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On a three-dimensional solvable system of difference equations

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Abstract: In this paper we solve the following system of difference equations

$$x_{n+1}=\frac{z_{n-1}}{a+by_nz_{n-1}},\quad y_{n+1}=\frac{x_{n-1}}{a+bz_nx_{n-1}},\quad z_{n+1}=\frac{y_{n-1}}{a+bx_ny_{n-1}},\quad n\in\mathbb{N}_0$$

where parameters a,b and initial values $x_{-1},x_0,y_{-1},y_0,z_{-1},z_0$ are nonzero real numbers, and give a representation of its general solution in terms of a specially chosen solutions to homogeneous linear difference equation with constant coefficients associated to the system.

Key words: System of difference equations, general solution, representation of solutions

1. Introduction

Finding closed-form formulas for solutions to difference equations and systems of difference equations has attracted considerable interest recently (see, for example, [1, 6, 8–23, 25–30, 32–36] and the related references therein).

The paper by Stevic [24] has considerably motivated this line of research. He gave a theoretical explanation for the formula of solutions of the following difference equation

$$x_{n+1}=\frac{x_{n-1}}{1+x_nx_{n-1}},\quad n\in\mathbb{N}_0. \tag{1.1}$$

Aloqeili [2] has obtained the solutions of the difference equation

$$x_{n+1}=\frac{x_{n-1}}{a-x_nx_{n-1}},\quad n\in\mathbb{N}_0. \tag{1.2}$$

Namely, Cinar [3–5] investigated the solutions of the following difference equations

$$x_{n+1}=\frac{x_{n-1}}{1+ax_nx_{n-1}},\quad x_{n+1}=\frac{x_{n-1}}{-1+x_nx_{n-1}},\quad x_{n+1}=\frac{ax_{n-1}}{1+bx_nx_{n-1}},\quad n\in\mathbb{N}_0. \tag{1.3}$$

In this paper we consider the following extension of the equations in (1.1), (1.2), and (1.3)

$$x_{n+1}=\frac{z_{n-1}}{a+by_nz_{n-1}},\quad y_{n+1}=\frac{x_{n-1}}{a+bz_nx_{n-1}},\quad z_{n+1}=\frac{y_{n-1}}{a+bx_ny_{n-1}},\quad n\in\mathbb{N}_0 \tag{1.4}$$

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where the parameters a, b and the initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}, z_0$ are nonzero real numbers.

Other extensions of the equations in (1.1), (1.2), and (1.3) have been studied in two-dimensional system, for example Elsayed in [7] obtained the form of the solutions of the following rational difference system

$$x_{n+1} = \frac{x_{n-1}}{\pm 1 + y_n x_{n-1}}, \quad y_{n+1} = \frac{y_{n-1}}{\pm 1 + x_n y_{n-1}}, \quad n \in \mathbb{N}_0. \tag{1.5}$$

The closed-form formulas of solutions are given and proven by induction.

Our objective is to show that system (1.4) is solvable by finding its closed-form formulas through an analytical approach, and to show that all the closed-form formulas obtained in [2–5] and [24] easily follow from the ones in our present paper.

2. Main results

Assume that $\{x_n, y_n, z_n\}_{n \geq -1}$ is a well-defined solution to system (1.4). Then from (1.4) we have

$$x_{n+1} = \frac{y_n z_{n-1}}{a y_n + b y_n^2 z_{n-1}}, \quad y_{n+1} = \frac{z_n x_{n-1}}{a z_n + b z_n^2 x_{n-1}}, \quad z_{n+1} = \frac{x_n y_{n-1}}{a x_n + b x_n^2 y_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.1}$$

Let

$$u_n = x_n y_{n-1}, \quad v_n = y_n z_{n-1}, \quad w_n = z_n x_{n-1}, \quad n \in \mathbb{N}_0. \tag{2.2}$$

Then system (2.1) can be written as

$$u_{n+1} = \frac{v_n}{a + b v_n}, \quad v_{n+1} = \frac{w_n}{a + b w_n}, \quad w_{n+1} = \frac{u_n}{a + b u_n}, \quad n \in \mathbb{N}_0. \tag{2.3}$$

If we use the last recurrence relation in (2.3) into the second one, we obtain

$$v_{n+1} = \frac{u_{n-1}}{a^2 + b(a+1)u_{n-1}}, \quad n \in \mathbb{N}_1. \tag{2.4}$$

We replace the equation (2.4) into the first recurrence relation in (2.3), we obtain:

$$u_{n+1} = \frac{u_{n-2}}{a^3 + b(a^2 + a + 1)u_{n-2}}, \quad n \in \mathbb{N}_2. \tag{2.5}$$

Let

$$u_n^{(j)} = u_{3n+j}, \tag{2.6}$$

where $n \in \mathbb{N}_0, j \in \{0, 1, 2\}$.
Using notation (2.6), we can write (2.5) as

$$u_{n+1}^{(j)} = \frac{u_n^{(j)}}{A + B u_n^{(j)}}, \quad n \in \mathbb{N}_0, \tag{2.7}$$

with

$$A = a^3, \quad B = b(a^2 + a + 1).$$

Equation (2.7) can be reduced to the equation:

$$\mathcal{H}_{n+1} = \frac{(1+A)\mathcal{H}_n - A}{\mathcal{H}_n}, \quad n \in \mathbb{N}_0, \tag{2.8}$$

by using the change of variable

$$u_n^{(j)} = \frac{1}{B}\left(\mathcal{H}_n - A\right), \quad n \in \mathbb{N}_0. \tag{2.9}$$

Now we consider that the difference equation (2.8) with the initial value $\mathcal{H}_0$ is nonzero real number.

Through an analytical approach, we put

$$\mathcal{H}_n = \frac{k_n}{k_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.10}$$

Then equation (2.8) becomes

$$k_{n+1} - (1+A)k_n + Ak_{n-1} = 0, \quad n \in \mathbb{N}_0. \tag{2.11}$$

Case $A \neq 1$**:**

Let $\{k_n\}_{n \geq -1}$ be the solution to equation (2.11) such that k_0 and k_{-1} are nonzero real numbers. The roots of the characteristic polynomial $P(\lambda) = \lambda^2 - (1+A)\lambda + A$ are $\lambda_1 = A$ and $\lambda_2 = 1$. Then the general solution to equation (2.11) can be written in the following form

$$k_n = c_1 + c_2 A^n, \quad n \in \mathbb{N}_0.$$

Using the initial values k_0 and k_{-1} with some calculations we get

$$\begin{aligned} c_1 &= \frac{k_0 - k_{-1}A}{1-A}, \\ c_2 &= \frac{A(k_{-1} - k_0)}{1-A}. \end{aligned}$$

Thus, the general solution of equation (2.11) is

$$k_n = \frac{1}{1-A}\left[k_0\left(1 - A^{n+1}\right) - Ak_{-1}\left(1 - A^n\right)\right], \quad n \in \mathbb{N}_0. \tag{2.12}$$

From all mentioned above, we see that the following theorem holds.

Theorem 2.1 *Let* $\{\mathcal{H}_n\}_{n \geq 0}$ *be a well-defined solution to the equation* (2.8)*. Then, for* $n \in \mathbb{N}_0$

$$\mathcal{H}_n = \frac{A(1-A^n) - \mathcal{H}_0(1-A^{n+1})}{A(1-A^{n-1}) - \mathcal{H}_0(1-A^n)}. \tag{2.13}$$

Then, from (2.9) we see that

$$\begin{aligned} u_n^{(j)} &= \frac{1}{B}\left(\mathcal{H}_n - A\right) \\ &= \frac{(A-1)u_0^{(j)}}{A^n(A-1) - B(1-A^n)u_0^{(j)}}, \end{aligned}$$

using

$$\sum_{i=0}^{n-1} A^i = \frac{A^n - 1}{A - 1}, \quad \mathcal{H}_0 = A + Bu_0^{(j)}$$

we get that the solutions of the difference equation (2.7) is

$$u_n^{(j)} = \frac{u_0^{(j)}}{A^n + B\left(\sum_{i=0}^{n-1} A^i\right) u_0^{(j)}}, \tag{2.14}$$

for each $j \in \{0, 1, 2\}$.

From all mentioned above by using (2.6), we see that the following corollary holds.

Corollary 2.2 *Let* $\{u_n\}_{n\geq 0}$ *be a well-defined solution to the equation* (2.5)*. Then, for* $n \in \mathbb{N}_0$

$$\begin{cases} u_{3n} = \dfrac{u_0}{A^n + B\left(\sum_{i=0}^{n-1} A^i\right) u_0}, \\[2ex] u_{3n+1} = \dfrac{u_1}{A^n + B\left(\sum_{i=0}^{n-1} A^i\right) u_1}, \\[2ex] u_{3n+2} = \dfrac{u_2}{A^n + B\left(\sum_{i=0}^{n-1} A^i\right) u_2}. \end{cases} \tag{2.15}$$

Let $\{u_n\}_{n\geq 0}$ be a well-defined solution to the equation in (2.5). From (2.3), we have

$$u_1 = \frac{v_0}{a + bv_0}.$$

On the other hand, from Corollary (2.2), we obtain

$$\begin{aligned} u_{3n+1} &= \frac{\dfrac{v_0}{a + bv_0}}{a^{3n} + b(a^2 + a + 1)\left(\sum_{i=0}^{n-1} a^{3i}\right) \dfrac{v_0}{a + bv_0}} \\ &= \frac{v_0}{a^{3n}(a + bv_0) + b\left((a^2 + a + 1)\left(\sum_{i=0}^{n-1} a^{3i}\right)\right) v_0}. \end{aligned}$$

Thus, we get

$$u_{3n+1} = \frac{v_0}{a^{3n+1} + b\left(a(a+1)\left(\sum_{i=0}^{n-1} a^{3i}\right) + \left(\sum_{i=0}^{n} a^{3i}\right)\right) v_0}.$$

Moreover, from (2.3), we have

$$u_2 = \frac{w_0}{a^2 + b(a+1)w_0}.$$

Thus, from Corollary (2.2), we get

$$\begin{aligned} u_{3n+2} &= \frac{\dfrac{w_0}{a^2 + b(a+1)w_0}}{A^n + B\left(\displaystyle\sum_{i=0}^{n-1} A^i\right)\dfrac{w_0}{a^2 + b(a+1)w_0}} \\ &= \frac{w_0}{a^{3n}(a^2 + b(a+1)w_0) + b\left((a(a+1)+1)\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right)\right)w_0}. \end{aligned}$$

Thus, we get

$$u_{3n+2} = \frac{w_0}{a^{3n+2} + b\left(a^2\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right) + (a+1)\left(\displaystyle\sum_{i=0}^{n} a^{3i}\right)\right)w_0}.$$

Similarly, we can find $v_{3n}, v_{3n+1}, v_{3n+2}, w_{3n}, w_{3n+1}$, and w_{3n+2}.

From the above consideration, we see that the following result holds.

Theorem 2.3 *Let* $\{u_n, v_n, w_n\}_{n\geq 0}$ *be a well-defined solution to the system* (2.3)*. Then, for* $n \in \mathbb{N}_0$

$$\begin{cases} u_{3n} = \dfrac{u_0}{a^{3n} + b(a^2+a+1)\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right)u_0}, \\ u_{3n+1} = \dfrac{v_0}{a^{3n+1} + b\left(a(a+1)\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right) + \left(\displaystyle\sum_{i=0}^{n} a^{3i}\right)\right)v_0}, \\ u_{3n+2} = \dfrac{w_0}{a^{3n+2} + b\left(a^2\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right) + (a+1)\left(\displaystyle\sum_{i=0}^{n} a^{3i}\right)\right)w_0}, \end{cases}$$

$$\begin{cases} v_{3n} = \dfrac{v_0}{a^{3n} + b(a^2+a+1)\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right)v_0}, \\ v_{3n+1} = \dfrac{w_0}{a^{3n+1} + b\left(a(a+1)\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right) + \left(\displaystyle\sum_{i=0}^{n} a^{3i}\right)\right)w_0}, \\ v_{3n+2} = \dfrac{u_0}{a^{3n+2} + b\left(a^2\left(\displaystyle\sum_{i=0}^{n-1} a^{3i}\right) + (a+1)\left(\displaystyle\sum_{i=0}^{n} a^{3i}\right)\right)u_0}, \end{cases}$$

$$\begin{cases} w_{3n} &= \dfrac{w_0}{a^{3n}+b(a^2+a+1)\left(\displaystyle\sum_{i=0}^{n-1}a^{3i}\right)w_0}, \\ w_{3n+1} &= \dfrac{u_0}{a^{3n+1}+b\left(a(a+1)\left(\displaystyle\sum_{i=0}^{n-1}a^{3i}\right)+\left(\displaystyle\sum_{i=0}^{n}a^{3i}\right)\right)u_0}, \\ w_{3n+2} &= \dfrac{v_0}{a^{3n+2}+b\left(a^2\left(\displaystyle\sum_{i=0}^{n-1}a^{3i}\right)+(a+1)\left(\displaystyle\sum_{i=0}^{n}a^{3i}\right)\right)v_0}. \end{cases}$$

Let

$$x_n = \frac{u_n}{y_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.16}$$

$$y_n = \frac{v_n}{z_{n-1}}, \quad n \in \mathbb{N}_0, \tag{2.17}$$

$$z_n = \frac{w_n}{x_{n-1}}, \quad n \in \mathbb{N}_0. \tag{2.18}$$

Using (2.18) and (2.17) in formula (2.16), after some calculations we get

$$x_{6n} = \frac{u_{6n}w_{6n-2}v_{6n-4}}{v_{6n-1}u_{6n-3}w_{6n-5}}x_{6n-6}, \quad n \in \mathbb{N}_0. \tag{2.19}$$

Moreover, using equalities (2.18) and (2.16) in formula (2.17), we obtain

$$y_{6n} = \frac{v_{6n}u_{6n-2}w_{6n-4}}{w_{6n-1}v_{6n-3}u_{6n-5}}y_{6n-6}, \quad n \in \mathbb{N}_0. \tag{2.20}$$

Similarly, using (2.16) and (2.17) in formula (2.18), we get

$$z_{6n} = \frac{w_{6n}v_{6n-2}u_{6n-4}}{u_{6n-1}w_{6n-3}v_{6n-5}}z_{6n-6}, \quad n \in \mathbb{N}_0. \tag{2.21}$$

Multiplying the equalities which are obtained in (2.19), (2.20), and (2.21) from 1 to n, respectively, it follows that

$$x_{6n} = x_0\prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right), \quad n \in \mathbb{N}, \tag{2.22}$$

$$y_{6n} = y_0\prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right), \quad n \in \mathbb{N}, \tag{2.23}$$

$$z_{6n} = z_0\prod_{i=1}^{n}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right), \quad n \in \mathbb{N}. \tag{2.24}$$

Using the equalities (2.22), (2.23), and (2.24) in (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n-1} &= \frac{w_{6n}}{z_{6n}}, \\ &= \frac{1}{z_0}\frac{w_0}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)w_0}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right), \end{aligned}$$

so

$$x_{6n-1} = x_{-1}\frac{1}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)z_0x_{-1}}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right). \tag{2.25}$$

Moreover,

$$\begin{aligned} y_{6n-1} &= \frac{u_{6n}}{x_{6n}}, \\ &= \frac{1}{x_0}\frac{u_0}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)u_0}\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right); \end{aligned}$$

hence

$$y_{6n-1} = y_{-1}\frac{1}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)x_0y_{-1}}\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right). \tag{2.26}$$

Similarly,

$$\begin{aligned} z_{6n-1} &= \frac{v_{6n}}{y_{6n}}, \\ &= \frac{1}{y_0}\frac{v_0}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)v_0}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right), \end{aligned}$$

so

$$z_{6n-1} = z_{-1}\frac{1}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)y_0z_{-1}}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right). \tag{2.27}$$

And

$$\begin{aligned} x_{6n+1} &= \frac{u_{6n+1}}{y_{6n}}, \\ &= \frac{1}{y_0}\frac{v_0}{a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)v_0}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right); \end{aligned}$$

thus, we have

$$x_{6n+1} = z_{-1}\frac{1}{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}} \prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right). \tag{2.28}$$

Moreover,

$$\begin{aligned} y_{6n+1} &= \frac{v_{6n+1}}{z_{6n}}, \\ &= \frac{1}{z_0}\frac{w_0}{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) w_0} \prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right); \end{aligned}$$

hence,

$$y_{6n+1} = x_{-1}\frac{1}{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}} \prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right). \tag{2.29}$$

Moreover,

$$\begin{aligned} z_{6n+1} &= \frac{w_{6n+1}}{x_{6n}}, \\ &= \frac{1}{x_0}\frac{u_0}{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) u_0} \prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right), \end{aligned}$$

so

$$z_{6n+1} = y_{-1}\frac{1}{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}} \prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right). \tag{2.30}$$

Using the equalities (2.28), (2.29), and (2.30) in formulas (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n+2} &= \frac{u_{6n+2}}{y_{6n+1}}, \\ &= \frac{w_0}{x_{-1}}\frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) w_0}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) w_0} \prod_{i=1}^{n}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right); \end{aligned}$$

hence, we have

$$x_{6n+2} = z_0 \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}} \prod_{i=1}^{n}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right). \tag{2.31}$$

Moreover,

$$\begin{aligned} y_{6n+2} &= \frac{v_{6n+2}}{z_{6n+1}}, \\ &= \frac{u_0}{y_{-1}} \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) u_0}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) u_0} \prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right); \end{aligned}$$

thus

$$y_{6n+2} = x_0 \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}} \prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right). \tag{2.32}$$

Similarly,

$$\begin{aligned} z_{6n+2} &= \frac{w_{6n+2}}{x_{6n+1}}, \\ &= \frac{v_0}{z_{-1}} \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) v_0}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) v_0} \prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right); \end{aligned}$$

hence

$$z_{6n+2} = y_0 \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}} \prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right). \tag{2.33}$$

Using the equalities (2.31), (2.32), and (2.33) in formulas (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n+3} &= \frac{u_{6n+3}}{y_{6n+2}}, \\ &= \frac{u_0\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)u_0\right)}{x_0\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)u_0\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)u_0\right)} \\ &\times \prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right); \end{aligned}$$

hence

$$\begin{aligned} x_{6n+3} &= y_{-1}\frac{\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)}{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)x_0y_{-1}\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)} \\ &\times \prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right). \end{aligned} \quad (2.34)$$

Moreover,

$$\begin{aligned} y_{6n+3} &= \frac{v_{6n+3}}{z_{6n+2}}, \\ &= \frac{v_0\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)}{y_0\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)v_0\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)} \\ &\times \prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right), \end{aligned}$$

so

$$\begin{aligned} y_{6n+3} &= z_{-1}\frac{\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)}{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)y_0z_{-1}\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)} \\ &\times \prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right). \end{aligned} \quad (2.35)$$

Moreover,

$$\begin{aligned} z_{6n+3} &= \frac{w_{6n+3}}{x_{6n+2}}, \\ &= \frac{w_0\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}\right)}{z_0\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)w_0\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}\right)} \\ &\times \prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right); \end{aligned}$$

hence

$$\begin{aligned} z_{6n+3} &= x_{-1}\frac{\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}\right)}{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)z_0x_{-1}\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}\right)} \\ &\times \prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right). \end{aligned} \tag{2.36}$$

Using the equalities (2.34), (2.35), and (2.36) in formulas (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n+4} &= \frac{u_{6n+4}}{y_{6n+3}}, \\ &= \frac{v_0\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)}{z_{-1}\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)} \\ &\times \frac{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)y_0z_{-1}\right)}{\left(a^{6n+4}+b\left(a(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)+\left(\sum_{i=0}^{2n+1}a^{3i}\right)\right)v_0\right)}\prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right); \end{aligned}$$

hence

$$x_{6n+4} = y_0 \frac{\left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}\right)}{\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}\right)}$$

$$\times \frac{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) y_0 z_{-1}\right)}{\left(a^{6n+4} + b\left(a(a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) + \left(\sum_{i=0}^{2n+1} a^{3i}\right)\right) y_0 z_{-1}\right)} \prod_{i=1}^{n} \left(\frac{v_{6i} u_{6i-2} w_{6i-4}}{w_{6i-1} v_{6i-3} u_{6i-5}}\right),$$

Similarly,

$$y_{6n+4} = \frac{v_{6n+4}}{z_{6n+3}},$$

$$= \frac{w_0 \left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}{x_{-1}\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}$$

$$\times \frac{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) z_0 x_{-1}\right)}{\left(a^{6n+4} + b\left(a(a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) + \left(\sum_{i=0}^{2n+1} a^{3i}\right)\right) w_0\right)} \prod_{i=1}^{n} \left(\frac{w_{6i} v_{6i-2} u_{6i-4}}{u_{6i-1} w_{6i-3} v_{6i-5}}\right),$$

so

$$y_{6n+4} = z_0 \frac{\left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}{\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}$$

$$\times \frac{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) z_0 x_{-1}\right)}{\left(a^{6n+4} + b\left(a(a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) + \left(\sum_{i=0}^{2n+1} a^{3i}\right)\right) z_0 x_{-1}\right)} \prod_{i=1}^{n} \left(\frac{w_{6i} v_{6i-2} u_{6i-4}}{u_{6i-1} w_{6i-3} v_{6i-5}}\right).$$

Moreover,

$$
\begin{aligned}
z_{6n+4} &= \frac{w_{6n+4}}{x_{6n+3}}, \\
&= \frac{u_0\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)}{y_{-1}\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)} \\
&\times \frac{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)x_0y_{-1}\right)}{\left(a^{6n+4}+b\left(a(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)+\left(\sum_{i=0}^{2n+1}a^{3i}\right)\right)x_0y_{-1}\right)}\prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right);
\end{aligned}
$$

thus

$$
\begin{aligned}
z_{6n+4} &= x_0\frac{\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)}{\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)} \\
&\times \frac{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)x_0y_{-1}\right)}{\left(a^{6n+4}+b\left(a(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)+\left(\sum_{i=0}^{2n+1}a^{3i}\right)\right)x_0y_{-1}\right)}\prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right).
\end{aligned}
$$

Using relationships in the Theorem (2.3) we obtain

$$
\begin{cases}
w_{3n} = \dfrac{w_0}{a^{3n}+b(a^2+a+1)\left(\sum_{i=0}^{n-1}a^{3i}\right)w_0}, \\
w_{3n-1} = \dfrac{v_0}{a^{3n-1}+b\left(a^2\left(\sum_{i=0}^{n-2}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{n-1}a^{3i}\right)\right)v_0}, \\
w_{3n-2} = \dfrac{u_0}{a^{3n-2}+b\left(a(a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)+\left(\sum_{i=0}^{n-1}a^{3i}\right)\right)u_0}, \\
w_{3n-3} = \dfrac{w_0}{a^{3n-3}+b(a^2+a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)w_0}, \\
w_{3n-4} = \dfrac{v_0}{a^{3n-4}+b\left(a^2\left(\sum_{i=0}^{n-3}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)\right)v_0}, \\
w_{3n-5} = \dfrac{u_0}{a^{3n-5}+b\left(a(a+1)\left(\sum_{i=0}^{n-3}a^{3i}\right)+\left(\sum_{i=0}^{n-2}a^{3i}\right)\right)u_0}.
\end{cases}
$$

Moreover,

$$\begin{cases} v_{3n} &= \dfrac{v_0}{a^{3n}+b(a^2+a+1)\left(\sum_{i=0}^{n-1}a^{3i}\right)v_0}, \\ v_{3n-1} &= \dfrac{u_0}{a^{3n-1}+b\left(a^2\left(\sum_{i=0}^{n-2}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{n-1}a^{3i}\right)\right)u_0}, \\ v_{3n-2} &= \dfrac{w_0}{a^{3n-2}+b\left(a(a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)+\left(\sum_{i=0}^{n-1}a^{3i}\right)\right)w_0}, \\ v_{3n-3} &= \dfrac{v_0}{a^{3n-3}+b(a^2+a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)v_0}, \\ v_{3n-4} &= \dfrac{u_0}{a^{3n-4}+b\left(a^2\left(\sum_{i=0}^{n-3}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)\right)u_0}, \\ v_{3n-5} &= \dfrac{w_0}{a^{3n-5}+b\left(a(a+1)\left(\sum_{i=0}^{n-3}a^{3i}\right)+\left(\sum_{i=0}^{n-2}a^{3i}\right)\right)w_0}. \end{cases}$$

And

$$\begin{cases} u_{3n} &= \dfrac{u_0}{a^{3n}+b(a^2+a+1)\left(\sum_{i=0}^{n-1}a^{3i}\right)u_0}, \\ u_{3n-1} &= \dfrac{w_0}{a^{3n-1}+b\left(a^2\left(\sum_{i=0}^{n-2}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{n-1}a^{3i}\right)\right)w_0}, \\ u_{3n-2} &= \dfrac{v_0}{a^{3n-2}+b\left(a(a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)+\left(\sum_{i=0}^{n-1}a^{3i}\right)\right)v_0}, \\ u_{3n-3} &= \dfrac{u_0}{a^{3n-3}+b(a^2+a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)u_0}, \\ u_{3n-4} &= \dfrac{w_0}{a^{3n-4}+b\left(a^2\left(\sum_{i=0}^{n-3}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{n-2}a^{3i}\right)\right)w_0}, \\ u_{3n-5} &= \dfrac{v_0}{a^{3n-5}+b\left(a(a+1)\left(\sum_{i=0}^{n-3}a^{3i}\right)+\left(\sum_{i=0}^{n-2}a^{3i}\right)\right)v_0}. \end{cases}$$

From all mentioned above and

$$u_0 = x_0y_{-1}, \quad v_0 = y_0z_{-1}, \quad w_0 = z_0x_{-1},$$

we see that the following result holds.

Theorem 2.4 *Let $\{x_n, y_n, z_n\}_{n\geq -1}$ be a well-defined solution to the system (1.4). Then, for $n \in \mathbb{N}_0$*

$$x_{6n-1} = x_{-1}\frac{1}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)z_0x_{-1}}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right),$$

$$x_{6n} = x_0\prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right),$$

$$x_{6n+1} = z_{-1}\frac{1}{a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right),$$

$$x_{6n+2} = z_0\frac{a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}}{a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}}\prod_{i=1}^{n}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right),$$

$$x_{6n+3} = y_{-1}\frac{\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)}{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)x_0y_{-1}\right)\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)x_0y_{-1}\right)}$$
$$\times\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right)$$

$$x_{6n+4} = y_0\frac{\left(a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)}{\left(a^{6n+2}+b\left(a^2\left(\sum_{i=0}^{2n-1}a^{3i}\right)+(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)\right)y_0z_{-1}\right)}$$
$$\times\frac{\left(a^{6n+3}+b(a^2+a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)y_0z_{-1}\right)}{\left(a^{6n+4}+b\left(a(a+1)\left(\sum_{i=0}^{2n}a^{3i}\right)+\left(\sum_{i=0}^{2n+1}a^{3i}\right)\right)y_0z_{-1}\right)}\prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right)$$

$$y_{6n-1} = y_{-1}\frac{1}{a^{6n}+b(a^2+a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)x_0y_{-1}}\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right),$$

$$y_{6n} = y_0\prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right),$$

$$y_{6n+1} = x_{-1}\frac{1}{a^{6n+1}+b\left(a(a+1)\left(\sum_{i=0}^{2n-1}a^{3i}\right)+\left(\sum_{i=0}^{2n}a^{3i}\right)\right)z_0x_{-1}}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right),$$

$$y_{6n+2} = x_0 \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}} \prod_{i=1}^{n}\left(\frac{u_{6i} w_{6i-2} v_{6i-4}}{v_{6i-1} u_{6i-3} w_{6i-5}}\right),$$

$$y_{6n+3} = z_{-1} \frac{\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}\right)}{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) y_0 z_{-1}\right)\left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}\right)}$$

$$\times \prod_{i=1}^{n}\left(\frac{w_{6i-1} v_{6i-3} u_{6i-5}}{v_{6i} u_{6i-2} w_{6i-4}}\right)$$

$$y_{6n+4} = z_0 \frac{\left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}{\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}$$

$$\times \frac{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) z_0 x_{-1}\right)}{\left(a^{6n+4} + b\left(a(a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) + \left(\sum_{i=0}^{2n+1} a^{3i}\right)\right) z_0 x_{-1}\right)} \prod_{i=1}^{n}\left(\frac{w_{6i} v_{6i-2} u_{6i-4}}{u_{6i-1} w_{6i-3} v_{6i-5}}\right).$$

$$z_{6n-1} = z_{-1} \frac{1}{a^{6n} + b(a^2+a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) y_0 z_{-1}} \prod_{i=1}^{n}\left(\frac{w_{6i-1} v_{6i-3} u_{6i-5}}{v_{6i} u_{6i-2} w_{6i-4}}\right),$$

$$z_{6n} = z_0 \prod_{i=1}^{n}\left(\frac{w_{6i} v_{6i-2} u_{6i-4}}{u_{6i-1} w_{6i-3} v_{6i-5}}\right),$$

$$z_{6n+1} = y_{-1} \frac{1}{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}} \prod_{i=1}^{n}\left(\frac{v_{6i-1} u_{6i-3} w_{6i-5}}{u_{6i} w_{6i-2} v_{6i-4}}\right),$$

$$z_{6n+2} = y_0 \frac{a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}}{a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) y_0 z_{-1}} \prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right),$$

$$z_{6n+3} = x_{-1} \frac{\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) z_0 x_{-1}\right)\left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) z_0 x_{-1}\right)}$$

$$\times \prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right),$$

$$z_{6n+4} = x_0 \frac{\left(a^{6n+1} + b\left(a(a+1)\left(\sum_{i=0}^{2n-1} a^{3i}\right) + \left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}\right)}{\left(a^{6n+2} + b\left(a^2\left(\sum_{i=0}^{2n-1} a^{3i}\right) + (a+1)\left(\sum_{i=0}^{2n} a^{3i}\right)\right) x_0 y_{-1}\right)}$$

$$\times \frac{\left(a^{6n+3} + b(a^2+a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) x_0 y_{-1}\right)}{\left(a^{6n+4} + b\left(a(a+1)\left(\sum_{i=0}^{2n} a^{3i}\right) + \left(\sum_{i=0}^{2n+1} a^{3i}\right)\right) x_0 y_{-1}\right)} \prod_{i=1}^{n}\left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}}\right).$$

Case $A = 1$**:**

Then equation (2.8) becomes

$$k_{n+1} - 2k_n + k_{n-1} = 0, \quad n \in \mathbb{N}_0. \tag{2.37}$$

Let $\{k_n\}_{n\geq -1}$ be the solution to equation (2.11) such that k_0 and k_{-1} are nonzero real numbers. The root of the characteristic polynomial $P(\lambda) = (\lambda - 1)^2$ is $\lambda_{1,2} = 1$. Then general solution to equation (2.11) can be written in the following form

$$k_n = c_1 + c_2 n, \quad n \in \mathbb{N}_0.$$

Using the initial values k_0 and k_{-1}, after some calculations, we get

$$c_1 = k_0,$$
$$c_2 = k_0 - k_{-1}.$$

Thus, the general solution of equation (2.37) is

$$k_n = k_0(n+1) - k_{-1}n, \quad n \in \mathbb{N}_0. \tag{2.38}$$

From all mentioned above we see that the following theorem holds.

Theorem 2.5 *Let* $\{\mathcal{H}_n\}_{n\geq 0}$ *be a well-defined solution to the equation* (2.8). *Then, for* $n \in \mathbb{N}_0$

$$\mathcal{H}_n = \frac{n - \mathcal{H}_0(n+1)}{(n-1) - \mathcal{H}_0 n}. \tag{2.39}$$

Then, from (2.9) we see that

$$u_n^{(j)} = \frac{1}{B}\left(\mathcal{H}_n - 1\right), \quad B = 3b.$$

So the solution of the equation (2.7) is

$$u_n^{(j)} = \frac{u_0^{(j)}}{1 + Bnu_0^{(j)}}, \quad n \in \mathbb{N}_0. \tag{2.40}$$

where $j \in \{0, 1, 2\}$.

From all mentioned above and $u_n^{(j)} = u_{3n+j}$ we see that the following result holds.

Corollary 2.6 *Let $\{u_n\}_{n\geq 0}$ be a well-defined solution to the equation (2.5). Then, for $n \in \mathbb{N}_0$*

$$\begin{cases} u_{3n} &= \dfrac{u_0}{1 + Bnu_0}, \\ u_{3n+1} &= \dfrac{u_1}{1 + Bnu_1}, \\ u_{3n+2} &= \dfrac{u_2}{1 + Bnu_2}. \end{cases} \tag{2.41}$$

Let $\{u_n\}_{n\geq 0}$ be the solution to equation (2.5). From (2.3), we have

$$u_1 = \frac{v_0}{a + bv_0}.$$

On the other hand, from Corollary (2.6), we obtain

$$u_{3n+1} = \frac{\dfrac{v_0}{a + bv_0}}{1 + Bn\dfrac{v_0}{a + bv_0}},$$

Thus, we get

$$u_{3n+1} = \frac{v_0}{1 + b(3n + 1)v_0}$$

Moreover, from (2.3), we have

$$u_2 = \frac{w_0}{a^2 + b(a + 1)w_0}.$$

So, from Corollary (2.6) we have

$$u_{3n+2} = \frac{\dfrac{w_0}{a^2 + b(a + 1)w_0}}{1 + Bn\dfrac{w_0}{a^2 + b(a + 1)w_0}}.$$

So, we get

$$u_{3n+2} = \frac{w_0}{1 + b(3n + 2)w_0}.$$

From the above consideration, we see that the following result holds.

Theorem 2.7 *Let $\{u_n, v_n, w_n\}_{n\geq 0}$ be a well-defined solution to the system (2.3). Then, for $n \in \mathbb{N}_0$*

$$\begin{cases} u_{3n} = \dfrac{u_0}{1+3bnu_0}, & v_{3n} = \dfrac{v_0}{1+3bnv_0} & w_{3n} = \dfrac{w_0}{1+3bnw_0}, \\ u_{3n+1} = \dfrac{v_0}{1+b(3n+1)v_0}, & v_{3n+1} = \dfrac{w_0}{1+b(3n+1)w_0}, & w_{3n+1} = \dfrac{u_0}{1+b(3n+1)u_0}, \\ u_{3n+2} = \dfrac{w_0}{1+b(3n+2)w_0}, & v_{3n+2} = \dfrac{u_0}{1+b(3n+2)u_0}, & w_{3n+2} = \dfrac{v_0}{1+b(3n+2)v_0}. \end{cases} \tag{2.42}$$

Using (2.18) and (2.17) in formula (2.16), after some calculations we have

$$x_{6n} = \frac{u_{6n}w_{6n-2}v_{6n-4}}{v_{6n-1}u_{6n-3}w_{6n-5}} x_{6n-6}, \quad n \in \mathbb{N}. \tag{2.43}$$

Moreover, using equalities (2.18) and (2.16) in formula (2.17), we obtain

$$y_{6n} = \frac{v_{6n}u_{6n-2}w_{6n-4}}{w_{6n-1}v_{6n-3}u_{6n-5}} y_{6n-6}, \quad n \in \mathbb{N}. \tag{2.44}$$

Similarly using (2.16) and (2.17) in (2.18), we get

$$z_{6n} = \frac{w_{6n}v_{6n-2}u_{6n-4}}{u_{6n-1}w_{6n-3}v_{6n-5}} z_{6n-6}, \quad n \in \mathbb{N}. \tag{2.45}$$

Multiplying the equalities which are obtained form (2.43), (2.20), and (2.45) from 1 to n, respectively, it follows that

$$x_{6n} = x_0 \prod_{i=1}^{n} \left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}} \right), \quad n \in \mathbb{N}_0, \tag{2.46}$$

$$y_{6n} = y_0 \prod_{i=1}^{n} \left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}} \right), \quad n \in \mathbb{N}_0, \tag{2.47}$$

$$z_{6n} = z_0 \prod_{i=1}^{n} \left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}} \right), \quad n \in \mathbb{N}_0. \tag{2.48}$$

Using the equalities (2.46), (2.47), and (2.48) in formulas (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n-1} &= \frac{w_{6n}}{z_{6n}} \\ &= \frac{1}{z_0} \frac{w_0}{1+6bnw_0} \prod_{i=1}^{n} \left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}} \right), \end{aligned}$$

so

$$x_{6n-1} = x_{-1} \frac{1}{(1+6bnz_0x_{-1})} \prod_{i=1}^{n} \left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}} \right). \tag{2.49}$$

Moreover,

$$\begin{aligned} y_{6n-1} &= \frac{u_{6n}}{x_{6n}} \\ &= \frac{1}{x_0} \frac{u_0}{1+6bnu_0} \prod_{i=1}^{n} \left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}} \right); \end{aligned}$$

hence,

$$y_{6n-1}=y_{-1}\frac{1}{(1+6bnx_0y_{-1})}\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right). \tag{2.50}$$

Similarly,

$$\begin{aligned} z_{6n-1} &= \frac{v_{6n}}{y_{6n}} \\ &= \frac{1}{y_0}\frac{v_0}{1+6bnv_0}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right), \end{aligned}$$

so

$$z_{6n-1}=z_{-1}\frac{1}{(1+6bny_0z_{-1})}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right). \tag{2.51}$$

Moreover,

$$\begin{aligned} x_{6n+1} &= \frac{u_{6n+1}}{y_{6n}} \\ &= \frac{1}{y_0}\frac{v_0}{1+b(6n+1)v_0}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right); \end{aligned}$$

thus

$$x_{6n+1}=z_{-1}\frac{1}{(1+b(6n+1)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right). \tag{2.52}$$

Moreover,

$$\begin{aligned} y_{6n+1} &= \frac{v_{6n+1}}{z_{6n}} \\ &= \frac{1}{z_0}\frac{w_0}{1+b(6n+1)w_0}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right); \end{aligned}$$

hence

$$y_{6n+1}=x_{-1}\frac{1}{(1+b(6n+1)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right). \tag{2.53}$$

Similarly,

$$\begin{aligned} z_{6n+1} &= \frac{w_{6n+1}}{x_{6n}} \\ &= \frac{1}{x_0}\frac{u_0}{1+b(6n+1)u_0}\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right), \end{aligned}$$

so

$$z_{6n+1}=y_{-1}\frac{1}{(1+b(6n+1)x_0y_{-1})}\prod_{i=1}^{n}\left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}}\right). \tag{2.54}$$

Using the equalities (2.52), (2.53), and (2.54) in (2.16), (2.17) and (2.18), we obtain

$$\begin{aligned} x_{6n+2} &= \frac{u_{6n+2}}{y_{6n+1}} \\ &= \frac{1}{x_{-1}} \frac{w_0}{1+b(6n+2)w_0} \frac{(1+b(6n+1)z_0x_{-1})}{1} \prod_{i=1}^{n} \left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}} \right), \end{aligned}$$

hence

$$x_{6n+2} = z_0 \frac{(1+b(6n+1)z_0x_{-1})}{(1+b(6n+2)z_0x_{-1})} \prod_{i=1}^{n} \left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}} \right). \tag{2.55}$$

Also,

$$\begin{aligned} y_{6n+2} &= \frac{v_{6n+2}}{z_{6n+1}} \\ &= \frac{1}{y_{-1}} \frac{u_0}{1+b(6n+2)u_0} \frac{(1+b(6n+1)x_0y_{-1})}{1} \prod_{i=1}^{n} \left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}} \right), \end{aligned}$$

so

$$y_{6n+2} = x_0 \frac{(1+b(6n+1)x_0y_{-1})}{(1+b(6n+2)x_0y_{-1})} \prod_{i=1}^{n} \left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}} \right). \tag{2.56}$$

Similarly,

$$\begin{aligned} z_{6n+2} &= \frac{w_{6n+2}}{x_{6n+1}} \\ &= \frac{1}{z_{-1}} \frac{v_0}{1+b(6n+2)v_0} \frac{(1+b(6n+1)y_0z_{-1})}{1} \prod_{i=1}^{n} \left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}} \right); \end{aligned}$$

hence

$$z_{6n+2} = y_0 \frac{(1+b(6n+1)y_0z_{-1})}{(1+b(6n+2)y_0z_{-1})} \prod_{i=1}^{n} \left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}} \right). \tag{2.57}$$

Using the equalities (2.55), (2.56), and (2.57) in (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n+3} &= \frac{u_{6n+3}}{y_{6n+2}} \\ &= \frac{1}{x_0} \frac{u_0}{1+b(6n+3)u_0} \frac{(1+b(6n+2)x_0y_{-1})}{(1+b(6n+1)x_0y_{-1})} \prod_{i=1}^{n} \left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}} \right), \end{aligned}$$

so

$$x_{6n+3} = y_{-1} \frac{(1+b(6n+2)x_0y_{-1})}{(1+b(6n+1)x_0y_{-1})(1+b(6n+3)x_0y_{-1})} \prod_{i=1}^{n} \left(\frac{v_{6i-1}u_{6i-3}w_{6i-5}}{u_{6i}w_{6i-2}v_{6i-4}} \right). \tag{2.58}$$

Also,

$$\begin{aligned} y_{6n+3} &= \frac{v_{6n+3}}{z_{6n+2}} \\ &= \frac{1}{y_0}\frac{v_0}{1+b(6n+3)v_0}\frac{(1+b(6n+2)y_0z_{-1})}{(1+b(6n+1)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right); \end{aligned}$$

thus

$$y_{6n+3} = z_{-1}\frac{(1+b(6n+2)y_0z_{-1})}{(1+b(6n+1)y_0z_{-1})(1+b(6n+3)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{w_{6i-1}v_{6i-3}u_{6i-5}}{v_{6i}u_{6i-2}w_{6i-4}}\right). \tag{2.59}$$

Also,

$$\begin{aligned} z_{6n+3} &= \frac{w_{6n+3}}{x_{6n+2}} \\ &= \frac{1}{z_0}\frac{w_0}{1+b(6n+3)w_0}\frac{(1+b(6n+2)z_0x_{-1})}{(1+b(6n+1)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right), \end{aligned}$$

so

$$z_{6n+3} = x_{-1}\frac{(1+b(6n+2)z_0x_{-1})}{(1+b(6n+1)z_0x_{-1})(1+b(6n+3)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{u_{6i-1}w_{6i-3}v_{6i-5}}{w_{6i}v_{6i-2}u_{6i-4}}\right). \tag{2.60}$$

Using the equalities (2.58), (2.59), and (2.60) in (2.16), (2.17), and (2.18), we obtain

$$\begin{aligned} x_{6n+4} &= \frac{u_{6n+4}}{y_{6n+3}} \\ &= \frac{1}{z_{-1}}\frac{v_0}{1+b(6n+4)v_0}\frac{(1+b(6n+1)y_0z_{-1})(1+b(6n+3)y_0z_{-1})}{(1+b(6n+2)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right); \end{aligned}$$

hence

$$x_{6n+4} = y_0\frac{(1+b(6n+1)y_0z_{-1})(1+b(6n+3)y_0z_{-1})}{(1+b(6n+2)y_0z_{-1})(1+b(6n+4)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{v_{6i}u_{6i-2}w_{6i-4}}{w_{6i-1}v_{6i-3}u_{6i-5}}\right); \tag{2.61}$$

hence

$$\begin{aligned} y_{6n+4} &= \frac{v_{6n+4}}{z_{6n+3}} \\ &= \frac{1}{x_{-1}}\frac{w_0}{1+b(6n+4)w_0}\frac{(1+b(6n+1)z_0x_{-1})(1+b(6n+3)z_0x_{-1})}{(1+b(6n+2)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right); \end{aligned}$$

hence

$$y_{6n+4} = z_0\frac{(1+b(6n+1)z_0x_{-1})(1+b(6n+3)z_0x_{-1})}{(1+b(6n+2)z_0x_{-1})(1+b(6n+4)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{w_{6i}v_{6i-2}u_{6i-4}}{u_{6i-1}w_{6i-3}v_{6i-5}}\right). \tag{2.62}$$

Also,

$$\begin{aligned} z_{6n+4} &= \frac{w_{6n+4}}{x_{6n+3}} \\ &= \frac{1}{y_{-1}} \frac{u_0}{1+b(6n+4)u_0} \frac{(1+b(6n+1)x_0y_{-1})(1+b(6n+3)x_0y_{-1})}{(1+b(6n+2)x_0y_{-1})} \prod_{i=1}^{n} \left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}} \right), \end{aligned}$$

so

$$z_{6n+4} = x_0 \frac{(1+b(6n+1)x_0y_{-1})(1+b(6n+3)x_0y_{-1})}{(1+b(6n+2)x_0y_{-1})(1+b(6n+4)x_0y_{-1})} \prod_{i=1}^{n} \left(\frac{u_{6i}w_{6i-2}v_{6i-4}}{v_{6i-1}u_{6i-3}w_{6i-5}} \right). \tag{2.63}$$

From Theorem (2) we conclude that

$$\begin{cases} u_{3n} &= \dfrac{u_0}{1+3bnu_0}, \quad v_{3n} = \dfrac{v_0}{1+3bnv_0}, \quad w_{3n} = \dfrac{w_0}{1+3bnw_0}, \\ u_{3n-1} &= \dfrac{w_0}{1+b(3n-1)w_0}, \quad v_{3n-1} = \dfrac{u_0}{1+b(3n-1)u_0}, \quad w_{3n-1} = \dfrac{v_0}{1+b(3n-1)v_0}, \\ u_{3n-2} &= \dfrac{v_0}{1+b(3n-2)v_0}, \quad v_{3n-2} = \dfrac{w_0}{1+b(3n-2)w_0}, \quad w_{3n-2} = \dfrac{u_0}{1+b(3n-2)u_0}, \\ u_{3n-3} &= \dfrac{u_0}{1+b(3n-3)u_0}, \quad v_{3n-3} = \dfrac{v_0}{1+b(3n-3)v_0}, \quad w_{3n-3} = \dfrac{w_0}{1+b(3n-3)w_0}, \\ u_{3n-4} &= \dfrac{w_0}{1+b(3n-4)w_0}, \quad v_{3n-4} = \dfrac{u_0}{1+b(3n-4)u_0}, \quad w_{3n-4} = \dfrac{v_0}{1+b(3n-4)v_0}, \\ v_{3n-5} &= \dfrac{v_0}{1+b(3n-5)v_0}, \quad v_{3n-5} = \dfrac{w_0}{1+b(3n-5)w_0}, \quad w_{3n-5} = \dfrac{u_0}{1+b(3n-5)u_0}. \end{cases}$$

From all mentioned above and

$$u_0 = x_0y_{-1}, \quad v_0 = y_0z_{-1}, \quad w_0 = z_0x_{-1},$$

we see that the following result holds.

Theorem 2.8 *Let $\{x_n, y_n, z_n\}_{n\geq -1}$ be a well-defned solution to the system (1.4) and $a = 1$. Then, for $n \in \mathbb{N}_0$*

$$\begin{aligned} x_{6n-1} &= \frac{x_{-1}}{(1+6bnz_0x_{-1})} \prod_{i=1}^{n} \left(\frac{(1+6biz_0x_{-1})(1+b(6i-2)z_0x_{-1})(1+b(6i-4)z_0x_{-1})}{(1+b(6i-1)z_0x_{-1})(1+(6i-3)z_0x_{-1})(1+(6i-5)z_0x_{-1})} \right), \\ x_{6n} &= x_0 \prod_{i=1}^{n} \left(\frac{(1+b(6i-1)x_0y_{-1})(1+b(6i-3)x_0y_{-1})(1+b(6i-5)x_0y_{-1})}{(1+6bix_0y_{-1})(1+b(6i-2)x_0y_{-1})(1+b(6i-4)x_0y_{-1})} \right), \\ x_{6n+1} &= \frac{z_{-1}}{(1+b(6n+1)y_0z_{-1})} \prod_{i=1}^{n} \left(\frac{(1+b6iy_0z_{-1})(1+b(6i-2)y_0z_{-1})(1+b(6i-4)y_0z_{-1})}{(1+b(6i-1)y_0z_{-1})(1+b(6i-3)y_0z_{-1})(1+b(6i-5)y_0z_{-1})} \right), \\ x_{6n+2} &= \frac{z_0(1+b(6n+1)z_0x_{-1})}{(1+b(6n+2)z_0x_{-1})} \prod_{i=1}^{n} \left(\frac{(1+b(6i-1)z_0x_{-1})(1+b(6i-3)z_0x_{-1})(1+b(6i-5)z_0x_{-1})}{(1+6biz_0x_{-1})(1+b(6i-2)z_0x_{-1})(1+b(6i-4)z_0x_{-1})} \right), \\ x_{6n+3} &= \frac{y_{-1}(1+b(6n+2)x_0y_{-1})}{(1+b(6n+1)x_0y_{-1})(1+b(6n+3)x_0y_{-1})} \prod_{i=1}^{n} \left(\frac{(1+6bix_0y_{-1})(1+b(6i-2)x_0y_{-1})(1+b(6i-4)x_0y_{-1})}{(1+b(6i-1)x_0y_{-1})(1+b(6i-3)x_0y_{-1})(1+b(6i-5)x_0y_{-1})} \right), \\ x_{6n+4} &= \frac{y_0(1+b(6n+1)y_0z_{-1})(1+b(6n+3)y_0z_{-1})}{(1+b(6n+2)y_0z_{-1})(1+b(6n+4)y_0z_{-1})} \prod_{i=1}^{n} \left(\frac{(1+b(6i-1)y_0z_{-1})(1+b(6i-3)y_0z_{-1})(1+b(6i-5)y_0z_{-1})}{(1+6biy_0z_{-1})(1+b(6i-2)y_0z_{-1})(1+b(6i-4)y_0z_{-1})} \right), \end{aligned}$$

$$y_{6n-1} = \frac{y_{-1}}{(1+6bnx_0y_{-1})}\prod_{i=1}^{n}\left(\frac{(1+6bix_0y_{-1})(1+b(6i-2)x_0y_{-1})(1+b(6i-4)x_0y_{-1})}{(1+b(6i-1)x_0y_{-1})(1+b(6i-3)x_0y_{-1})(1+b(6i-5)x_0y_{-1})}\right),$$

$$y_{6n} = y_0\prod_{i=1}^{n}\left(\frac{(1+b(6i-1)y_0z_{-1})(1+b(6i-3)y_0z_{-1})(1+b(6i-5)y_0z_{-1})}{(1+6biy_0z_{-1})(1+b(6i-2)y_0z_{-1})(1+b(6i-4)y_0z_{-1})}\right),$$

$$y_{6n+1} = \frac{x_{-1}}{(1+b(6n+1)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{(1+6biz_0x_{-1})(1+b(6i-2)z_0x_{-1})(1+b(6i-4)z_0x_{-1})}{(1+b(6i-1)z_0x_{-1})(1+b(6i-3)z_0x_{-1})(1+b(6i-5)z_0x_{-1})}\right),$$

$$y_{6n+2} = \frac{x_0(1+b(6n+1)x_0y_{-1})}{(1+b(6n+2)x_0y_{-1})}\prod_{i=1}^{n}\left(\frac{(1+b(6i-1)x_0y_{-1})(1+b(6i-3)x_0y_{-1})(1+b(6i-5)x_0y_{-1})}{(1+6bix_0y_{-1})(1+b(6i-2)x_0y_{-1})(1+b(6i-4)x_0y_{-1})}\right),$$

$$y_{6n+3} = \frac{z_{-1}(1+b(6n+2)y_0z_{-1})}{(1+b(6n+1)y_0z_{-1})(1+b(6n+3)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{(1+6biy_0z_{-1})(1+b(6i-2)y_0z_{-1})(1+b(6i-4)y_0z_{-1})}{(1+b(6i-1)y_0z_{-1})(1+b(6i-3)y_0z_{-1})(1+b(6i-5)y_0z_{-1})}\right),$$

$$y_{6n+4} = \frac{z_0(1+b(6n+1)z_0x_{-1})(1+b(6n+3)z_0x_{-1})}{(1+b(6n+2)z_0x_{-1})(1+b(6n+4)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{(1+b(6i-1)z_0x_{-1})(1+b(6i-3)z_0x_{-1})(1+b(6i-5)z_0x_{-1})}{(1+6biz_0x_{-1})(1+b(6i-2)z_0x_{-1})(1+b(6i-4)z_0x_{-1})}\right),$$

$$z_{6n-1} = \frac{z_{-1}}{(1+6bny_0z_{-1})}\prod_{i=1}^{n}\left(\frac{(1+6biy_0z_{-1})(1+b(6i-2)y_0z_{-1})(1+b(6i-4)y_0z_{-1})}{(1+b(6i-1)y_0z_{-1})(1+b(6i-3)y_0z_{-1})(1+b(6i-5)y_0z_{-1})}\right),$$

$$z_{6n} = z_0\prod_{i=1}^{n}\left(\frac{(1+b(6i-1)z_0x_{-1})(1+b(6i-3)z_0x_{-1})(1+b(6i-5)z_0x_{-1})}{(1+6biz_0x_{-1})(1+b(6i-2)z_0x_{-1})(1+b(6i-4)z_0x_{-1})}\right),$$

$$z_{6n+1} = \frac{y_{-1}}{(1+b(6n+1)x_0y_{-1})}\prod_{i=1}^{n}\left(\frac{(1+6bix_0y_{-1})(1+b(6i-2)x_0y_{-1})(1+b(6i-4)x_0y_{-1})}{(1+b(6i-1)x_0y_{-1})(1+b(6i-3)x_0y_{-1})(1+b(6i-5)x_0y_{-1})}\right),$$

$$z_{6n+2} = \frac{y_0(1+b(6n+1)y_0z_{-1})}{(1+b(6n+2)y_0z_{-1})}\prod_{i=1}^{n}\left(\frac{(1+b(6i-1)y_0z_{-1})(1+b(6i-3)y_0z_{-1})(1+b(6i-5)y_0z_{-1})}{(1+6biy_0z_{-1})(1+b(6i-2)y_0z_{-1})(1+b(6i-4)y_0z_{-1})}\right),$$

$$z_{6n+3} = \frac{x_{-1}(1+b(6n+2)z_0x_{-1})}{(1+b(6n+1)z_0x_{-1})(1+b(6n+3)z_0x_{-1})}\prod_{i=1}^{n}\left(\frac{(1+6biz_0x_{-1})(1+b(6i-2)z_0x_{-1})(1+b(6i-4)z_0x_{-1})}{(1+b(6i-1)z_0x_{-1})(1+b(6i-3)z_0x_{-1})(1+b(6i-5)z_0x_{-1})}\right),$$

$$z_{6n+4} = \frac{x_0(1+b(6n+1)x_0y_{-1})(1+b(6n+3)x_0y_{-1})}{(1+b(6n+2)x_0y_{-1})(1+b(6n+4)x_0y_{-1})}\prod_{i=1}^{n}\left(\frac{(1+b(6i-1)x_0y_{-1})(1+b(6i-3)x_0y_{-1})(1+b(6i-5)x_0y_{-1})}{(1+6bix_0y_{-1})(1+b(6i-2)x_0y_{-1})(1+b(6i-4)x_0y_{-1})}\right),$$

3. Summary and recommendations

In this work, we have successfully established in a constructive way the closed-form solution of system of rational difference equations

$$x_{n+1} = \frac{z_{n-1}}{a+by_nz_{n-1}}, \quad y_{n+1} = \frac{x_{n-1}}{a+bz_nx_{n-1}}, \quad z_{n+1} = \frac{y_{n-1}}{a+bx_ny_{n-1}}, \quad n \in \mathbb{N}_0$$

where parameters a, b and initial values $x_{-1}, x_0, y_{-1}, y_0, z_{-1}$ and z_0 are real numbers.
The results in this paper can be extended to the following system of difference equations

$$x^{(1)}_{n+1} = \frac{x^{(3)}_{n-1}}{a+bx^{(2)}_n x^{(3)}_{n-1}}, \quad x^{(2)}_{n+1} = \frac{x^{(4)}_{n-1}}{a+bx^{(3)}_n x^{(4)}_{n-1}}, \quad \ldots, \quad x^{(p)}_{n+1} = \frac{x^{(p-1)}_{n-1}}{a+bx^{(1)}_n x^{(p-1)}_{n-1}}, \quad n, p \in \mathbb{N}_0$$

where the parameters a, b and the initial values $x^{(j)}_{-1}$ and $x^{(j)}_0$, $j = \overline{0,p}$ are nonzero real numbers.

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References

[1] Akrour Y, Touafek N, Halim Y. On a system of difference equations of second order solved in closed-form. Miskolc Mathematical Notes 2019; 20 (2): 701-717. doi: 10.18514/MMN.2019.2923

[2] Aloqeili M. Dynamics of a rational difference equation. Applied Mathematics and Computation 2006; 176 (1): 768-774. doi: 10.1016/j.amc.2005.10.024

[3] Cinar C. On the positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-1}}{1 + ax_n x_{n-1}}$. Applied Mathematics and Computation 2004; 158 (3): 809-812. doi: 10.1016/j.amc.2003.08.140

[4] Cinar C. On the positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-1}}{-1 + x_n x_{n-1}}$. Applied Mathematics and Computation 2004; 158 (1): 813-816. doi: 10.1016/j.amc.2003.08.122

[5] Cinar C. On the positive solutions of the difference equation $x_{n+1} = \dfrac{x_{n-1}}{-1 + bx_n x_{n-1}}$. Applied Mathematics and Computation 2004; 158 (1): 793-797. doi: 10.1016/j.amc.2003.08.139

[6] Elsayed EM. On a system of two nonlinear difference equations of order two. Proceedings of the Jangjeon Mathematical Society 2015; 18: 353-368.

[7] Elsayed EM. Solutions of rational difference systems of order two. Mathematical and Computer Modelling 2012; 55 (1): 378-384. doi: 10.1016/j.mcm.2011.08.012

[8] Elsayed EM, Ibrahim TF. Periodicity and solutions for some systems of nonlinear rational difference equations. Hacettepe Journal of Mathematics and Statistics 2015; 44: 1361-1390. doi: 10.15672/HJMS.2015449653

[9] Elsayed EM. Solution for systems of difference equations of rational form of order two. Computational and Applied Mathematics 2014; 33: 751-765. doi: 10.1007/s40314-013-0092-9

[10] Halim Y. Global character of systems of rational difference equations. Electronic Journal of Mathematical Analysis and Applications 2015; 3: 204-214.

[11] Halim Y. Form and periodicity of solutions of some systems of higher-order difference equations. Mathematical Sciences Letters 2016; 5: 79-84. doi: 10.18576/msl/050111

[12] Halim Y. A system of difference equations with solutions associated to Fibonacci numbers. International Journal of Difference Equations 2016; 11: 65-77.

[13] Halim Y, Touafek N, Yazlik Y. Dynamic behavior of a second-order nonlinear rational difference equation. Turkish Journal of Mathematics 2015; 39: 1004-1018. doi:10.3906/mat-1503-80

[14] Halim Y, Bayram M. On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences. Mathematical Methods in the Applied Sciences 2016; 39: 2974-2982. doi: 10.1002/mma.3745

[15] Halim Y, Rabago JTF. On some solvable systems of difference equations with solutions associated to Fibonacci numbers. Electronic Journal of Mathematical Analysis and Applications 2017; 5: 166-178.

[16] Halim Y, Rabago JFT. On the solutions of a second-order difference equations in terms of generalized Padovan sequences. Mathematica Slovaca 2018; 68 (3): 625-638. doi: 10.1515/ms-2017-0130

[17] Rabago JTF, Halim Y. Supplement to the paper of Halim, Touafek and Elsayed: Part I. Dynamics of Continuous, Discrete and Impulsive Systems Series A: Mathematical Analysis 2017; 24: 121-131.

[18] Kara M, Yazlik Y. Solvability of a system of nonlinear difference equations of higher order. Turkish Journal of Mathematics 2019; 43 (3): 1533-1565. doi: 10.3906/mat-1902-24

[19] Kara M, Yazlik Y. On the system of difference equations $x_n = \frac{x_{n-2}y_{n-3}}{y_{n-1}(a_n+b_nx_{n-2}y_{n-3})}, y_n = \frac{y_{n-2}x_{n-3}}{x_{n-1}(\alpha_n+\beta_ny_{n-2}x_{n-3})}$. Journal of Mathematical Extension 2020; 14 (1): 41-59.

[20] Kara M, Yazlik Y, Tollu DT. Solvability of a system of higher order nonlinear difference equations. Hacettepe Journal of Mathematics & Statistics 2020; doi: 10.15672/HJMS.xx

[21] Khelifa A, Halim Y, Berkal M. Solutions of a system of two higher-order difference equations in terms of Lucas sequence. Universal Journal of Mathematics and Applications 2019; 2(4): 202-211. doi: 10.32323/ujma.610399

[22] Stević S. On a third-order system of difference equations. Applied Mathematics and Computation 2012; 218: 7649-7654. doi: 10.1016/j.amc.2012.01.034

[23] Stević S. On some solvable systems of difference equations. Applied Mathematics and Computation 2012; 218 (9): 5010-5018. doi: 10.1016/j.amc.2011.10.068

[24] Stević S. More on a rational recurrence relation. Applied Mathematics E-Notes 2004; 4 (1): 80-85.

[25] Stević S. On a system of difference equations, Applied Mathematics and Computation 2011; 218 (7): 3372-3378. doi: 10.1016/j.amc.2011.08.079

[26] Stevic S.On a system of difference equations with period two coefficients. Applied Mathematics and computation 2011; 218 (8): 4317-4324. doi: 10.1016/j.amc.2011.10.005

[27] Tollu DT, Yazlik Y, Taskara N. On fourteen solvable systems of difference equations. Applied Mathematics and Computation 2014; 233: 310-319. doi: 10.1016/j.amc.2014.02.001

[28] Tollu DT, Yazlik Y, Taskara N. On the solutions of two special types of Riccati difference equation via Fibonacci numbers. Advances in Differential Equations 2013; 174: 7pages. doi: 10.1186/1687-1847-2013-174

[29] Touafek N. On some fractional systems of difference equations. Iranian Journal of Mathematical Sciences and Informatics 2014; 9: 303-305. doi: 10.7508/ijmsi.2014.02.007

[30] Touafek N. On a second order rational difference equation. Hacettepe Journal of Mathematics and Statistics 2012; 41: 867-874.

[31] Touafek N, Elsayed EM. On the periodicity of some systems of nonlinear difference equations. Bulletin Mathématique de la Société des Sciences Mathématiques de Roumanie. Nouvelle Série 2021; 55: 217-224.

[32] Touafek N, Elsayed EM. On the solutions of systems of rational difference equations. Mathematical and Computer Modelling 2012; 55(7-8): 1987-1997. doi: 10.1016/j.mcm.2011.11.058

[33] Yazlik Y, Tollu DT, Taskara N. On the solutions of difference equation systems with Padovan numbers. Applied Mathematics 2013; 12: 15-20. doi: 10.4236/am.2013.412A1002

[34] Yazlik Y, Tollu DT, Taskara N. Behaviour of solutions for a system of two higher-order difference equations. Journal of Science and Arts 2018; 45 (4): 813-826.

[35] Yazlik Y, Kara M. On a solvable system of difference equations of higher-order with period two coefficients. Communications Faculty of Sciences University of Ankara Series A1 Mathematics and Statistics 2019; 68 (2): 1675-1693.

[36] Yazlik Y, Kara M. Beşinci mertebeden fark denklem sisteminin çözülebilirliği üzerine. Eskişehir Teknik Üniversitesi Bilim ve Teknoloji Dergisi B-Teorik Bilimler 2019; 7 (1): 29-45 (in Turkish).