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Oscillatory and asymptotic behavior of third-order nonlinear differential equations with a superlinear neutral term

Said R. GRACE[1], **Irena JADLOVSKÁ**[2,*], **Ercan TUNÇ**[3]
[1]Department of Engineering Mathematics, Faculty of Engineering, Cairo University, Giza, Egypt
[2]Department of Mathematics and Theoretical Informatics,
Faculty of Electrical Engineering and Informatics, Technical University of Košice,
Košice, Slovakia
[3]Department of Mathematics, Faculty of Arts and Sciences, Gaziosmanpaşa University, Tokat, Turkey

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Abstract: Sufficient conditions are derived for all solutions of a class of third-order nonlinear differential equations with a superlinear neutral term to be either oscillatory or convergent to zero asymptotically. Examples illustrating the results are included and some suggestions for further research are indicated.

Key words: Oscillation, third-order, asymptotic behavior, neutral differential equation

1. Introduction

In this paper, we study the oscillatory and asymptotic behavior of the solutions of the third-order nonlinear differential equation with a superlinear neutral term

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + q(t)x^{\delta}(\sigma(t)) = 0, \quad t \geq t_0 > 0, \tag{1.1}$$

where $z(t) = x(t) + p(t)x^{\beta}(\tau(t))$. In the sequel, we assume that:

(C_1) α, β, and δ are the ratios of odd positive integers with $\beta \geq 1$;

(C_2) $r, p, q : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions with $r(t) > 0$, $p(t) \geq 1$, $p(t) \not\equiv 1$ for large t, $q(t) \geq 0$, and $q(t)$ is not identically zero for large t;

(C_3) $\tau, \sigma : [t_0, \infty) \to \mathbb{R}$ are real-valued continuous functions such that $\tau(t) \leq t$, $\sigma(t) \leq t$, τ is strictly increasing, and $\lim_{t\to\infty} \tau(t) = \lim_{t\to\infty} \sigma(t) = \infty$;

(C_4) $h(t) := \tau^{-1}(\sigma(t)) \leq t$ and $\lim_{t\to\infty} h(t) = \infty$, where τ^{-1} is the inverse function of τ.

We let

$$I_1(v, u) = \int_u^v r^{-1/\alpha}(s)ds, \quad v \geq u \geq t_0,$$

*Correspondence: irena.jadlovska@tuke.sk
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and assume that

$$I_1(t,t_0) \to \infty \quad \text{as } t \to \infty. \tag{1.2}$$

By a solution of equation (1.1), we mean a function $x \in C([t_x,\infty),\mathbb{R})$ for some $t_x \geq t_0$ with $z \in C^2([t_x,\infty),\mathbb{R})$, $r(z'')^\alpha \in C^1([t_x,\infty),\mathbb{R})$, and which satisfies (1.1) on $[t_x,\infty)$. We only consider those solutions of (1.1) that exist on some half-line $[t_x,\infty)$ and satisfy the condition

$$\sup\{|x(t)| : T_1 \leq t < \infty\} > 0 \;\text{ for any }\; T_1 \geq t_x;$$

moreover, we tacitly assume that (1.1) possesses such solutions. Such a solution $x(t)$ of (1.1) is said to be *oscillatory* if it has arbitrarily large zeros on $[t_x,\infty)$, i.e. for any $t_1 \in [t_x,\infty)$ there exists a $t_2 \geq t_1$ such that $x(t_2)=0$; otherwise, it is called nonoscillatory, i.e. if it is eventually positive or eventually negative. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

The qualitative analysis of neutral differential equations, i.e. equations in which the highest-order derivative of the unknown function appears both with and without deviating arguments, is not only of theoretical interest but also has significant practical importance. This is due to the fact that such equations find numerous applications in natural sciences and technology. For instance, the equations of this type appear in the study of electric networks containing lossless transmission lines (as in high-speed computers where such lines are used to interconnect switching circuits), in the study of vibrating masses attached to an elastic bar, and in the solution of variational problems with time delays; see [14] for additional applications.

The problem of establishing sufficient conditions for the oscillatory and asymptotic behavior of solutions of third-order neutral differential and dynamic equations has been the subject of intensive investigations during the past decades. We refer to the papers [2–13, 15, 18–21, 23–27, 29] as well as the references cited therein as examples of recent results on this topic. Most of the literature, however, is focused on equations with linear neutral term (i.e. $\beta=1$), and very few results are available for equations with nonlinear neutral term (i.e. $\beta \neq 1$); see [10] for a sublinear neutral term (i.e. $\beta<1$), and see [28] for a superlinear neutral term (i.e. $\beta>1$). To the best of our knowledge, there are no papers at the present time dealing with third-order differential equations with superlinear neutral term except [28], where equation (1.1) was considered in the case when $r(t)=1$ and $\alpha=1$. Motivated by these observations, our aim in this paper is to obtain sufficient conditions under which every solution of equation (1.1) either oscillates or converges to zero as $t\to\infty$. New oscillation criteria are established via a comparison with first-order delay differential equations whose oscillatory characters are known as well as by using an integral criterion. We wish to point out that the results of this paper can be applied to the case where $p(t)\to\infty$ as $t\to\infty$ for $\beta>1$, and to the cases where $p(t)$ is a bounded function and/or $p(t)\to\infty$ as $t\to\infty$ for $\beta=1$.

2. Main results

We begin with the following lemmas that are essential in the proofs of our theorems. For simplicity in what follows, it will be convenient to set:

$$I_2(t,t_{**}) = \int_{t_{**}}^{t} I_1(s,t_*)ds \quad \text{for } t \geq t_{**} \geq t_*, \quad \text{where } t_* \in [t_0,\infty),$$

and throughout this paper, we assume that, for every positive constants k and l,

$$P_1(t) := \frac{1}{p(\tau^{-1}(t))}\left[1 - \left(\frac{I_2(\tau^{-1}(\tau^{-1}(t)), t_{**})}{I_2(\tau^{-1}(t), t_{**})}\right)^{1/\beta} \frac{k^{\frac{1}{\beta}-1}}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right] \geq 0 \tag{2.1}$$

and

$$P_2(t) := \frac{1}{p(\tau^{-1}(t))}\left(1 - \frac{l^{\frac{1}{\beta}-1}}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right) \geq 0 \tag{2.2}$$

for all sufficiently large t.

Remark 2.1 *It is useful to note that since*

$$P(t, t_{**}) := \frac{I_2(\tau^{-1}(t), t_{**})}{I_2(t, t_{**})} \frac{1}{p(\tau^{-1}(t))} \geq \frac{1}{p(\tau^{-1}(t))},$$

then the condition

$$\lim_{t\to\infty} P(t, t_{**}) = 0 \quad \text{for } \beta > 1$$
$$P(t, t_{**}) < 1 \quad \text{for } \beta = 1$$

ensures the positivity of the functions P_1 and P_2.

Lemma 2.2 *Let conditions (C_1)–(C_3) and (1.2) hold and assume that x is an eventually positive solution of equation (1.1). Then there exists a $t_1 \in [t_0, \infty)$ such that the corresponding function z satisfies one of the following two cases:*

(I) $z(t) > 0$, $z'(t) > 0$, $z''(t) > 0$, *and* $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \leq 0$,

(II) $z(t) > 0$, $z'(t) < 0$, $z''(t) > 0$, *and* $\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \leq 0$,

for $t \geq t_1$.

Proof The proof is straightforward; hence, we omit the details. □

Lemma 2.3 *Let conditions (C_1)–(C_4) and (1.2) hold and assume that x is an eventually positive solution of equation (1.1) with $z(t)$ satisfying case (I) of Lemma 2.2. Then $z(t)$ satisfies the inequality*

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + q(t)P_1^{\delta/\beta}(\sigma(t))z^{\delta/\beta}(h(t)) \leq 0 \tag{2.3}$$

for large t.

Proof Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. It follows from the definition of z that

$$x^{\beta}(\tau(t)) = \frac{1}{p(t)}(z(t) - x(t)) \leq \frac{z(t)}{p(t)},$$

from which and the fact that $\tau(t) \le t$ is strictly increasing, it is easy to see that

$$x(\tau^{-1}(t)) \le \frac{z^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}.$$

Using this in the definition of z, we obtain

$$\begin{aligned} x^{\beta}(t) &= \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - x(\tau^{-1}(t))\right] \\ &\ge \frac{1}{p(\tau^{-1}(t))}\left[z(\tau^{-1}(t)) - \frac{z^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right]. \end{aligned} \tag{2.4}$$

Since $r(t)\left(z''(t)\right)^{\alpha}$ is nonincreasing on $[t_1, \infty)$, we see that

$$z'(t) = z'(t_1) + \int_{t_1}^{t} \frac{\left(r(s)\left(z''(s)\right)^{\alpha}\right)^{1/\alpha}}{r^{1/\alpha}(s)} ds \ge \left(r(t)\left(z''(t)\right)^{\alpha}\right)^{1/\alpha} I_1(t, t_1). \tag{2.5}$$

From (2.5), we have for all $t \ge t_2 := t_1 + 1$ that

$$\left(\frac{z'(t)}{I_1(t,t_1)}\right)' = \frac{r^{-1/\alpha}(t)[r^{1/\alpha}(t)z''(t)I_1(t,t_1) - z'(t)]}{\left(I_1(t,t_1)\right)^2} \le 0,$$

i.e. $z'(t)/I_1(t,t_1)$ is nonincreasing for $t \ge t_2$. Using the fact that $z'(t)/I_1(t,t_1)$ is nonincreasing for $t \ge t_2$, we obtain

$$z(t) = z(t_2) + \int_{t_2}^{t} \frac{z'(s)}{I_1(s,t_1)} I_1(s,t_1) ds \ge \frac{z'(t)}{I_1(t,t_1)} \int_{t_2}^{t} I_1(s,t_1) ds = \frac{I_2(t,t_2)}{I_1(t,t_1)} z'(t) \quad \text{for } t \ge t_2;$$

thus, we have for all $t \ge t_3 := t_2 + 1$ that

$$\left(\frac{z(t)}{I_2(t,t_2)}\right)' = \frac{z'(t)I_2(t,t_2) - z(t)I_1(t,t_1)}{\left(I_2(t,t_2)\right)^2} \le 0,$$

i.e. $z(t)/I_2(t,t_2)$ is nonincreasing for $t \ge t_3$. Now, since $\tau(t) \le t$ and τ is strictly increasing, we see that τ^{-1} is increasing and $t \le \tau^{-1}(t)$. Thus,

$$\tau^{-1}(t) \le \tau^{-1}(\tau^{-1}(t)). \tag{2.6}$$

Since $z(t)/I_2(t,t_2)$ is nonincreasing, it follows from (2.6) that

$$\frac{I_2(\tau^{-1}(\tau^{-1}(t)), t_2) z(\tau^{-1}(t))}{I_2(\tau^{-1}(t), t_2)} \ge z(\tau^{-1}(\tau^{-1}(t))).$$

Using this in (2.4) yields

$$x^{\beta}(t) \ge \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1 - \left(\frac{I_2(\tau^{-1}(\tau^{-1}(t)), t_2)}{I_2(\tau^{-1}(t), t_2)}\right)^{1/\beta} \frac{z^{\frac{1}{\beta}-1}(\tau^{-1}(t))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right] \tag{2.7}$$

for $t \geq t_3$. Since $z(t)$ is positive and increasing for $t \geq t_3$, there exist a $t_4 \in [t_3, \infty)$ and a constant $c > 0$ such that

$$z(t) \geq c \quad \text{for } t \geq t_4. \tag{2.8}$$

From (2.7) and (2.8) we observe that

$$x^{\beta}(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1 - \left(\frac{I_2(\tau^{-1}(\tau^{-1}(t)), t_2)}{I_2(\tau^{-1}(t), t_2)}\right)^{1/\beta} \frac{c^{\frac{1}{\beta}-1}}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right] = P_1(t)z(\tau^{-1}(t))$$

for $t \geq t_4$, and so

$$x^{\beta}(\sigma(t)) \geq P_1(\sigma(t))z(\tau^{-1}(\sigma(t))) \quad \text{for } t \geq t_5,$$

where $\sigma(t) \geq t_4$ for $t \geq t_5$ for some $t_5 \geq t_4$. Using this in (1.1) gives

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \leq -q(t)P_1^{\delta/\beta}(\sigma(t))z^{\delta/\beta}(h(t)) \quad \text{for } t \geq t_5, \tag{2.9}$$

i.e. inequality (2.3) holds. This completes the proof of Lemma 2.3. □

Lemma 2.4 *Let conditions (C_1)–(C_4) and (1.2) hold and assume that x is an eventually positive solution of equation (1.1) with $z(t)$ satisfying case (II) of Lemma 2.2. Then $z(t)$ either satisfies the inequality*

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' + q(t)P_2^{\delta/\beta}(\sigma(t))z^{\delta/\beta}(h(t)) \leq 0 \tag{2.10}$$

for large t or $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$.

Proof Let $x(t)$ be an eventually positive solution of (1.1) such that $x(t) > 0$, $x(\tau(t)) > 0$ and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$. Proceeding as in the proof of Lemma 2.3, we again see that (2.4) and (2.6) hold. Since $z'(t) < 0$, it follows from (2.6) that

$$z\left(\tau^{-1}(t)\right) \geq z\left(\tau^{-1}(\tau^{-1}(t))\right).$$

Substituting the last inequality into (2.4) yields

$$x^{\beta}(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1 - \frac{z^{\frac{1}{\beta}-1}(\tau^{-1}(t))}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right]. \tag{2.11}$$

Since $z(t)$ satisfies case (II) of Lemma 2.2, there exists a constant κ such that

$$\lim_{t\to\infty} z(t) = \kappa < \infty.$$

(i) if $\kappa > 0$, then there exists a $t_2 \geq t_1$ such that

$$z(t) \geq \kappa \quad \text{for } t \geq t_2. \tag{2.12}$$

It follows from (2.12) that

$$z^{\frac{1}{\beta}-1}(t) \leq \kappa^{\frac{1}{\beta}-1}.$$

Using this in (2.11), we obtain

$$x^{\beta}(t) \geq \frac{z(\tau^{-1}(t))}{p(\tau^{-1}(t))}\left[1 - \frac{\kappa^{\frac{1}{\beta}-1}}{p^{1/\beta}(\tau^{-1}(\tau^{-1}(t)))}\right] = P_2(t)z(\tau^{-1}(t)).$$

Using this in (1.1) gives

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \leq -q(t)P_2^{\delta/\beta}(\sigma(t))z^{\delta/\beta}(h(t)) \tag{2.13}$$

for $t \geq t_3$ for some $t_3 \geq t_2$, i.e., inequality (2.10) holds.

(ii) If $\kappa = 0$, then $\lim_{t\to\infty} z(t) = 0$. Since $0 < x(t) \leq z(t)$ on $[t_1, \infty)$, $\lim_{t\to\infty} x(t) = 0$. This completes the proof of the lemma. □

Theorem 2.5 *Let conditions* $(C_1) - (C_4)$ *and* (1.2) *hold. If, for all sufficiently large* $t_* \in [t_0, \infty)$, *and for some* $t_{**} \in (t_*, \infty)$,

$$\int_{t_{**}}^{\infty} q(s)P_1^{\delta/\beta}(\sigma(s))ds = \infty, \tag{2.14}$$

and

$$\int_{t_0}^{\infty} q(s)P_2^{\delta/\beta}(\sigma(s))ds = \infty, \tag{2.15}$$

then every solution $x(t)$ *of equation* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \geq t_1$ for some $t_1 \geq t_0$, and assume (2.1) and (2.2) hold for $t \geq t_1$. The proof if $x(t)$ is eventually negative is similar, so we omit the details of that case here as well as in the remaining proofs in this paper. Then from Lemma 2.2, $z(t)$ satisfies either case (I) or case (II) for $t \geq t_1$.

First, we consider case (I). Then from Lemma 2.3, we see that inequalities (2.8) and (2.9) hold for $t \geq t_5$. Using (2.8) in (2.9) gives

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \leq -c^{\delta/\beta}q(t)P_1^{\delta/\beta}(\sigma(t)) \quad \text{for } t \geq t_5. \tag{2.16}$$

An integration of (2.16) from t_5 to t yields

$$r(t)\left(z''(t)\right)^{\alpha} \leq r(t_5)\left(z''(t_5)\right)^{\alpha} - c^{\delta/\beta}\int_{t_5}^{t} q(s)P_1^{\delta/\beta}(\sigma(s))ds \to -\infty \quad \text{as } t \to \infty,$$

which contradicts the fact that $r(t)\left(z''(t)\right)^{\alpha}$ is positive.

Next, we consider case (II). Then from Lemma 2.4, we again have case (i) or case (ii). In case (i), we see that (2.12) and (2.13) hold for $t \geq t_3$. Using (2.12) in (2.13) yields

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)' \leq -\kappa^{\delta/\beta}q(t)P_2^{\delta/\beta}(\sigma(t)) \quad \text{for } t \geq t_3. \tag{2.17}$$

An integration of (2.17) from t_3 to t yields

$$r(t)\left(z''(t)\right)^{\alpha} \leq r(t_3)\left(z''(t_3)\right)^{\alpha} - \kappa^{\delta/\beta}\int_{t_3}^{t} q(s)P_2^{\delta/\beta}(\sigma(s))ds \to -\infty \quad \text{as } t \to \infty,$$

which again contradicts the fact that $r(t)\left(z''(t)\right)^{\alpha}$ is positive.

In case (ii), as in Lemma 2.4, we see that $x(t)\to 0$ as $t\to\infty$. This completes the proof. □

Next, we establish a new oscillation criterion for equation (1.1) via a comparison with first-order delay differential equations whose oscillatory characters are known.

Theorem 2.6 *Let conditions* $(C_1)-(C_4)$ *and* (1.2) *be satisfied. Suppose that there exist continuous functions* $\eta,\xi:[t_0,\infty)\to\mathbb{R}$ *such that* $h(t)\le\eta(t)\le\xi(t)\le t$ *for* $t\ge t_0$. *If the first-order delay differential equations*

$$w'(t)+q(t)P_1^{\delta/\beta}(\sigma(t))I_2^{\delta/\beta}(h(t),t_0)w^{\delta/\alpha\beta}(h(t))=0 \tag{2.18}$$

and

$$y'(t)+q(t)P_2^{\delta/\beta}(\sigma(t))\left[(\eta(t)-h(t))\,I_1(\xi(t),\eta(t))\right]^{\delta/\beta}y^{\delta/\alpha\beta}(\xi(t))=0 \tag{2.19}$$

are oscillatory, then every solution $x(t)$ *of equation* (1.1) *is either oscillatory or satisfies* $\lim_{t\to\infty}x(t)=0$.

Proof Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t)>0$, $x(\tau(t))>0$, and $x(\sigma(t))>0$ for $t\ge t_1$ for some $t_1\ge t_0$, and assume (2.1) and (2.2) hold for $t\ge t_1$. Then from Lemma 2.2, $z(t)$ satisfies either case (I) or case (II) for $t\ge t_1$.

First, we consider case (I). Proceeding as in the proof of Lemma 2.3, we again arrive at (2.5) for $t\ge t_1$ and (2.9) for $t\ge t_5$. An integration of (2.5) from t_1 to t gives

$$z(t)\ge\left(\int_{t_1}^{t}I_1(s,t_1)ds\right)\left(r(t)\left(z''(t)\right)^{\alpha}\right)^{1/\alpha}=I_2(t,t_1)\left(r(t)\left(z''(t)\right)^{\alpha}\right)^{1/\alpha},$$

and so

$$z(h(t))\ge I_2(h(t),t_1)\left(r(h(t))\left(z''(h(t))\right)^{\alpha}\right)^{1/\alpha}\quad\text{for } t\ge t_2,$$

where $h(t)\ge t_1$ for $t\ge t_2$ for some $t_2\ge t_1$. Using this in (2.9) and taking $\lim_{t\to\infty}h(t)=\infty$ into account, we see that

$$\left(r(t)\left(z''(t)\right)^{\alpha}\right)'+q(t)P_1^{\delta/\beta}(\sigma(t))I_2^{\delta/\beta}(h(t),t_1)\left(r(h(t))\left(z''(h(t))\right)^{\alpha}\right)^{\delta/\alpha\beta}\le 0 \tag{2.20}$$

for $t\ge t_5$. Letting $w(t)=r(t)\left(z''(t)\right)^{\alpha}$, we see that w is a positive solution of the first-order delay differential inequality

$$w'(t)+q(t)P_1^{\delta/\beta}(\sigma(t))I_2^{\delta/\beta}(h(t),t_1)w^{\delta/\alpha\beta}(h(t))\le 0. \tag{2.21}$$

The function $w(t)$ is decreasing on $[t_5,\infty)$, and so by [22, Theorem 1], there exists a positive solution of equation (2.18). This contradicts the fact that equation (2.18) is oscillatory.

Next, we consider case (II). Then from Lemma 2.4, we again have case (i) or case (ii). In case (i), we again see that (2.13) holds for $t\ge t_3$. Since case (II) holds, for $v\ge u\ge t_3$, we have

$$z(u)=z(v)+\int_u^v -z'(s)ds\ge(v-u)(-z'(v)). \tag{2.22}$$

Setting $u = h(t)$ and $v = \eta(t)$ in (2.22), we obtain

$$z(h(t)) \geq (\eta(t) - h(t))\,(-z'(\eta(t))\,. \tag{2.23}$$

Since $z'(t) < 0$, and $r(t)(z''(t))^{\alpha}$ is decreasing, we have

$$\begin{aligned} -z'(u) \geq z'(v) - z'(u) &= \int_u^v r^{-1/\alpha}(s)\left(r^{1/\alpha}(s)z''(s)\right)ds \\ &\geq I_1(v,u)\left[r(v)(z''(v))^{\alpha}\right]^{1/\alpha}; \end{aligned}$$

hence,

$$-z'(u) \geq I_1(v,u)\left[r(v)(z''(v))^{\alpha}\right]^{1/\alpha}.$$

Letting $u = \eta(t)$ and $v = \xi(t)$ in the last inequality, we have

$$-z'(\eta(t)) \geq I_1(\xi(t),\eta(t))\left[r(\xi(t))(z''(\xi(t)))^{\alpha}\right]^{1/\alpha}. \tag{2.24}$$

Combining (2.23) and (2.24) yields

$$z(h(t)) \geq (\eta(t) - h(t))\,I_1(\xi(t),\eta(t))\left[r(\xi(t))(z''(\xi(t)))^{\alpha}\right]^{1/\alpha}. \tag{2.25}$$

Now using (2.25) in (2.13) gives

$$y'(t) + q(t)P_2^{\delta/\beta}(\sigma(t))\left[(\eta(t) - h(t))\,I_1(\xi(t),\eta(t))\right]^{\delta/\beta} y^{\delta/\alpha\beta}(\xi(t)) \leq 0, \tag{2.26}$$

where $y(t) = r(t)(z''(t))^{\alpha} > 0$. As in case (I), we see that there exists a positive solution of equation (2.19), which contradicts the fact that equation (2.19) is oscillatory.

In case (ii), as in Lemma 2.4, we see that $x(t) \to 0$ as $t \to \infty$. This completes the proof. □

It is well known from [17] (see also [1, Lemma 2.2.9] that if

$$\liminf_{t\to\infty} \int_{\mu(t)}^{t} W(s)ds > \frac{1}{e}, \tag{2.27}$$

then the first-order delay differential equation

$$x'(t) + W(t)x(\mu(t)) = 0 \tag{2.28}$$

is oscillatory, where $W, \mu \in C([t_0,\infty),\mathbb{R})$ with $W(t) \geq 0$, $\mu(t) \leq t$, and $\lim_{t\to\infty}\mu(t) = \infty$.

Thus, from Theorem 2.6, we have the following oscillation result for equation (1.1) in the case when $\delta = \alpha\beta$.

Corollary 2.7 *Let $\delta = \alpha\beta$ and conditions $(C_1) - (C_4)$ and (1.2) hold. Assume that there exist continuous functions $\eta, \xi : [t_0,\infty) \to \mathbb{R}$ such that $h(t) \leq \eta(t) \leq \xi(t)) \leq t$ for $t \geq t_0$. If*

$$\liminf_{t\to\infty} \int_{h(t)}^{t} q(s)P_1^{\delta/\beta}(\sigma(s))I_2^{\delta/\beta}(h(s),t_0)ds > \frac{1}{e} \tag{2.29}$$

and

$$\liminf_{t\to\infty}\int_{\xi(t)}^{t} q(s)P_2^{\delta/\beta}(\sigma(s))\left[(\eta(s)-h(s))\,I_1(\xi(s),\eta(s))\right]^{\delta/\beta}ds > \frac{1}{e}, \tag{2.30}$$

then every solution $x(t)$ of equation (1.1) either oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof In view of (2.27) and (2.28), the proof follows from (2.18), (2.19), and Theorem 2.6; we omit the details. □

In the case when $\delta < \alpha\beta$, by Theorem 2.6, we have the following result.

Corollary 2.8 *Let $\delta < \alpha\beta$ and conditions $(C_1)-(C_4)$ and (1.2) hold. Assume that there exist continuous functions $\eta, \xi : [t_0, \infty) \to \mathbb{R}$ such that $h(t) \le \eta(t) \le \xi(t)) \le t$ for $t \ge t_0$. If, for all sufficiently large $t_* \in [t_0, \infty)$, and for some $t_{**} \in (t_*, \infty)$,*

$$\int_{t_{**}}^{\infty} q(s)P_1^{\delta/\beta}(\sigma(s))I_2^{\delta/\beta}(h(s), t_0)ds = \infty, \tag{2.31}$$

and

$$\int_{t_0}^{\infty} q(s)P_2^{\delta/\beta}(\sigma(s))\left[(\eta(s)-h(s))\,I_1(\xi(s),\eta(s))\right]^{\delta/\beta}ds = \infty, \tag{2.32}$$

then every solution $x(t)$ of equation (1.1) either oscillates or satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof Let $x(t)$ be a nonoscillatory solution of equation (1.1), say $x(t) > 0$, $x(\tau(t)) > 0$, and $x(\sigma(t)) > 0$ for $t \ge t_1$ for some $t_1 \ge t_0$, and assume (2.1) and (2.2) hold for $t \ge t_1$. Proceeding as in the proof of Theorem 2.6, we again see that $z(t)$ satisfies either case (I) or case (II) for $t \ge t_1$. In case (I), we again arrive at (2.21) for $t \ge t_5$. Using the fact that $w(t) = r(t)\,(z''(t))^{\alpha}$ is positive and decreasing, and noting that $h(t) \le t$, we have

$$w(h(t)) \ge w(t)$$

Thus, inequality (2.21) can be written as

$$w'(t) + q(t)P_1^{\delta/\beta}(\sigma(t))I_2^{\delta/\beta}(h(t), t_1)w^{\delta/\alpha\beta}(t) \le 0,$$

or

$$\frac{w'(t)}{w^{\delta/\alpha\beta}(t)} + q(t)P_1^{\delta/\beta}(\sigma(t))I_2^{\delta/\beta}(h(t), t_1) \le 0 \quad \text{for } t \ge t_5. \tag{2.33}$$

An integration of (2.33) from t_5 to ∞ gives

$$\int_{t_5}^{\infty} q(s)P_1^{\delta/\beta}(\sigma(s))I_2^{\delta/\beta}(h(s), t_1)ds \le \frac{w^{1-\frac{\delta}{\alpha\beta}}(t_5)}{1-\frac{\delta}{\alpha\beta}} < \infty,$$

which contradicts (2.31). Using the similar arguments, the remainder of proof follows from inequality (2.26), $\xi(t) \le t$ and case (ii) in Theorem 2.6; we omit the details. □

We conclude this paper with the following examples and remarks to illustrate the above results. Our first example deals with the equation with a superlinear neutral term in the case where $p(t) \to \infty$ as $t \to \infty$, and the second example is concerned with the equation with a linear neutral term in the case where p is a constant function.

Example 2.9 *Consider the third-order differential equation with a superlinear neutral term*

$$\left(\frac{1}{t^{1/3}}\left(z''(t)\right)^{1/3}\right)' + \frac{8t}{3}x^3\left(\frac{t}{3}\right) = 0, \quad t \geq 2, \tag{2.34}$$

with

$$z(t) = x(t) + 4tx^3\left(\frac{t}{2}\right).$$

Here $r(t) = 1/t^{1/3}$, $p(t) = 4t$, $q(t) = 8t/3$, $\tau(t) = t/2$, $\sigma(t) = t/3$, $\alpha = 1/3$, $\beta = 3$, *and* $\delta = 3$. *Then it is easy to see that conditions* $(C_1) - (C_4)$ *and* (1.2) *hold,*

$$I_1(t, t_*) = I_1(t, t_0) = I_1(t, 2) = (t^2 - 4)/2,$$

$$I_2(\tau^{-1}(t), t_{**}) = I_2(2t, 3) = (8t^3 - 24t + 9)/6,$$

$$I_2(\tau^{-1}(\tau^{-1}(t)), t_{**}) = I_2(4t, 3) = (64t^3 - 48t + 9)/6$$

and

$$P_2(t) = \frac{1}{8t}\left[1 - \frac{l^{\frac{1}{3}-1}}{(16t)^{1/3}}\right].$$

Since

$$\frac{64t^3 - 48t + 9}{128t^3 - 384t + 144} \leq \frac{177}{272} \quad \textit{for } t \geq 3,$$

we have

$$P_1(t) \geq \frac{1}{8t}\left[1 - \left(\frac{177}{272}\right)^{1/3}\frac{k^{\frac{1}{3}-1}}{t^{1/3}}\right] \quad \textit{for } t \geq 3.$$

Thus, it follows from (2.14) *and* (2.15) *that*

$$\int_{t_{**}}^{\infty} q(s)P_1^{\delta/\beta}(\sigma(s))ds \geq \int_3^{\infty}\left[1 - \left(\frac{177}{272}\right)^{1/3}\frac{3^{1/3}}{k^{2/3}s^{1/3}}\right]ds = \infty,$$

and

$$\int_{t_0}^{\infty} q(s)P_2^{\delta/\beta}(\sigma(s))ds = \int_2^{\infty}\left(1 - \frac{3^{1/3}}{l^{2/3}(16s)^{1/3}}\right)ds = \infty,$$

i.e. conditions (2.14) *and* (2.15) *hold, respectively. Thus, by Theorem 2.5, any solution* $x(t)$ *of equation* (2.34) *is either oscillatory or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Example 2.10 *Consider the third-order differential equation with a linear neutral term*

$$\left(\frac{1}{t^{1/5}}\left(z''(t)\right)^{1/5}\right)' + (1+t^2)x^{1/5}\left(\frac{t}{8}\right) = 0, \quad t \geq 2, \tag{2.35}$$

with

$$z(t) = x(t) + 16x\left(\frac{t}{2}\right).$$

Here $r(t) = 1/t^{1/5}$, $p(t) = 16$, $q(t) = 1 + t^2$, $\tau(t) = t/2$, $\sigma(t) = t/8$, $\alpha = 1/5$, $\beta = 1$, *and* $\delta = 1/5$. *Then it is easy to see that conditions* $(C_1) - (C_4)$ *and* (1.2) *hold,*

$$I_1(t,2) = (t^2 - 4)/2,$$

$$I_2(h(t),2) = (t^3 - 192t + 1024)/384,$$

$$P_2(t) = 15/256$$

and

$$P_1(t) \geq 95/4352.$$

Letting $\eta(t) = t/3$ *and* $\xi(t) = t/2$, *we get*

$$I_1(\xi(t),\eta(t)) = 5t^2/72.$$

Thus, it follows from (2.31) *and* (2.32) *that*

$$\int_{t_{**}}^{\infty} q(s)P_1^{\delta/\beta}(\sigma(s))I_2^{\delta/\beta}(h(s),t_0)ds$$

$$\geq \left(\frac{95}{4352 \times 384}\right)^{1/5} \int_3^{\infty} (1+s^2)(s^3 - 192s + 1024)^{1/5} ds = \infty$$

and

$$\int_{t_0}^{\infty} q(s)P_2^{\delta/\beta}(\sigma(s))\left[(\eta(s) - h(s))\, I_1(\xi(s),\eta(s))\right]^{\delta/\beta} ds$$

$$= \left(\frac{75}{256 \times 864}\right)^{1/5} \int_2^{\infty} (1+s^2)s^{3/5} ds = \infty,$$

i.e. conditions (2.31) *and* (2.32) *hold, respectively. Thus, by Corollary 2.8, any solution* $x(t)$ *of equation* (2.35) *either oscillates or satisfies* $\lim_{t\to\infty} x(t) = 0$.

Remark 2.11 *It will be of interest to study equation* (1.1) *under condition*

$$I_1(t,t_0) := \int_{t_0}^{t} r^{-1/\alpha}(s)ds < \infty \quad \text{as } t \to \infty.$$

Remark 2.12 *The results of this paper are presented in a form that can be extended to higher-order equations of the form*

$$\left(r(t)\left(z^{(n-1)}(t)\right)^{\alpha}\right)' + q(t)x^{\delta}(\sigma(t)) = 0, \quad t \geq t_0 > 0,$$

where $n \geq 3$ *is an odd natural number,* α, β, δ, r, p, q, σ, τ, *and* z *are defined as in this paper.*

Remark 2.13 *It would be of interest to study equation* (1.1) *in the case where* $p(t) \leq -1$ *with* $p(t) \not\equiv -1$ *for large* t.

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