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Solvability and maximal regularity results for a differential equation with diffusion coefficient

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Abstract: We consider a second-order differential equation with rapidly growing intermediate coefficients. We obtain a solvability result in the cases that the diffusion coefficient of equation is unbounded or it tends to zero at the infinity. Under additional conditions, we prove the L_p- maximal regularity estimate for the solution of this equation.

Key words: Second-order differential equation, unbounded intermediate coefficient, existence of a solution, uniqueness, maximal regularity estimate

1. Introduction

In this paper, we consider the following equation:

$$-\rho(x)\left(\rho(x)y'\right)' + r(x)y' + s(x)y = F(x), \tag{1.1}$$

where $x \in \mathbb{R} = (-\infty, +\infty)$, all the coefficient functions are defined on $\mathbb{R}$, $\rho(x)$ is a positive and twice continiously differentiable function, $r(x)$ is a continiously differentiable function, and $s(x)$ is a continuous function and $F \in L_p := L_p(\mathbb{R})$, $1 < p < +\infty$.

Let $C_0^{(2)}(\mathbb{R})$ be the set of twice continuously differentiable functions with compact support. We define the operator $\tilde{l}_0$ on $C_0^{(2)}(\mathbb{R})$ as $\tilde{l}_0 y := -\rho(x)\left(\rho(x)y'\right)' + r(x)y' + s(x)y$. We denote the closure of the operator $\tilde{l}_0$ by $\tilde{l}$ in the space L_p. By solution of the equation (1.1) we mean a function $y \in D(\tilde{l})$ such that $\tilde{l}y = F$.

In this work, we study questions of the existence and uniqueness of the solutions of (1.1) and conditions, which for a solution $y(x)$ of (1.1) the following estimate holds:

$$\|\rho(\rho y')'\|_p + \|ry'\|_p + \|sy\|_p \le C\|F\|_p, \tag{1.2}$$

where $\|\cdot\|_p$ is the norm in L_p. If the estimate (1.2) holds, then we call that the solution $y(x)$ of (1.1) is maximally L_p-regular, and call (1.2) is an maximal L_p-regularity estimate.

In the applications of well-known projection methods (e.g., Fourier or Laplace transformations) to multidimensional differential equations and with coefficients depending on a single variable, we usually obtain ordinary differential equations. Therefore, the investigation of solvability questions for the one-dimensional equation (1.1) is important for the study of partial differential equations with unbounded coefficients. Since

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equation (1.1) with $\rho(x) \geq \delta > 0$ and its multidimensional generalizations are used in quantum mechanics and stochastic analysis, they have been studied intensively (see [2] and references therein). In [3, Ch.9, Theorem 2.4], [15, Ch.10.17], [12, Ch.7, Theorem 6], the authors considered the case that $s(x)$ is positive, and the growth at infinity of the absolute value of the intermediate coefficient $r(x)$ is limited by some power of $s(x)$. In [6–8, 10], it is assumed that the coefficient $r(x)$ is independent of the function $s(x)$, but $r(x)$ cannot grow faster than $|x|\ln|x| \quad (|x| \gg 1)$. The question arises whether there exists a unique solution of equation (1.1), if $|r(x)|$ grows more rapidly than $|x|\ln|x| \quad (|x| \gg 1)$ and cannot be controlled by the coefficient $s(x)$. It is also interesting to consider the case when the coefficient $\rho(x)$ in the leading term of the equation (1.1) tends to zero as $x \to +\infty$ or $x \to -\infty$.

If $\rho(x) = 1$ and $|r(x)|$ does not depend on the coefficient $s(x)$ and grows rapidly, then equation (1.1) has unique solution and an estimate of the maximal regularity for the solution holds (see [14, Theorem 1.1]). If coefficient $s(x)$ does not have a lower bound and the growth of $|s(x)|$ depends on the growth of $|r(x)|$, then equation (1.1) is well-posed ([13, Lemma 2.5]. The above gives that rapid and independent growth of the absolute value of the intermediate coefficient $r(x)$ have close relationship with the well-posedness of equation (1.1).

In contrast with [13, 14], in the current paper we consider the equation (1.1) with the coefficient $\rho(x)$ in the leading term. The study of (1.1) is not only of theoretical interest. It is known that an operator l above arises as generator of the transition semigroup of a stochastic Ornstein-Uhlenbeck process that determines a Brownian motion with a variable covariance matrix connected with $\rho(x)$. Studying (1.1) with the coefficient $\rho(x)$, we overcame new difficulties compared to [14], such as the choice and estimation of the linear functional in Theorem 3.1, as well as the construction of the operators B_λ and M_λ, and the estimation of the norm B_λ in Theorem 3.4. Furthermore, if $\rho(x)$ tends to zero at infinity, then we may consider the degeneracy case. For example, by Theorem 3.5, the following equation

$$-\frac{1}{3+x^2}\left(\frac{1}{3+x^2}y'\right)' + \left(17+3x^2\right)^{10} y' - x^7 y = f(x), \quad x \in \mathbb{R},$$

is uniquely solvable in L_2. Under additional conditions on $\rho(x)$ and $r(x)$, we can obtain the maximal regularity estimate of the solution y of the equation (1.1). We remark that the question of maximal regularity is an important tool in the theory of nonlinear PDEs (see, e.g., [1, 5]).

2. One-weighted integral inequality

Let $g(x)$ and $h(x) \neq 0$ be given continuous functions, and $q = \frac{p}{p-1}$. We denote

$$\tilde{\alpha}_{g,h}(t) := \|g\|_{L_p(0,t)} \left\|h^{-1}\right\|_{L_q(t,+\infty)} \ (t > 0),$$

$$\tilde{\beta}_{g,h}(\tau) := \|g\|_{L_p(\tau,0)} \left\|h^{-1}\right\|_{L_q(-\infty,\tau)} \ (\tau < 0),$$

$$\alpha_{g,h} := \sup_{t>0} \tilde{\alpha}_{g,h}(t), \ \beta_{g,h} := \sup_{\tau<0} \tilde{\beta}_{g,h}(\tau), \ \gamma_{g,h} := \max\left(\alpha_{g,h}, \beta_{g,h}\right).$$

Lemma 2.1 *If $g(x)$ and $h(x) \neq 0$ are continuous functions with $\gamma_{g,h} < +\infty$, then*

$$\int_{-\infty}^{+\infty} |g(x)y(x)|^p \, dx \leq C \int_{-\infty}^{+\infty} |h(x)y'(x)|^p \, dx, \quad \forall y \in C_0^{(1)}(\mathbb{R}).$$

Moreover, if C is the smallest constant for which this inequality holds, then

$$(\min(\alpha_{g,h}, \beta_{g,h}))^p \leq C \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \gamma_{g,h}\right)^p.$$

Proof Let $y \in C_0^{(1)}(\mathbb{R})$. We define

$$y_1 := y\mathcal{X}_{[0,+\infty)}, \quad y_2 := y\mathcal{X}_{(-\infty,0]}, \quad g_1 := g\mathcal{X}_{[0,+\infty)},$$
$$g_2 := g\mathcal{X}_{(-\infty,0]}, \quad h_1 := h\mathcal{X}_{[0,+\infty)}, \quad h_2 := h\mathcal{X}_{(-\infty,0]},$$

where $\mathcal{X}_A$ is the characteristic function of the set A. By [11, Theorem 2], we have that

$$\int_0^{+\infty} |g_1(x)y_1(x)|^p dx \leq C_1 \int_0^{+\infty} |h_1(x)y_1'(x)|^p dx, \tag{2.1}$$

and if C_1 is the smallest constant for which this inequality holds, then

$$(\alpha_{g_1,h_1})^p \leq C_1 \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \alpha_{g_1,h_1}\right)^p. \tag{2.2}$$

Let $t > 0, \tau = -t$. Then $\tilde{\alpha}_{g_2(-x),h_2(-x)}(t) = \tilde{\beta}_{g_2(x),h_2(x)}(\tau) \leq \gamma_{g,h} < +\infty$. Applying (2.1), we obtain that

$$\int_0^{+\infty} |g_2(-x)y_2(-x)|^p dx \leq C_2 \int_0^{+\infty} |h_2(-x)y_2'(-x)|^p dx,$$

i.e.

$$\int_{-\infty}^{0} |g_2(x)y_2(x)|^p dx \leq C_2 \int_{-\infty}^{0} |h_2(x)y_2'(x)|^p dx. \tag{2.3}$$

If C_2 is the smallest constant for which this inequality holds, then (2.2) gives that

$$(\beta_{g_2,h_2})^p \leq C_2 \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \beta_{g_2,h_2}\right)^p. \tag{2.4}$$

Putting (2.1) and (2.3) together, we get

$$\int_{-\infty}^{+\infty} |g(x)y(x)|^p \, dx \leq C \int_{-\infty}^{+\infty} |h(x)y'(x)|^p \, dx.$$

From (2.2) and (2.4), it follows that $(\min(\alpha_{g,h}, \beta_{g,h}))^p \leq C \leq \left(p^{\frac{1}{p}} q^{\frac{1}{q}} \gamma_{g,h}\right)^p$. □

3. Main results

We consider the following equation

$$-\rho(x)\left(\rho(x)y'\right)' + r(x)y' = f(x), \tag{3.1}$$

where $f \in L_p$. Let $l_0 y := -\rho(x)\left(\rho(x)y'\right)' + r(x)y'$ for $y \in C_0^{(2)}(\mathbb{R})$. We denote the closure of the operator l_0 by l in the space L_p. By solution of equation (3.1) we mean a function $y \in D(l)$ such that $ly = f$.

Theorem 3.1 *Let* $1 < p < +\infty$, *and* $\rho(x) > 0$ *be twice continuously differentiable function, and* $r(x) \geq 1$ *be continuously differentiable function. Suppose that*

$$1 \leq \frac{r(x)}{\rho^2(x)} \leq C\left(\frac{r(x)}{\rho(x)}\right)^p, \tag{3.2}$$

$$\gamma_{1,\left(\frac{r}{\rho^2}\right)^{1/p}\rho} < +\infty, \tag{3.3}$$

and there exists $\xi \in \mathbb{R}$ *such that*

$$\sup_{x<\xi}\left\{\rho(x)\exp\left(-\int_x^{\xi}\frac{r(t)}{\rho^2(t)}dt\right)\right\} < +\infty. \tag{3.4}$$

Then equation (3.1) *has a unique solution* y *for any* $f \in L_p$ *and the following estimate holds*

$$\left\|\left(\frac{r}{\rho^2}\right)^{1/p}\rho y'\right\|_p + \|y\|_p \leq C\|f\|_p.$$

Proof Let $y \in D(l_0)$. Set $z(x) := y'(x)$ and $Lz := l_0 y = -\rho(x)(\rho(x)z)' + r(x)z$. Let $\gamma \in \mathbb{R}$, $\gamma > -1$. Using integration by parts it is easy to verify that

$$\begin{aligned}\int_{\mathbb{R}} z(x)[(\rho(x)z(x))^2]^{\gamma/2}Lz\,dx &= \int_{\mathbb{R}} z(x)\left[(\rho(x)z(x))^2\right]^{\gamma/2}\left[-\rho(x)(\rho(x)z(x))' + r(x)z(x)\right]dx \\ &= \int_{\mathbb{R}} r(x)\rho^{\gamma}(x)\left[z^2(x)\right]^{\frac{\gamma}{2}+1}dx.\end{aligned} \tag{3.5}$$

Let $\gamma = p-2$. Then $\gamma + 1 = \frac{\gamma+2}{q}$, where $q = \frac{p}{p-1}$. Applying Hölder's inequality and (3.2), we obtain that

$$\int_{\mathbb{R}} z(x)[(\rho(x)z)^2]^{\gamma/2}Lz\,dx \leq \left(\int_{\mathbb{R}}\left|r^{-\frac{1}{q}}(x)\rho^{\frac{\gamma}{p}}(x)Lz\right|^p dx\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}}\left[r^{\frac{1}{q}}(x)\rho^{\frac{\gamma}{q}}(x)|z(x)|^{\frac{\gamma+2}{q}}\right]^q dx\right)^{\frac{1}{q}},$$

i. e.

$$\int_{\mathbb{R}} z(x)[(\rho z)^2]^{\gamma/2}Lz\,dx \leq \left\|\left(\frac{r}{\rho^2}\right)^{-\frac{1}{q}}\frac{1}{\rho}Lz\right\|_p\left(\int_{\mathbb{R}} r(x)\rho^{\gamma}(x)|z(x)|^{\gamma+2}dx\right)^{\frac{1}{q}}. \tag{3.6}$$

From (3.5) and (3.6) it follows that

$$\left\|\left(\frac{r}{\rho^2}\right)^{\frac{1}{p}}\rho y'\right\|_p \le \left\|\left(\frac{r}{\rho^2}\right)^{-\frac{1}{q}}\frac{1}{\rho}ly\right\|_p . \tag{3.7}$$

By (3.2),

$$\frac{1}{\rho(x)}\left[\frac{r(x)}{\rho^2(x)}\right]^{-\frac{1}{q}} = \left(\frac{r(x)}{\rho^2(x)}\right)^{\frac{1}{p}}\frac{\rho(x)}{r(x)} \le C^{1/p},$$

and so

$$\left\|\left(\frac{r}{\rho^2}\right)^{-\frac{1}{q}}\frac{1}{\rho}ly\right\|_p \le C^{1/p}\left\|ly\right\|_p . \tag{3.8}$$

Applying (3.3) and Lemma 2.1, we obtain

$$\|y\|_p \le C_1\left\|\left(\frac{r}{\rho^2}\right)^{\frac{1}{p}}\rho y'\right\|_p . \tag{3.9}$$

Using (3.7), (3.8), and (3.9), we get that

$$\left\|\left(\frac{r}{\rho^2}\right)^{\frac{1}{p}}\rho y'\right\|_p + \|y\|_p \le C_2\left\|ly\right\|_p . \tag{3.10}$$

Now, we suppose that $y \in D(l)$. Then there exists a sequence $\{y_n(x)\}_{n=1}^{\infty} \subset C_0^{(2)}(\mathbb{R})$ such that $\|y_n - y\|_p \to 0$ and $\|ly_n - ly\|_p \to 0$ $(n \to +\infty)$. Let $W_p^1(r,\rho)$ be the completion of the linear space

$$\left\{ y \in L_p : \|y\|_{W_p^1(r,\rho)} := \left\|\left(\frac{r}{\rho^2}\right)^{\frac{1}{p}}\rho y'\right\|_p + \|y\|_p < +\infty \right\}$$

with respect to the norm $\|y\|_{W_p^1(r,\rho)}$. Hence, from (3.10) it follows that

$$\|y_n\|_{W_p^1(r,\rho)} \le C_2\|ly_n\|_p, \quad n = 1, 2, \ \dots . \tag{3.11}$$

(3.11) implies that the sequence $\{y_n\}_{n=1}^{+\infty}$ is a Cauchy sequence. Since $W_p^1(r,\rho)$ is the Banach space, we deduce that $y(x) \in W_p^1(r,\rho)$ and $\|y_n - y\|_{W_p^1(r,\rho)} \to 0$ $(n \to +\infty)$. Letting $n \to +\infty$ in (3.11), we get

$$\|y\|_{W_p^1(r,\rho)} \le C_2\|ly\|_p,$$

i.e. (3.10) holds for all $y \in D(l)$. Hence, the operator l is invertible and $R(l)$ is closed.

Next we prove that $R(l) = L_p$. We denote the adjoint operator of l by l^*. Let $R(l) \ne L_p$. Since $R(l) = N(l^*)^{\perp} = \{w(x) \in L_p : \langle w, y^*\rangle = 0 \ \forall y^* \in N(l^*)\}$, there exists a nonzero element $v(x) \in L_p \backslash R(l)$ such that $l^*v = \rho(x)(\rho(x)v)' + r(x)v = 0$ [16, p. 205, Theorem]. Assume that $x_0 \in \mathbb{R}$ and $v(x_0) \ne 0$. Since the last equation is shift invariant, we can choose $x_0 = \xi$. It is easy to verify that

$$\rho(x)(\rho(x)v)' + r(x)v = C. \tag{3.12}$$

Hence, $v(x) \in C^{(1)}_{loc}(\mathbb{R})$.

Let $C \neq 0$. Since equation (3.12) is linear, without loss of generality, we can assume that $C = -1$ and $v(x_0) > 0$. Then

$$\left(\rho(x)v(x)\exp\int\limits_{x_0}^{x}\frac{r(t)}{\rho^2(t)}dt\right)' < 0.$$

Therefore,

$$\rho(x)\exp\left(-\int\limits_{x}^{x_0}\frac{r(t)}{\rho^2(t)}dt\right)v(x) > \rho(x_0)v(x_0), \quad x < x_0.$$

By (3.4), there exists $K > 0$ such that

$$v(x) > \rho(x_0)v(x_0)\frac{1}{K}, \quad x < x_0.$$

It follows that $v(x) \notin L_p$.

If $C = 0$, then

$$\rho(x)v(x)\exp\int\limits_{x_0}^{x}\frac{r(t)}{\rho^2(t)}dt = C_1.$$

It is clear that $C_1 \neq 0$. Thus, for $x < x_0$,

$$|v(x)| = \frac{|C_1|}{\rho(x)}\exp\left(-\int\limits_{x_0}^{x}\frac{r(t)}{\rho^2(t)}dt\right) > \frac{|C_1|}{K}.$$

Hence, for $x < x_0$ the inequality $|v(x)| \geq C_4 > 0$ holds, consequently $v \notin L_p$. This is contradiction, so we obtain that $R(l) = L_p$. □

Example 3.2 *Let* $r(x) = (1+x^2)^{\frac{n}{2}}$ *and* $\rho(x) = (1+x^2)^{\frac{k}{2}}$ *in* (3.1) , $k, n > 0$.

We check the conditions of Theorem 3.1. $\left(\frac{r(x)}{\rho^2(x)}\right)^{1/p}\rho(x) = (1+x^2)^{m/2}$ where $m = \frac{n-2k}{p} + k$. Then

$$\begin{aligned}\tilde{\alpha}_{1,\left(\frac{r}{\rho^2}\right)^{1/p}\rho}(t) = \tilde{\beta}_{1,\left(\frac{r}{\rho^2}\right)^{1/p}\rho}(-t) &= t^{1/p}\left(\int\limits_t^{+\infty}\frac{dx}{(1+x^2)^{\frac{mq}{2}}}\right)^{\frac{1}{q}} \\ &\leq t^{1/p}\left(\frac{1}{(1+t^2)^{\frac{mq}{2}-\frac{1}{2}-\epsilon}}\right)^{\frac{1}{q}}\left(\int\limits_t^{+\infty}\frac{dx}{(1+x^2)^{\frac{1}{2}+\epsilon}}\right)^{\frac{1}{q}} \\ &\leq \frac{C}{(1+t^2)^{\frac{m}{2}-\frac{1}{q}\left(\frac{1}{2}+\epsilon\right)-\frac{1}{2p}}},\end{aligned}$$

where $\epsilon > 0$. Thus, $\sup\limits_{t>0}\tilde{\alpha}_{1,\left(\frac{r}{\rho^2}\right)^{1/p}\rho}(t) = \sup\limits_{\tau<0}\tilde{\beta}_{1,\left(\frac{r}{\rho^2}\right)^{1/p}\rho}(\tau) < +\infty$, if $\frac{m}{2} - \frac{1}{q}\left(\frac{1}{2}+\epsilon\right) - \frac{1}{2p} \geq 0$. The last inequality is equivalent to $m \geq 1 + \frac{2\epsilon}{q}$. Hence, if $n > 2k + p - kp$, then (3.3) holds. Also, if $n \geq 2k$, then (3.2) and (3.4) hold.

Therefore, if $k \geq 1$ and $n \geq 2k$, or $0 \leq k < 1$ and $n > 2k + p(1-k)$, then the equation

$$-(1+x^2)^{\frac{k}{2}}\left((1+x^2)^{\frac{k}{2}}y'\right)' + (1+x^2)^{\frac{n}{2}}y' = f_0(x)$$

for any $f_0 \in L_p$ has a unique solution $y(x)$ and the following estimate holds

$$\left\|(1+x^2)^{\frac{n-2k}{2p}+\frac{k}{2}}y'\right\|_p + \|y\|_p \leq C\,\|f_0\|_p\,.$$

Example 3.3 *Consider the following differential equation in* L_2

$$-\rho_0(x)\left(\rho_0(x)y'\right)' + \left(4x^2+3\right)^2 y' = f(x), \tag{3.13}$$

where $x \in \mathbb{R}$, $f \in L_2$, *and*

$$\rho_0(x) = \begin{cases} \frac{2x^2-x+1}{(1+x^2)(1-x)}, & -\infty < x < 0, \\ 1+x^2, & 0 \leq x < +\infty. \end{cases}$$

The function $\rho_0(x)$ is twice continuously differentiable, since $\lim\limits_{x\to 0-}\rho_0(x) = \lim\limits_{x\to 0+}\rho_0(x) = \rho_0(0) = 1$, $\lim\limits_{x\to 0-}\rho_0'(x) = \lim\limits_{x\to 0+}\rho_0'(x) = \rho_0'(0) = 0$, and $\lim\limits_{x\to 0-}\rho_0''(x) = \lim\limits_{x\to 0+}\rho_0''(x) = \rho_0''(0) = 2$.

If $r_0(x) := (4x^2+3)^2$, then in the case $p = 2$, $\rho_0(x)$ and $r_0(x)$ satisfy (3.2), (3.3), and (3.4). Indeed, we have that

$$\frac{r_0(x)}{\rho_0^2(x)} = \begin{cases} \left(\frac{(4x^2+3)(1+x^2)(1-x)}{2x^2-x+1}\right)^2, & -\infty < x < 0, \\ \left(\frac{4x^2+3}{1+x^2}\right)^2, & 0 \leq x < +\infty. \end{cases}$$

Since $\left(\frac{r_0(x)}{\rho_0^2(x)}\right)' < 0$ for $x < 0$ and $\left(\frac{r_0(x)}{\rho_0^2(x)}\right)' > 0$ for $x > 0$, we obtain that $\frac{r_0(x)}{\rho_0^2(x)} \geq \frac{r_0(0)}{\rho_0^2(0)}(0) = 9$ for any $x \in \mathbb{R}$. On the other hand, since $p = 2$ and $r_0(x) \geq 3$, we deduce that (3.2) holds.

Notice that $\left(\frac{r_0(x)}{\rho_0^2(x)}\right)^{1/2}\rho_0(x) = 4x^2+3$ is even function. From

$$\beta_{1,4x^2+3}(t) \leq \sup_{t>0}\left(\frac{t}{1+t^2}\right)^{\frac{1}{2}}\left(\int\limits_t^{+\infty}\frac{dx}{x^2+1}\right)^{\frac{1}{2}} \leq \frac{1}{\sqrt{2}}\sqrt{\frac{\pi}{2}} < +\infty,$$

we obtain that (3.3) holds. For any $\xi < 0$, one easily checks that

$$\begin{aligned} &\sup_{x<\xi}\frac{2x^2-x+1}{(1+x^2)(1-x)}\exp\left(-\int\limits_x^{\xi}\left(\frac{(4t^2+3)(1+t^2)(1-t)}{2t^2-t+1}\right)^2 dt\right) \\ &\leq \frac{3}{2}\sup_{x<\xi}\exp\left(-\int\limits_x^{\xi}dt\right) \\ &= \frac{3}{2}\sup_{x<\xi}e^{-\xi+x} = \frac{3}{2}, \end{aligned}$$

i.e., (3.4) holds.

Therefore, the equation (3.13) is uniquely solvable and for its solution $y(x)$, $\left\|(4x^2+3)^2y'\right\|_2 + \|y\|_2 \leq C\,\|f\|_2$ holds.

Theorem 3.4 *Let* $1 < p < +\infty$, $\rho(x) > 0$ *be twice continuously differentiable function, and* $r(x) \geq 1$ *be continuously differentiable function. Suppose that* (3.2), (3.3), *and* (3.4) *hold. If* $\theta r^{\frac{1}{p-2}}(x) \leq \rho(x) < +\infty$ *for* $p > 2$ *and* $\theta > 0$, *and there exist* $C > 1$ *and* $C_1 > 1$ *such that*

$$C^{-1} \leq \frac{\rho(x)}{\rho(\nu)} \leq C, \quad C_1^{-1} \leq \frac{r(x)}{r(\nu)} \leq C_1, \; x, \nu \in \mathbb{R} : |x-\nu| \leq 1, \tag{3.14}$$

then for the solution $y(x)$ *of the equation* (3.1) *the following estimate holds*

$$\|-\rho(\rho y')'\|_p + \|ry'\|_p \leq C_0 \|f\|_p . \tag{3.15}$$

Proof Let $\lambda \geq 0$. First we consider the differential operator $l_{\lambda,0}y = -\rho(x)\left(\rho(x)y'\right)' + [r(x)+\lambda]\, y'$ with $D(l_{\lambda,0}) = C_0^{(2)}(\mathbb{R})$. Let $z = y'$ and $l_{\lambda,0}y = L_{\lambda,0}z = -\rho(x)\left(\rho(x)z\right)' + [r(x)+\lambda]\, z$. We denote the closure in L_p of the operator $L_{\lambda,0}$ by L_λ. By Theorem 3.1 L_λ is invertible, and its inverse L_λ^{-1} is defined on whole L_p. The following inequality holds for $z \in D(L_\lambda)$ (see (3.7)):

$$\left\| \left(\frac{r+\lambda}{\rho^2}\right)^{\frac{1}{p}} \rho z \right\|_p \leq \left\| \left(\frac{r+\lambda}{\rho^2}\right)^{-\frac{1}{q}} \frac{1}{\rho} L_\lambda z \right\|_p , \tag{3.16}$$

and by (3.8)

$$\|z\|_p \leq C_1 \|L_\lambda z\|_p . \tag{3.17}$$

Let $\Delta_j := (j-1, j+1)$ $(j \in \mathbb{Z})$, and $\{\varphi_j(x)\}_{j=-\infty}^{+\infty}$ be a sequence in $C_0^\infty(\Delta_j)$ such that

$$0 \leq \varphi_j(x) \leq 1 \; (j \in \mathbb{Z}), \qquad \sum_{j=-\infty}^{+\infty} \varphi_j^2(x) = 1. \tag{3.18}$$

We denote $z_j(x) = \varphi_j(x)z(x)$ and $\|\cdot\|_{p,\Delta_j} = \|\cdot\|_{L_p(\Delta_j)}$, where $j \in \mathbb{Z}$. By (3.16) and (3.14), we have that

$$\|L_\lambda z_j\|_{p,\Delta_j} \geq \frac{\inf\limits_{x\in\Delta_j} \left[\left(\frac{r(x)+\lambda}{\rho^2(x)}\right)^{\frac{1}{p}} \rho(x) \right]}{\sup\limits_{x\in\Delta_j} \left[\left(\frac{r(x)+\lambda}{\rho^2(x)}\right)^{-\frac{1}{q}} \frac{1}{\rho(x)} \right]} \|z_j\|_{p,\Delta_j}$$

$$\geq C_2 \sup_{x\in\Delta_j} \frac{\left(\frac{r(x)+\lambda}{\rho^2(x)}\right)^{\frac{1}{p}} \rho(x)}{\left(\frac{r(x)+\lambda}{\rho^2(x)}\right)^{-\frac{1}{q}} \frac{1}{\rho(x)}} \|z_j\|_{p,\Delta_j} = C_2 \sup_{x\in\Delta_j} (r_j(x)+\lambda) \|z_j\|_p . \tag{3.19}$$

For any $f \in L_p$, we define

$$B_\lambda f := -\sum_{j=-\infty}^{+\infty} \rho^2(x)\varphi_j'(x) L_\lambda^{-1} \varphi_j f, \qquad M_\lambda f := \sum_{j=-\infty}^{+\infty} \varphi_j(x) L_\lambda^{-1} \varphi_j f.$$

It is clear that for any $x \in \mathbb{R}$, the above two series have at most two nonzero terms. Hence, B_λ and M_λ are well-defined.

We consider the operator $L_\lambda M_\lambda$. Using the equality $L_\lambda(gz) = g(x)L_\lambda z - \rho^2(x)g'(x)z$ and the properties (3.18) of $\varphi_j(x)$ $(j \in \mathbb{Z})$, we get

$$L_\lambda M_\lambda f = \sum_{j=-\infty}^{+\infty} L_\lambda(\varphi_j L_\lambda^{-1}\varphi_j f) = \sum_{j=-\infty}^{+\infty} \left(\varphi_j(x)L_\lambda L_\lambda^{-1}\varphi_j f - \rho^2(x)\varphi_j'(x)L_\lambda^{-1}\varphi_j f\right) = (E + B_\lambda)f,$$

where E is the identity map on L_p. So,

$$L_\lambda M_\lambda = E + B_\lambda. \tag{3.20}$$

On the other hand,

$$\begin{aligned}
\|B_\lambda f\|_p^p &= \int_{-\infty}^{+\infty} \left| \sum_{j=-\infty}^{+\infty} \rho^2(x)\varphi_j'(x)L_\lambda^{-1}\varphi_j f \right|^p dx \\
&\le \sum_{j=-\infty}^{+\infty} \int_{\Delta_j} \left|\rho^2(x)\varphi_j'(x)L_\lambda^{-1}\varphi_j f\right|^p dx \\
&\le \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left| \sum_{j=-\infty}^{+\infty} \rho^2(x)\varphi_j'(x)L_\lambda^{-1}\varphi_j f \right|^p dx \\
&= \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \big|\rho^2(x)\varphi_{k-1}'(x)L_\lambda^{-1}\varphi_{k-1} f \\
&\qquad + \rho^2(x)\varphi_k'(x)L_\lambda^{-1}\varphi_k f + \rho^2(x)\varphi_{k+1}'(x)L_\lambda^{-1}\varphi_{k+1} f\big|^p dx \\
&\le C_3 \sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left|\rho^2(x)\varphi_k'(x)L_\lambda^{-1}\varphi_k f\right|^p dx.
\end{aligned}$$

Using (3.19), we obtain that

$$\left\|\rho^2\varphi_k' L_\lambda^{-1}\varphi_k f\right\|_{p,\Delta_k} \le \frac{C_4 \sup\limits_{x\in\Delta_k} \rho^2(x)}{\inf\limits_{x\in\Delta_k} (r(x)+\lambda)} \|\varphi_k f\|_{p,\Delta_k} \le \frac{C_5}{1+\lambda}\|\varphi_k f\|_{p,\Delta_k}. \tag{3.21}$$

By properties of the function $\varphi_k(x)$ $(k \in \mathbb{Z})$, it follows that

$$\sum_{k=-\infty}^{+\infty} \int_{\Delta_k} \left|\rho^2(x)\varphi_k'(x)L_\lambda^{-1}\varphi_k f\right|^p dx \le \frac{C_6}{1+\lambda} \|f\|_p^p.$$

Thus, $\|B_\lambda\| \to 0$ as $\lambda \to \infty$. Therefore, there exists $\lambda_0 > 0$ such that $\|B_\lambda\| \le \frac{1}{2}$ for $\lambda \ge \lambda_0$. Hence, from (3.20) it follows that

$$L_\lambda^{-1} = M_\lambda(E + B_\lambda)^{-1}, \ \left\|(E + B_\lambda)^{-1}\right\| \le 2, \ \lambda \ge \lambda_0. \tag{3.22}$$

Now, we prove the estimate (3.15). According to (3.19) and (3.22), we get

$$\begin{aligned}\|(r+\lambda)z\|_p^p &= \left\|(r+\lambda)L_\lambda^{-1}f\right\|_p^p\\ &\le 2\left\|(r+\lambda)M_\lambda f\right\|_p^p\\ &\le C_7\sum_{j=-\infty}^{+\infty}\left\|(r+\lambda)L_\lambda^{-1}\varphi_j f\right\|_{p,\Delta_j}^p\\ &\le C_7\sum_{j=-\infty}^{+\infty}\sup_{x\in\Delta_j}(r(x)+\lambda)^p\left\|L_\lambda^{-1}\varphi_j f\right\|_{p,\Delta_j}^p\\ &\le C_7\sum_{j=-\infty}^{+\infty}\sup_{x\in\Delta_j}(r(x)+\lambda)^p\frac{1}{\inf\limits_{x\in\Delta_j}(r(x)+\lambda)^p}\left\|\varphi_j f\right\|_{p,\Delta_j}^p\\ &\le C_8\sum_{j=-\infty}^{+\infty}\left\|\varphi_j f\right\|_{p,\Delta_j}^p\\ &\le C_9\|f\|_p^p, \qquad \forall z\in D(L_\lambda).\end{aligned}$$

Therefore,

$$\left\|-\rho(\rho z)'\right\|_p+\|(r+\lambda)z\|_p\le C_{10}\|f\|_p.$$

Taking $z(x)=y'(x)$, by estimate (3.17), we obtain (3.15). □

For the equation (1.1), we have the following result.

Theorem 3.5 *Let $1<p<\infty$, and functions $\rho(x)$ and $r(x)$ satisfy the conditions of Theorem 3.4. If $s(x)$ is a continuous function such that $\gamma_{s,r}<+\infty$, then for any $F\in L_p$, the equation (1.1) has a unique solution $y(x)$, which satisfies the following inequality:*

$$\left\|-\rho\left(\rho y'\right)'\right\|_p+\|ry'\|_p+\|sy\|_p\le C\|F\|_p, \tag{3.23}$$

where C depends only on $\gamma_{s,r}$ and p.

Proof

For $a>0$, we denote

$$\tilde{y}(t)=y(at),\quad \tilde{\rho}(t)=\rho(at),\quad \tilde{r}(t)=r(at),\quad \tilde{s}(t)=s(at),\quad \tilde{f}(t)=a^{-1}F(at).$$

The change of variable $x\mapsto at$ changes the equation (1.1) to the following:

$$-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y}')'+\tilde{r}(t)\tilde{y}'+a^{-1}\tilde{s}(t)\tilde{y}=\tilde{f}(t). \tag{3.24}$$

We denote the closure in L_p of the differential operator $-\tilde{\rho}(t)(\tilde{\rho}(t)\tilde{y}')'+\tilde{r}(t)\tilde{y}$ defined on $\tilde{y}\in C_0^2(\mathbb{R})$ by l_a. It is easy to show that $\tilde{\rho}(t)$ and $\tilde{r}(t)$ satisfy the conditions of Theorem 3.4. Therefore,

$$\|-\tilde{\rho}(\tilde{\rho}\tilde{y}')'\|_p+\|\tilde{r}\tilde{y}'\|_p\le C_{l_a}\|l_a\tilde{y}\|_p,\quad \tilde{y}\in D(l_a). \tag{3.25}$$

According to Lemma 2.1,

$$\left\|a^{-1}\tilde{s}\tilde{y}\right\|_p\le a^{-1}p^{\frac{1}{p}}q^{\frac{1}{q}}\gamma_{s,r}\|\tilde{r}\tilde{y}'\|_p\le a^{-1}p^{\frac{1}{p}}q^{\frac{1}{q}}\gamma_{s,r}C_{l_a}\|l_a\tilde{y}\|_p. \tag{3.26}$$

Set $a = \frac{1}{2}p^{\frac{1}{p}}q^{\frac{1}{q}}\gamma_{s,r}C_{l_a}$. By [9, Ch. 4, Theorem 1.16], we get that the operator $l_a + a^{-1}\tilde{s}(x)E$ is bounded invertible. Using (3.24) and (3.25), we obtain that

$$\|-\tilde{\rho}(\tilde{\rho}\tilde{y}')'\|_p + \|\tilde{r}\tilde{y}'\|_p + \left\|a^{-1}\tilde{s}\tilde{y}\right\|_p \le \left(C_{l_a} + \frac{1}{2}\right)\|l_a\tilde{y}\|_p .$$

It follows from (3.26) that

$$\|l_a\tilde{y}\|_p \le \left\|\left(l_a + a^{-1}\tilde{s}E\right)\tilde{y}\right\|_p + \left\|a^{-1}\tilde{s}\tilde{y}\right\|_p \le \left\|\left(l_a + a^{-1}\tilde{s}E\right)\tilde{y}\right\|_p + \frac{1}{2}\|l_a\tilde{y}\|_p .$$

The above two inequalities imply that

$$\|-\tilde{\rho}(\tilde{\rho}\tilde{y}')'\|_p + \|\tilde{r}\tilde{y}'\|_p + \left\|a^{-1}\tilde{s}\tilde{y}\right\|_p \le C\left\|\tilde{f}\right\|_p .$$

Making the change of variable $t \mapsto a^{-1}x$, we get the desired estimate (3.23). □

Example 3.6 *We consider the following equation in* L_2

$$-\frac{x^2+5}{(x^2+3)^2}\left(\frac{x^2+5}{(x^2+3)^2}y'\right)' + (x^2+3)^2y' - 5xy = f(x). \tag{3.27}$$

Let $r_1(x) = (x^2+3)^2$, $\rho_1(x) = \frac{x^2+5}{(x^2+3)^2}$ and $s_1(x) = -5x$.

1) $\frac{r_1(x)}{\rho_1^2(x)} = \frac{(x^2+3)^6}{(x^2+5)^2} \ge 1$, $r_1(x) \ge 1$ and $p = 2$. Hence, (3.2) holds.

2) Since $r^{\frac{1}{2}}(x) = x^2+3$ is even function, and

$$\sup_{\tau<0}\beta_{1,r_1}(\tau) = \sup_{t>0}\alpha_{1,r_1}(t) \le \sup_{t>0}\left(\frac{t}{3+t^2}\right)^{\frac{1}{2}}\left(\int_t^{+\infty}\frac{dx}{x^2+3}\right)^{\frac{1}{2}} \le \frac{\sqrt{\pi}}{2} < +\infty,$$

(3.3) holds.

3) We have that

$$\sup_{x<\xi}\rho_1(x)\exp\left(-\int_x^{\xi}\frac{r_1(x)}{\rho_1^2(x)}dt\right) = \sup_{x<\xi}\frac{x^2+5}{x^2+3}\exp\left(-\int_x^{\xi}\frac{(t^2+3)^6}{(t^2+5)^2}dt\right) \le 2\sup_{x<\xi}\exp\left(-\int_x^{\xi}dt\right) = 2.$$

Therefore, (3.4) holds.

4) If $|x-\nu| < 1$, then

$$\frac{r_1(x)}{r_1(\nu)} = \left[\frac{x^2+3}{\nu^2+3}\right]^2 \le \sup\left(\frac{(\nu+1)^2+3}{\nu^2+3}\right)^2 \le 3.$$

Consequently, $\frac{r_1(\nu)}{r_1(x)} \ge \frac{1}{2}$. Furthermore, since

$$\frac{\rho_1(x)}{\rho_1(\nu)} = \frac{\left[1+\frac{2}{x^2+3}\right]\frac{2}{x^2+3}}{\left[1+\frac{2}{\nu^2+3}\right]\frac{2}{\nu^2+3}},$$

we have that

$$\frac{\nu^2+3}{2(x^2+3)} \le \frac{\rho_1(x)}{\rho_1(\nu)} \le \frac{2(\nu^2+3)}{x^2+3}.$$

It follows that

$$\frac{1}{6} \le \frac{\rho_1(x)}{\rho_1(\nu)} \le 6.$$

Thus, (3.14) holds.

5) It is clear that $\rho_1(x)$ is a bounded function. We calculate $\gamma_{s_1,r}$. Notice that $\|-5x\|_{L_2(0,t)} = \|-5x\|_{L_2(-t,0)}$ for $t>0$, and $(x^2+3)^2$ is even function. Hence,

$$\begin{aligned}\sup_{\tau<0}\tilde{\beta}_{s_1,r_1}(\tau) &= \sup_{t>0}\tilde{\alpha}_{s_1,r_1}(t) = \sup_{t>0}\left(\int_0^t(-5x)^2dx\right)^{\frac{1}{2}}\left(\int_t^{+\infty}\frac{dx}{(x^2+3)^4}\right)^{\frac{1}{2}} \\ &\le 5\sup_{t>0} t^{\frac{3}{2}}\left(\int_t^{+\infty}\frac{dx}{(x^2+3)^4}\right)^{\frac{1}{2}} \\ &\le 5\sup_{t>0}\frac{t^{\frac{3}{2}}}{(1+t^2)^{\frac{3}{2}}}\left(\int_t^{+\infty}\frac{dx}{1+x^2}\right)^{\frac{1}{2}} \le 5\sqrt{\frac{\pi}{2}},\end{aligned}$$

i.e., $\gamma_{s_1,r_1} \le 5\sqrt{\frac{\pi}{2}}$.

Thus, the coefficients of equation (3.27) satisfy the conditions of Theorem 3.5. Hence, for each $f \in L_2$ there exists a unique solution y of equation (3.27), which satisfies the following:

$$\left\|-\frac{x^2+5}{x^2+3}^2\left(\frac{(x^2+5}{(x^2+3)^2}y'\right)'\right\|_2 + \left\|(x^2+3)^2y'\right\|_2 + \|5xy\|_2 \le C\|f\|_2.$$

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