\sqrt{n}} \in]0,1]$ with $\frac{1}{\sqrt{n}} < \delta$ and $n \in F$, we get $\dfrac{n\,x^2}{3+n\,x^2} = \dfrac{1}{2} > \dfrac{1}{5}$. However, it is not difficult to see that the sequence $(g_n)_n$ is neither uniformly nor relatively uniformly convergent to $g(x) = 0$ (see also [28]).

3. The main results

In this section we prove our main Korovkin-type theorem.

Theorem 3.1 *Let* a_r, e_r, $r = 0,1,\ldots,m$, *be elements of* $\mathcal{C}_c(G)$, *satisfying the conditions* (P1)*and* (P2). *Let* $(L_\lambda)_{\lambda\in\Lambda}$ *be a family of positive linear operators acting from* $\mathcal{C}_c(G)$ *into itself and* σ *is the scale function (possibly unbounded).*

Then $(L_\lambda(e_r))_\lambda$ $(\mathcal{F},\mathcal{H},\sigma_r)$*-converges to* e_r *for every* $r = 0,1,\ldots,m$ *if and only if* $(L_\lambda(f))_\lambda$ $(\mathcal{F},\mathcal{H},\sigma)$*-converges to* f *for each* $f \in \mathcal{C}_c(G)$, *where*

$$\sigma(t) = \max\{|\sigma_r(t)| : r = 0,1,\ldots,m\}, \quad t \in G. \tag{3.1}$$

Proof First, we begin the "if" part. Our hypothesis is that $(L_\lambda(f))_\lambda$, $(\mathcal{F},\mathcal{H},\sigma)$-converges to f for each $f \in \mathcal{C}_c(G)$, which means that: $\forall f \in C_c(G)$, $\forall \varepsilon > 0$, $\exists$ $F \in \mathcal{F}$ and $H \in \mathcal{H}$ such that $|L_\lambda(f)(t) - f(t)| \le \varepsilon\sigma(t)$,

$\forall\lambda \in F$ and $t \in H$. Since $e_r \in C_c(G)$, $r = 0, ..., m$, if we choose $\varepsilon = \frac{\varepsilon^* |\sigma_r(t)|}{\sigma(t)}$, then we get $\forall \varepsilon^* > 0$, $\exists\ F \in \mathcal{F}$ and $H \in \mathcal{H}$ such that $|L_\lambda(e_r)(t) - e_r(t)| \leq \varepsilon^* |\sigma_r(t)|$, $r = 0, ..., m$, $\forall\lambda \in F$ and $t \in H$.

We now turn to the "only if" part. Pick arbitrarily $f \in \mathcal{C}_c(G)$. As G is equipped with the uniformity $\mathcal{Y}$, f is uniformly continuous and bounded on G.

Choose arbitrarily $\varepsilon > 0$. Without loss of generality, we can and do assume $0 < \varepsilon \leq 1$. Thanks to the uniform continuity of f, there is $U \in \mathcal{Y}$ such that

$$|f(s) - f(t)| \leq \frac{\varepsilon}{4}$$

for each $s, t \in G$ with $(s,t) \in U$. In correspondence with ε and U, let $\eta > 0$ satisfy the condition (P2). It is

$$|f(s) - f(t)| \leq 2\,S \leq \frac{2\,S}{\eta}\,P_s(t) \tag{3.2}$$

for all $s, t \in G$ with $(s,t) \notin U$. Thus, for every $s, t \in G$ we get

$$|f(s) - f(t)| \leq \frac{\varepsilon}{4} + \frac{2\,S}{\eta}\,P_s(t), \tag{3.3}$$

that is

$$-\frac{\varepsilon}{4} - \frac{2\,S}{\eta}\,P_s(t) \leq f(s) - f(t) \leq \frac{\varepsilon}{4} + \frac{2\,S}{\eta}\,P_s(t). \tag{3.4}$$

Let $S = \sup_{t\in G} |f(t)|$ and $N = \sup_{t \in G, r=0,1,\ldots,m} |a_r(t)|$. By $(\mathcal{F}, \mathcal{H}, \sigma_r)$-convergence to e_r of $(L_\lambda(e_r))_\lambda$, for every $r = 0, 1, \ldots, m$ there are $F_r \in \mathcal{F}$ and $H_r \in \mathcal{H}$ with

$$|L_\lambda(e_r)(s) - f(s)| \leq |\sigma_r(s)| \cdot \min\Big\{\varepsilon, \frac{\varepsilon}{4\,S}, \frac{\varepsilon\,\eta}{8\,N\,S\,(m+1)}\Big\} \tag{3.5}$$

whenever $\lambda \in F_r$ and $s \in H_r$.

Set $F = \bigcap_{r=0}^{m} F_r$, $H = \bigcap_{r=0}^{m} H_r$. As $\mathcal{F}$ and $\mathcal{H}$ are filters, we get that $F \in \mathcal{F}$ and $H \in \mathcal{H}$. Pick arbitrarily $s \in \mathcal{F}$ and $\lambda \in H$. Since the operators L_λ are linear and positive and $P_s(s) = 0$, from (3.4) we obtain

$$\begin{aligned} -\ & \frac{\varepsilon}{4}\,L_\lambda(e_0)(s) - \frac{2\,S}{\eta}\,L_\lambda(P_s)(s) + P_s(s) \leq \\ \leq\ & L_\lambda(f)(s) - f(s)\,L_\lambda(e_0)(s) \leq \\ \leq\ & \frac{\varepsilon}{4}\,L_\lambda(e_0)(s) + \frac{2\,S}{\eta}(L_\lambda P_s)(s). \end{aligned} \tag{3.6}$$

As $P_s(s) = \sum_{r=0}^{m} a_r(s)e_r(s)$, from (3.6) we get

$$\begin{aligned} - & \frac{\varepsilon}{4} L_\lambda(e_0)(s) - \frac{2\,S}{\eta} \sum_{r=0}^{m} a_r(s)\,(L_\lambda(e_r)(s) - e_r(s)) \leq \\ \leq & L_\lambda(f)(s) - f(s) + f(s)\,e_0(s) - f(s)\,L_\lambda(e_0)(s) \leq \\ \leq & \frac{\varepsilon}{4} L_\lambda(e_0)(s) + \frac{2\,S}{\eta} \sum_{r=0}^{m} a_r(s)\,(L_\lambda(e_r)(s) - e_r(s)). \end{aligned} \tag{3.7}$$

From (3.7) we obtain

$$\begin{aligned} |L_\lambda(f)(s) - f(s)| & \leq |f(s)|\,|L_\lambda(e_0)(s) - e_0(s)| + \frac{\varepsilon}{4}\,|L_\lambda(e_0)(s) - e_0(s)| + \\ & + \frac{\varepsilon}{4}\,e_0(s) + \frac{2\,N\,S}{\eta} \sum_{r=0}^{m} |L_\lambda(e_r)(s) - e_r(s)|. \end{aligned} \tag{3.8}$$

From (3.5) and (3.8) we deduce

$$\begin{aligned} \frac{|L_\lambda(f)(s) - f(s)|}{\sigma(s)} & \leq |f(s)|\,\frac{|L_\lambda(e_0)(s) - e_0(s)|}{|\sigma_r(s)|} + \frac{\varepsilon}{4}\,\frac{|L_\lambda(e_0)(s) - e_0(s)|}{|\sigma_r(s)|} + \\ & + \frac{\varepsilon}{4}\,e_0(s) + \frac{2\,N\,S}{\eta} \sum_{r=0}^{m} \frac{|L_\lambda(e_r)(s) - e_r(s)|}{|\sigma_r(s)|} \leq \\ & \leq S\,\frac{\varepsilon}{4\,S} + \frac{2\,\varepsilon}{4} + (m+1)\,\frac{2\,N\,S}{\eta}\,\frac{\varepsilon\,\eta}{8\,N\,S\,(m+1)} = \varepsilon. \end{aligned} \tag{3.9}$$

This ends the proof. □

4. Applications

We now deal with some examples, which show that our main Korovkin type approximation theorem is stronger than the corresponding classical ones.

Example 4.1 Let $G = [0, 1]$, $\mathcal{H} \subset \mathcal{P}(G)$ be the set of all neighborhoods of $x_0 = 0$, $\mathcal{F} = \mathcal{F}_d$, $F \in \mathcal{F}$ and consider the following linear positive operators:

$$M_n\,(f)\,(x) = \int_G K_n\,(t)\,f\,(tx)\,dt,\ n \in \mathbb{N},\ x \in G,$$

for every $f \in C\,(G)$, where $K_n\,(t) = (n+1)\,t^n$ (see [39]). Then, using the operators M_n, we define the sequence of positive linear operators $\mathbb{T} := (T_n)_n$ on $C_c\,(G)$ as follows:

$$T_n\,(f)\,(x) = (1 + g_n\,(x))\,M_n\,(f)\,(x)\,,$$

where $(g_n)_n$ is the same as in (2.4). Choose $\sigma_i\,(x) = \sigma\,(x)$ $(i = 0, 1, 2)$, where σ is the same as in (2.5). Let $e_0\,(x) = a_2\,(x) = 1$, $e_1\,(x) = x$, $e_2\,(x) = a_0\,(x) = x^2$, $a_1\,(x) = -2x$. We now claim that $(T_n\,(e_i))_n$

$(\mathcal{F}_d, \mathcal{H})$-converges to e_i with respect to the scale function σ_i, $i = 0, 1, 2$. Now, observe that

$$\begin{aligned} |T_n(e_0)(x) - e_0(x)| &= g_n(x), \\ |T_n(e_1)(x) - e_1(x)| &\leq \frac{1 + g_n(x)}{n+2} + g_n(x), \\ |T_n(e_2)(x) - e_2(x)| &\leq \frac{2(1 + g_n(x))}{n+3} + g_n(x). \end{aligned}$$

As we know from Example 2.3, $(g_n)_n$ $(\mathcal{F}_d, \mathcal{H})$-converges to $g(x) = 0$ with respect to the scale function σ. Hence, our claim is true for $i = 0, 1, 2$ and from our main theorem, we get that $(T_n(f))_n$ $(\mathcal{F}_d, \mathcal{H})$-converges to f with respect to the scale function σ. However, $(T_n(e_0))_n$ is neither uniformly nor relatively uniformly convergent to e_0. Hence, the classical and relative Korovkin theorems do not work for our sequence $(T_n)_n$ (It is illustrated for the function $f(x) = x^3 + 1$ in Figure 1).

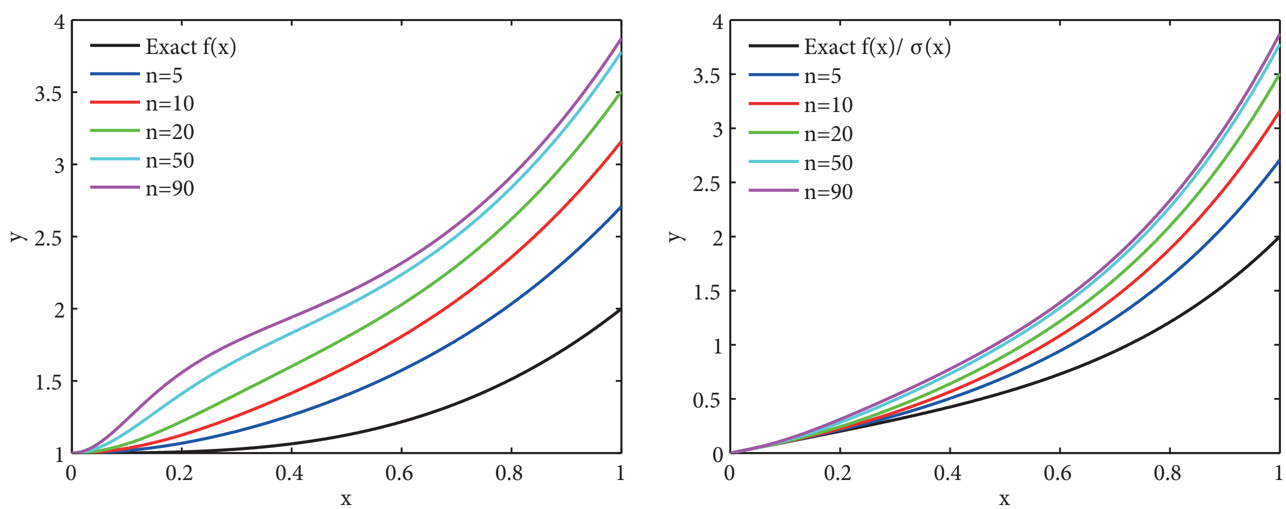

Figure 1. (Left) The operators $T_n(f)(x)$ do not $(\mathcal{F}_d, \mathcal{H})$-converge, and hence $T_n(f)(x)$ do not $\mathcal{F}_d$-converge on $[0,1]$, but (Right) the operators $T_n(f)(x)$ with the scale function $\sigma(x)$, $\sigma(x) = \frac{1}{x}$ if $0 < x \leq 1$ and $\sigma(x) = 1$ if $x = 0$, $(\mathcal{F}_d, \mathcal{H})$-converge to the function $f(x) = x^3 + 1$.

Example 4.2 Let $G = [0,1]^2$, $\Lambda = \mathbb{N}$, $\mathcal{H} \subset \mathcal{P}(G)$ be the set of all neighborhoods of $(x_0, y_0) = (0, 0)$, $\mathcal{F} = \mathcal{F}_d$, $F \in \mathcal{F}$ and consider the following bivariate Kantorovich-type operators, defined by

$$K_n(f)(x,y) = (n+1)^2 \sum_{k,j=0, k+j \leq n} p_{n,k,j}(x,y) \int_{k/(n+1)}^{(k+1)/(n+1)} \int_{j/(n+1)}^{(j+1)/(n+1)} f(s,t)\, ds dt,$$

for every $f \in C_c(G)$, $n \in \mathbb{N}$, where

$$\begin{aligned} p_{n,k,j}(x,y) &= \frac{n!}{k!j!(n-k-j)!} x^k y^j (1-x-y)^{n-k-j}, \\ k, j &\geq 0,\ k+j \leq n,\ x, y \geq 0,\ x+y \leq 1 \end{aligned}$$

(see also [45]). Then, using the operators K_n, we define the sequence of positive linear operators $\mathbb{L} := (L_n)_n$ on $C_c(G)$ as follows:

$$L_n(f)(x,y) = (1 + g_n(x,y)) K_n(f)(x,y), \tag{4.1}$$

for every $f \in C_c(G)$, $n \in \mathbb{N}$ and $x, y \in [0,1]$, where $g_n : G \to \mathbb{R}$ is defined by

$$g_n(x,y) = \begin{cases} 2n\,x^2 y^2, & n \notin F \\ \dfrac{n\,x^2y^2}{3 + n\,x^2y^2}, & n \in F \end{cases}. \tag{4.2}$$

Choose $\sigma_i(x,y) = \sigma(x,y)$ $(i = 0,1,2,3)$, where

$$\sigma(x,y) = \begin{cases} 1, & x = 0 \text{ or } y = 0 \\ \dfrac{1}{xy}, & (x,y) \in]0,1] \times]0,1] \end{cases}. \tag{4.3}$$

Let $e_0(x,y) = a_3(x,y) = 1$, $e_1(x,y) = x$, $e_2(x,y) = y$, $e_3(x,y) = a_0(x,y) = x^2 + y^2$, $a_1(x,y) = -2x$, $a_2(x,y) = -2y$. We claim that the sequence $(L_n(e_i))_n$ $(\mathcal{F}_d, \mathcal{H})$-converges to e_i with respect to the scale function σ_i, $i = 0,1,2,3$. From [45], we see that

$$\begin{aligned}
L_n(e_0)(x,y) &= 1 + g_n(x,y), \\
L_n(e_1)(x,y) &= (1 + g_n(x,y))\left(\frac{n\,x}{n+1} + \frac{1}{2(n+1)}\right), \\
L_n(e_2)(x,y) &= (1 + g_n(x,y))\left(\frac{n\,y}{n+1} + \frac{1}{2(n+1)}\right), \\
L_n(e_3)(x,y) &= (1 + g_n(x,y))\left(\frac{n(n-1)}{(n+1)^2}\left(x^2 + y^2\right) + \frac{2n(x+y)}{(n+1)^2} + \frac{2}{3(n+1)^2}\right).
\end{aligned}$$

Hence, we can see that

$$\left|\frac{L_n(e_0)(x,y) - e_0(x,y)}{\sigma_0(x,y)}\right| = \left|\frac{g_n(x,y)}{\sigma(x,y)}\right|.$$

Let $0 < \varepsilon < 1$ be given and choose $\delta_0 = \sqrt{\varepsilon}$. Let $n \in F$, $x, y \in [0,1]$ be with $|x| < \delta_0$, $|y| < \delta_0$. Then,

$$\left|\frac{g_n(x,y)}{\sigma(x,y)}\right| \le \frac{n\,x^3\,y^3}{3 + n\,x^2y^2} < x\,y < \delta_0^2 = \varepsilon.$$

Moreover, choose $\delta_1 = \sqrt{\frac{\varepsilon}{2}}$. Let $n \in F$, $x, y \in [0,1]$ be with $|x| < \delta_1$, $|y| < \delta_1$. We have

$$\begin{aligned}
&\left|\frac{L_n(e_1)(x,y) - e_1(x,y)}{\sigma_1(x,y)}\right| \\
= &\left|\frac{1}{\sigma(x,y)}\left((1 + g_n(x,y))\left(\frac{n\,x}{n+1} + \frac{1}{2(n+1)}\right) - x\right)\right| \\
\le &\frac{x^2y}{2} + \frac{3}{2}\left|\frac{g_n(x,y)}{\sigma(x,y)}\right| < \frac{\delta_1^2}{2} + \frac{3\delta_1^2}{2} = 2\delta_1^2 = \varepsilon.
\end{aligned}$$

Similarly, choose $\delta_2 = \sqrt{\frac{\varepsilon}{2}}$. Let $n \in F$, $x, y \in [0,1]$ be with $|x| < \delta_2$, $|y| < \delta_2$. We get

$$\left| \frac{L_n(e_2)(x,y) - e_2(x,y)}{\sigma_2(x,y)} \right| < 2\delta_2^2 = \varepsilon.$$

Finally, choose $\delta_3 = \sqrt{\frac{3\varepsilon}{38}}$. Let $n \in F$, $x, y \in [0,1]$ be with $|x| < \delta_3$, $|y| < \delta_3$. Then,

$$\begin{aligned} & \left| \frac{L_n(e_3)(x,y) - e_3(x,y)}{\sigma_3(x,y)} \right| \\ = & \left| \frac{(1+g_n(x,y))}{\sigma(x,y)} \left(\frac{n(n-1)}{(n+1)^2}\left(x^2+y^2\right) + \frac{2\,n(x+y)}{(n+1)^2} + \frac{2}{3(n+1)^2} \right) - \frac{\left(x^2+y^2\right)}{\sigma(x,y)} \right| \\ \leq & \ 3\left(x^3y + xy^3\right) + \frac{20}{3}\left| \frac{g_n(x,y)}{\sigma(x,y)} \right| < 6\delta_3^2 + \frac{20\delta_3^2}{3} = \frac{38\delta_3^2}{3} = \varepsilon. \end{aligned}$$

Hence our claim is true for $i = 0,1,2,3$ and from our main theorem, we get that $(L_n(f))_n$ $(\mathcal{F}_d, \mathcal{H})$-converges to f with respect to the scale function σ defined in (4.3). However, the sequence $(L_n(e_0))_n$ is neither uniformly nor relatively uniformly convergent to e_0. Hence, the classical and relative Korovkin theorems do not work for our sequence $(L_n)_n$ (It is illustrated for the function $f(x,y) = x^3 + y^3 + 1$ in Figure 2)

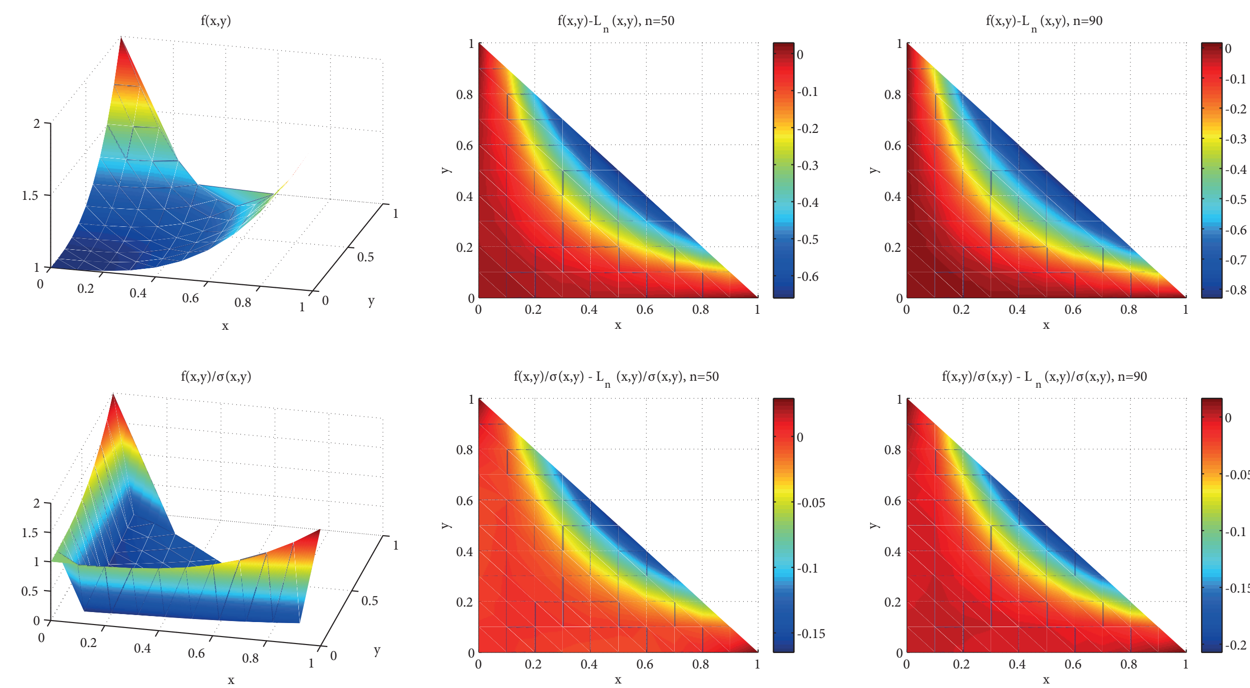

Figure 2. Plots of $f(x,y) = x^3 + y^3 + 1$ (Top) and f/σ (Bottom) and difference between f and L_n (Top) and f/σ and L_n/σ (Bottom) for $n = 50, 90$ (from the left to the right).

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