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Faber polynomial coefficients for certain subclasses of analytic and biunivalent functions

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Abstract: In this paper, we introduce and investigate two new subclasses of analytic and bi-univalent functions defined in the open unit disc. We use the Faber polynomial expansions to find upper bounds for the nth $(n \geq 3)$ Taylor-Maclaurin coefficients $|a_n|$ of functions belong to these new subclasses with $a_k = 0$ for $2 \leq k \leq n-1$, also we find non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$. The results, which are presented in this paper, would generalize those in related earlier works of several authors.

Key words: Faber polynomial, univalent functions, bi-univalent functions, coefficient bounds

1. Introduction

Faber polynomials, which were introduced by Faber in 1903 [22], play an important role in the theory of functions of a complex variable and in different areas of mathematics. Given a function $h(z)$ of the form

$$h(z) = z + b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots,$$

consider the expansion

$$\frac{\varsigma h^{'}(\zeta)}{h(\zeta) - w} = \sum_{n=0}^{\infty} \Psi_n(w) \zeta^{-n},$$

valid for all ζ in some neighborhood of ∞. The function $\Psi_n(w) = w^n + \sum_{k=1}^{n} a_{nk} w^{n-k}$ is a polynomial of degree n, called the nth Faber polynomial with respect to the function $h(z)$. In particular,

$$\begin{aligned} \Psi_0(w) &= 1, \qquad \Psi_1(w) = w - b_0, \\ \Psi_2(w) &= w^2 - 2b_0 w + (b_0^2 - 2b_1), \\ \Psi_3(w) &= w^3 - 3b_0 w^2 + (3b_0^2 - 3b_1)w + (b_0^3 + 3b_1 b_0 - 3b_2). \end{aligned}$$

Let $\Psi_n(0) = F_n(b_0, b_1, \dots, b_n), n \geq 0$, see ([21, page 118]). Let A denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

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2010 *AMS Mathematics Subject Classification:* 30C45; 30C50; 30C55; 30C80 which are analytic in the open unit disc $U=\{z : z\in\mathbb{C}$ and $|z|<1\}$, $\mathbb{C}$ being, as usual, the set of complex numbers. We also denote by S the subclass of all functions in A which are univalent in U. Recently, Airault and Ren [2, page 344] introduced the generalized Faber polynomials F_j^k ($j\geq 0, k$ is an integer) associated with the univalent function f of the form (1.1), by

$$\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^k = 1-\sum_{j=2}^{\infty} F_{j-1}^{k+j-1}(a_2,a_3,...,a_j)z^{j-1}. \tag{1.2}$$

They showed that those Faber polynomials are linked to the coefficients in the asymptotic expansion of the function $\left(\frac{f(z)}{z}\right)^p$,

$$\left(\frac{f(z)}{z}\right)^p = 1+\sum_{j=2}^{\infty} K_{j-1}^{p}(a_2,a_3,...,a_j)z^{j-1}. \tag{1.3}$$

Also in [1, page 184] Airault and Bouali, showed that

$$\frac{zf'(z)}{f(z)} = 1-\sum_{j=2}^{\infty} F_{j-1}(a_2,a_3,...,a_j)z^{j-1}, \tag{1.4}$$

where the first few terms of the generalized Faber polynomials $F_{j-1}^k(a_2,a_3,...,a_j)$, $j\geq 2$, are given by (e.g. see [2, page 351])

$$\begin{aligned}
F_1^k &= -ka_2, \qquad F_2^k = \frac{k(3-k)}{2}a_2^2-ka_3,\\
F_3^k &= \frac{k(4-k)(k-5)}{3!}a_2^3+k(4-k)a_2a_3-ka_4,\\
F_4^k &= \frac{k(5-k)(k-6)(k-7)}{4!}a_2^4+\frac{k(5-k)(k-6)}{2!}a_2^2a_3-k(5-k)a_2a_4\\
&\quad +\frac{k(5-k)}{2}a_3^2-ka_5\\
F_5^k &= \frac{k(6-k)(k-7)(k-8)(k-9)}{5!}a_2^5+\frac{k(6-k)(k-7)(k-8)}{3!}a_2^3a_3\\
&\quad +\frac{k(6-k)(k-7)}{2}a_2^2a_4+\frac{k(6-k)(k-7)}{2}a_2a_3^2+k(6-k)a_3a_4\\
&\quad +k(6-k)a_2a_5-ka_6.
\end{aligned} \tag{1.5}$$

Note that, the nth Faber polynomial $F_n = F_n^n$ (see [2, page 350] and [8, page 52]) and $F_n^{n+j} = -\left(1+\frac{n}{j}\right) K_n^j$ (see [2, page 352]), where the coefficients $K_n^p(a_2, a_3, ..., a_n)$ are given by,

$$\begin{aligned}
K_1^p &= pa_2, \qquad K_2^p = \frac{p(p-1)}{2}a_2^2 + pa_3, \\
K_3^p &= p(p-1)a_2a_3 + pa_4 + \frac{p(p-1)(p-2)}{3!}a_2^3, \\
K_4^p &= p(p-1)a_2a_4 + pa_5 + \frac{p(p-1)}{2}a_3^2 + \frac{p(p-1)(p-2)}{2}a_2^2a_3 + \frac{p!}{(p-4)!4!}a_2^4, \\
&\vdots \\
K_n^p &= \frac{p!}{(p-n)!n!}a_2^n + \frac{p!}{(p-n+1)!(n-2)!}a_2^{n-2}a_3 + \frac{p!}{(p-n+2)!(n-3)!}a_2^{n-3}a_4 \\
&+ \frac{p!}{(p-n+3)!(n-4)!}a_2^{n-4}\left[a_5 + \frac{p-n+3}{2}a_3^2\right] \\
&+ \frac{p!}{(p-n+4)!(n-5)!}a_2^{n-4}\left[a_6 + (p-n+3)a_3a_4\right] + \sum_{j\geq 6}^{\infty} a_2^{n-j}V_j \qquad (1.6)
\end{aligned}$$

and V_j is homogeneous polynomial of degree j in the variables $a_3, ..., a_n$, see ([2, page 349] and [1, pages 183 and 205]). If f and g are analytic functions in U, we say that f is subordinate to g, written $f(z) \prec g(z)$ if there exists a Schwarz function φ, which (by definition) is analytic in U with $\varphi(0) = 0$ and $|\varphi(z)| < 1$ for all $z \in U$, such that $f(z) = g(\varphi(z))$, $z \in U$. Furthermore, if the function g is univalent in U, then we have the following equivalence

$$f(z) \prec g(z)(z \in U) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

The Koebe one-quarter theorem [21, page 31] ensures the range of every function of the class S contains the disc $\{w : |w| < \frac{1}{4}\}$. Thus, every univalent function $f \in S$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z \qquad (z \in U)$$

and

$$f(f^{-1}(\omega)) = \omega \qquad (|\omega| < \frac{1}{4}).$$

In fact, the coefficients of inverse function $g = f^{-1}$ are given by (see [1, page 185])

$$\begin{aligned}
g(\omega) &= f^{-1}(\omega) = w + \sum_{n=2}^{\infty} \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n)\omega^n \\
&= w - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^2 - 5a_2a_3 + a_4)\omega^4 + \ldots .
\end{aligned}$$

A function $f \in A$ is said to be bi-univalent in U if f and f^{-1} are univalent in U. Let σ denote the class of bi-univalent functions in U given by (1.1). In 1985 Louis de Branges [9] proved the celebrated Bieberbach Conjecture which states that, for each $f(z) \in S$ given by the Taylor–Maclaurin series expansion (1.1), the following coefficient inequality holds true:

$$|a_n| \leq n \qquad (n = 2, 3, 4, \ldots).$$

The class of analytic bi-univalent functions was first introduced and studied by Lewin [31], where it was proved that $|a_2| < 1.51$. Subsequently, Brannan and Clunie [10] improved Lewin's result to $|a_2| \leq \sqrt{2}$. Brannan and Taha [12] and Taha [46] considered certain subclasses of bi-univalent functions, similar to the familiar subclasses of univalent functions consisting of strongly starlike and convex functions. They introduced bi-starlike functions and bi-convex functions and found nonsharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$. For further historical account of functions in the class σ, see the work by Srivastava et al. [43] (see also [3, 4, 6, 11–15, 19, 23, 24, 27, 29, 30, 32, 34–37, 40, 42, 44, 45, 47–49]).

2. Coefficient estimates for the class $B_\sigma(p, \lambda, \tau, \varphi)$

In the sequel, it is assumed that φ is an analytic function with positive real part in the unit disc U, satisfying $\varphi(0) = 1, \varphi^{'}(0) > 0,$ and $\varphi(U)$ is symmetric with respect to the real axis. Such a function has a Taylor series of the form

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + ... \ (B_1 > 0). \tag{2.1}$$

Suppose that $u(z)$ and $v(z)$ are analytic in the unit disc U with $u(0) = v(0) = 0$, $|u(z)| < 1$, $|v(z)| < 1$, and suppose that

$$u(z) = b_1 z + \sum_{n=2}^{\infty} b_n z^n, \quad v(z) = c_1 z + \sum_{n=2}^{\infty} c_n z^n \quad (z \in U). \tag{2.2}$$

It is well known that (see Duren [21, page 265])

$$|b_n| \leq 1, |c_n| \leq 1 \qquad n = 2, 3, \dots \ . \tag{2.3}$$

By a simple calculation, we have

$$\begin{aligned} \varphi(u(z)) &= 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}(b_1, b_2, ..., b_n, B_1, B_1, B_2, B_3, ..., B_n) z^n \\ &= 1 + B_1 b_1 z + (B_1 b_2 + B_2 b_1^2) z^2 + ... \ (z \in U), \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} \varphi(v(\omega)) &= 1 - B_1 \sum_{n=1}^{\infty} K_n^{-1}(c_1, c_2, ..., c_n, B_1, B_2, B_3, ..., B_n) w^n \\ &= 1 + B_1 c_1 \omega + (B_1 c_2 + B_2 c_1^2) \omega^2 + ... \ (\omega \in U). \end{aligned} \tag{2.5}$$

In general (see [20, page 649]), the coefficients $K_n^p(k_1, k_2, ..., k_n, B_1, B_2, B_3, ..., B_n)$ are given by

$$\begin{aligned} & K_n^p(k_1, k_2, ..., k_n, B_1, B_2, B_3, ..., B_n) \\ = \quad & \frac{p!}{(p-n)!n!}k_1^n \frac{(-1)^{n+1}B_n}{B_1} + \frac{p!}{(p-n+1)!(n-2)!}k_1^{n-2}k_2 \frac{(-1)^n B_{n-1}}{B_1} \\ & + \frac{p!}{(p-n+2)!(n-3)!}k_1^{n-3}k_3 \frac{(-1)^{n-1}B_{n-2}}{B_1} \\ & + \frac{p!}{(p-n+3)!(n-4)!}k_1^{n-4}[k_4 \frac{(-1)^{n-2}B_{n-3}}{B_1} + \frac{p-n+3}{2}k_2^2 k_3 \frac{(-1)^{n-1}B_{n-2}}{B_1}] \\ & + \sum_{j \geq 5}^{\infty} k_1^{n-j} X_j, \end{aligned}$$

where X_j is a homogeneous polynomial of degree j in the variables $k_2, ..., k_n$.

Definition 2.1 *A function $f(z) \in A$ is said to be in the class $B(p, \lambda, \tau, \varphi)$ $(p > 0, 0 \leq \lambda \leq 1, \tau \in \mathbb{C} \backslash \{0\})$ if it satisfies*

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{f(z)}{z})^p + \lambda \frac{zf'(z)}{f(z)}(\frac{f(z)}{z})^p - 1] \prec \varphi(z) \ (z \in U).$$

We note that:

1. The class $B(\alpha, 1, 1, 1 + \mu z) = B(\alpha, \mu)$ $(\alpha, \mu > 0)$ was introduced and studied by Ponnusamy [38] and Yang [50];
2. the class $B(\alpha, \lambda, 1, 1 + \mu z)$ $(\alpha > 0)$ was studied by Ponnusamy and Rajasekaran [39], Darwish et al. [18], and Prajapat and Agarwal [41];
3. the class $B(\alpha, \lambda, 1, \frac{1+Az}{1+Bz}) = B(\lambda, \alpha, A, B)$ $(-1 \leq B \leq 1, A \neq B)$ was introduced and studied by Liu [33].
4. $B(\alpha, 1, 1, \frac{1+z}{1-z})$ is the subclass of Bazilevic functions [7];

Definition 2.2 *A function $f \in \sigma$ given by (1.1) is said to be in the class $B_\sigma(p, \lambda, \tau, \varphi)$ $(p > 0, 0 \leq \lambda \leq 1, \tau \in \mathbb{C})$ if both f and its inverse map $g = f^{-1}$ are in $B(p, \lambda, \tau, \varphi)$.*

Note that:

1. The class $B_\sigma(1, \lambda, 1, \varphi) = H^\sigma(\lambda, \varphi)$ was introduced and studied by Goyal and Kumar [25] and [51];
2. the class $B_\sigma(\alpha, \lambda, 1, \left(\frac{1+z}{1-z}\right)^\upsilon) = N_\sigma^\alpha(\upsilon, \lambda)$ was introduced and studied by Srivastava et al. [42];
3. the class $B_\sigma(\alpha, \lambda, 1, \varphi) = H_\sigma^{\alpha,\lambda}(\varphi)$ was introduced and studied by Bulut [16];
4. the class $B_\sigma(1, \lambda, 1, \left(\frac{1+z}{1-z}\right)^\upsilon) = B_\sigma(\upsilon, \lambda)$ was introduced and studied by Frasin and Aouf [23].

Unless otherwise mentioned, we shall assume in the remainder of this section that $p > 0, 0 \leq \lambda \leq 1$ and $\tau \in \mathbb{C} \backslash \{0\}$.

Theorem 2.3 *Let the function* $f \in \sigma$ *given by (1.1) be in the class* $B_{\sigma}(p, \lambda, \tau, \varphi)$. *If* $a_k = 0$ *for* $2 \le k \le n-1$, *then*

$$|a_n| \le \frac{|\tau|\, B_1}{[p + \lambda(n-1)]}, \qquad n \ge 3.$$

Proof Since both functions f and its inverse map $g = f^{-1}$ are in $B(\alpha, \lambda, \varphi)$, by the definition of subordination, there are analytic functions $u, v : U \to U$ given by (2.2) such that

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{f(z)}{z})^p + \lambda \frac{zf^{'}(z)}{f(z)}(\frac{f(z)}{z})^p - 1] = \varphi(u(z)) \quad (z \in U) \tag{2.6}$$

and

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{g(w)}{w})^p + \lambda \frac{wg^{'}(w)}{g(w)}(\frac{g(w)}{w})^p - 1] = \varphi(v(w)) \quad (z \in U). \tag{2.7}$$

It follows from (1.2) and (1.3) that

$$\begin{aligned} & 1 + \frac{1}{\tau}[(1-\lambda)(\frac{f(z)}{z})^p + \lambda \frac{zf^{'}(z)}{f(z)}(\frac{f(z)}{z})^p - 1] \\ = \; & 1 + \frac{1}{\tau}\{(1-\lambda)[1 + \sum_{j=2}^{\infty} K_{j-1}^{p}(a_2, a_3, ..., a_j) z^{j-1}] \\ & + \lambda[1 - \sum_{j=2}^{\infty} F_{j-1}^{p+j-1}(a_2, a_3, ..., a_j) z^{j-1}] - 1\} \\ = \; & 1 + \frac{1}{\tau}\sum_{j=2}^{\infty}[(1-\lambda)K_{j-1}^{p}(a_2, a_3, ..., a_j) - \lambda F_{j-1}^{p+j-1}(a_2, a_3, ..., a_j)] z^{j-1} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} & 1 + \frac{1}{\tau}[(1-\lambda)(\frac{g(w)}{w})^p + \lambda \frac{wg^{'}(w)}{g(w)}(\frac{g(w)}{w})^p - 1] \\ = \; & 1 + \frac{1}{\tau}\sum_{j=2}^{\infty}[(1-\lambda)K_{j-1}^{p}(d_2, d_3, ..., d_j) - \lambda F_{j-1}^{p+j-1}(d_2, d_3, ..., d_j)] w^{j-1}, \end{aligned} \tag{2.9}$$

where $d_n = \frac{1}{n} K_{n-1}^{-n}(a_2, a_3, ..., a_n)$. Comparing the corresponding coefficients of (2.8) and (2.4) gives

$$\begin{aligned} & (1-\lambda)K_{n-1}^{p}(a_2, a_3, ..., a_n) - \lambda F_{n-1}^{p+n-1}(a_2, a_3, ..., a_n) \\ = \; & -\tau B_1 K_{n-1}^{-1}(b_1, b_2, ..., b_{n-1}, B_1, B_2, B_3, ..., B_{n-1}) \end{aligned} \tag{2.10}$$

Similarly, comparing the corresponding coefficients of (2.9) and (2.5) yields

$$\begin{aligned} & (1-\lambda)K_{n-1}^{p}(d_2, d_3, ..., d_n) - \lambda F_{n-1}^{p+n-1}(d_2, d_3, ..., d_n) \\ = \; & -\tau B_1 K_{n-1}^{-1}(c_1, c_2, ..., c_{n-1}, B_1, B_2, B_3, ..., B_{n-1}). \end{aligned} \tag{2.11}$$

Since $a_k = 0$ for $2 \le k \le n-1$, by using $d_n = -a_n$, $K_{n-1}^{p} = pa_n$ and $F_{n-1}^{p+n-1} = -(p+n-1)a_n$ in (2.10) and (2.11), we have

$$[p + \lambda(n-1)]a_n = \tau B_1 b_{n-1} \tag{2.12}$$

and

$$-[p + \lambda(n-1)]a_n = \tau B_1 c_{n-1}. \tag{2.13}$$

By using (2.3), we conclude that

$$|a_n| \le \frac{|\tau|\, B_1}{[p + \lambda(n-1)]},$$

this completes the proof. □

To prove our next theorem, we shall need the following lemma.

Lemma 2.4 *[20] Let the function $\Phi(z) = \sum\limits_{n=1}^{\infty} \Phi_n z^n$ be a Schwarz function with $|\Phi(z)| < 1$, $z \in U$. Then for $-\infty < \rho < \infty$.*

$$\left|\Phi_2 + \rho\Phi_1^2\right| \le \begin{cases} 1 - (1-\rho)\left|\Phi_1^2\right| & \rho > 0 \\ 1 - (1+\rho)\left|\Phi_1^2\right| & \rho \le 0 \end{cases}$$

Theorem 2.5 *If the function $f \in \sigma$ given by (1.1) be in the class $B_\sigma(p, \lambda, \tau, \varphi)$, then*

$$|a_2| \le \begin{cases} \frac{|\tau|B_1\sqrt{2B_1}}{\sqrt{(p+2\lambda)(p+1)|\tau|B_1^2 + 2(p+\lambda)^2(B_1+B_2)}} & (B_2 \le 0, B_1 + B_2 \ge 0) \\ \frac{|\tau|B_1\sqrt{2B_1}}{\sqrt{(p+2\lambda)(p+1)|\tau|B_1^2 + 2(p+\lambda)^2(B_1-B_2)}} & (B_2 > 0, B_1 - B_2 \ge 0) \end{cases} \tag{2.14}$$

and

$$\left|a_3 - a_2^2\right| \le \begin{cases} \frac{|\tau|B_1}{(p+2\lambda)} & (B_1 \ge |B_2|) \\ \frac{|\tau B_2|}{(p+2\lambda)} & (B_1 < |B_2|). \end{cases} \tag{2.15}$$

Proof Let $f \in B_\sigma(p, \lambda, \tau, \varphi)$. Then there are analytic functions $u, v : U \to U$ given by (2.2) such that (2.6) and (2.7) are satisfied. Replacing $n = 2$ and 3 in (2.10) and (2.11), respectively, we find that

$$(p + \lambda)a_2 = \tau B_1 b_1, \tag{2.16}$$

$$(p + 2\lambda)[\frac{(p-1)}{2}a_2^2 + a_3] = \tau[B_1 b_2 + B_2 b_1^2], \tag{2.17}$$

$$-(p + \lambda)a_2 = \tau B_1 c_1, \tag{2.18}$$

$$(p + 2\lambda)[\frac{(p+3)}{2}a_2^2 - a_3] = \tau[B_1 c_2 + B_2 c_1^2], \tag{2.19}$$

It follows from (2.16) and (2.18) that

$$b_1 = -c_1. \tag{2.20}$$

Adding (2.17) to (2.19) leads to

$$(p+2\lambda)(p+1)a_2^2 = \tau B_1(b_2+c_2) + \tau B_2\left(b_1^2+c_1^2\right) \tag{2.21}$$

or

$$\left|a_2^2\right| \leq \frac{|\tau|\, B_1}{(p+2\lambda)(p+1)}\left(\left|b_2+\frac{B_2}{B_1}b_1^2\right| + \left|c_2+\frac{B_2}{B_1}c_1^2\right|\right). \tag{2.22}$$

Case 1. We suppose that $B_2 \leq 0$, then for $\rho = \frac{B_2}{B_1} \leq 0$ and $B_1+B_2 \geq 0$ applying Lemma 2.4 and (2.20), we get

$$\left|a_2^2\right| \leq \frac{2\,|\tau|\, B_1}{(p+2\lambda)(p+1)}\left(1-[\frac{B_1+B_2}{B_1}]\,|b_1|^2\right). \tag{2.23}$$

Thus, by considering (2.16) and (2.23), we obtain

$$|a_2| \leq \frac{|\tau|\, B_1\sqrt{2B_1}}{\sqrt{(p+2\lambda)(p+1)\,|\tau|\, B_1^2 + 2(p+\lambda)^2(B_1+B_2)}}. \tag{2.24}$$

Case 2. Let $B_2 > 0$, then for $\rho = \frac{B_2}{B_1} > 0$ and $B_1 - B_2 \geq 0$ using Lemma 2.4 and (2.20) for (2.22), we obtain

$$\left|a_2^2\right| \leq \frac{2\,|\tau|\, B_1}{(p+2\lambda)(p+1)}\left(1-[\frac{B_1-B_2}{B_1}]\,|b_1|^2\right). \tag{2.25}$$

It follows from (2.25) and (2.16), that

$$|a_2| \leq \frac{|\tau|\, B_1\sqrt{2B_1}}{\sqrt{(p+2\lambda)(p+1)\,|\tau|\, B_1^2 + 2(p+\lambda)^2(B_1-B_2)}}. \tag{2.26}$$

From (2.24) and (2.26) we obtain the desired estimate of $|a_2|$ given by (2.14). Next, from (2.17) and (2.19), we have

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|\, B_1}{2(p+2\lambda)}\left(\left|b_2+\frac{B_2}{B_1}b_1^2\right| + \left|c_2+\frac{B_2}{B_1}c_1^2\right|\right). \tag{2.27}$$

If $B_2 \leq 0$, then for $\rho = \frac{B_2}{B_1} \leq 0$ applying Lemma 2.4 we get

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|\, B_1}{2(p+2\lambda)}\left(\left[1-\frac{B_1+B_2}{B_1}\,|b_1|^2\right] + \left[1-\frac{B_1+B_2}{B_1}\,|c_1|^2\right]\right). \tag{2.28}$$

If $B_1+B_2 \geq 0$ then (2.28) gives

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|\, B_1}{(p+2\lambda)}.$$

Let $B_1+B_2 < 0$; thus, from (2.3) and (2.28)

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|\, B_1}{(p+2\lambda)}\left[1-\frac{B_1+B_2}{B_1}\right] = -\frac{|\tau|\, B_2}{(p+2\lambda)}.$$

If $B_2 > 0$, then for $\rho = \frac{B_2}{B_1} > 0$ applying Lemma 2.4 to (2.27) we get

$$\left|a_3-a_2^2\right| \leq \frac{|\tau|\, B_1}{2(p+2\lambda)}\left(\left[1-\frac{B_1-B_2}{B_1}\,|b_1|^2\right] + \left[1-\frac{B_1-B_2}{B_1}\,|c_1|^2\right]\right). \tag{2.29}$$

If $B_1 - B_2 \geq 0$, then (2.29) gives

$$\left|a_3 - a_2^2\right| \leq \frac{|\tau| B_1}{(p+2\lambda)}.$$

If $B_1 - B_2 < 0$, then from (2.3) and (2.29) we have

$$\left|a_3 - a_2^2\right| \leq \frac{|\tau| B_1}{(p+2\lambda)}\left[1 - \frac{B_1 - B_2}{B_1}\right] = \frac{|\tau| B_2}{(p+2\lambda)}.$$

Which is the second part of assertion (2.15). This completes the proof of Theorem 2.5. □

If we set

$$\varphi(z) = \left(\frac{1+z}{1-z}\right)^{\gamma} = 1 + 2\gamma z + 2\gamma^2 z^2 + ...(0 < \gamma \leq 1, z \in U)$$

in Definition 2.2 of the bi-univalent function class $B_\sigma(p,\lambda,\tau,\varphi)$, we obtain a new class $B_\sigma(p,\lambda,\tau,\gamma)$ given by Definition 2.6 below.

Definition 2.6 *Let $0 < \gamma \leq 1$. A function $f \in \sigma$ given by (1.1) is said to be in the class $B_\sigma(p,\lambda,\tau,\gamma)$, if the following conditions are satisfied:*

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{f(z)}{z})^p + \lambda\frac{zf^{'}(z)}{f(z)}(\frac{f(z)}{z})^p - 1] \prec \left(\frac{1+z}{1-z}\right)^{\gamma} \ (z \in U)$$

and

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{g(w)}{w})^p + \lambda\frac{wg^{'}(w)}{g(w)}(\frac{g(w)}{w})^p - 1] \prec \left(\frac{1+\omega}{1-\omega}\right)^{\gamma} \ (\omega \in U),$$

where $g = f^{-1}$.

Using the parameter setting of Definition 2.6 in the Theorem 2.5, we get the following corollary.

Corollary 2.7 *Let $0 < \gamma \leq 1$. If the function $f \in \sigma$ given by (1.1) be in the class $B_\sigma(p,\lambda,\tau,\gamma)$, then*

$$|a_2| \leq \frac{2\,|\tau|\,\gamma}{\sqrt{(p+2\lambda)(p+1)\,|\tau|\,\gamma + (p+\lambda)^2(1-\gamma)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{2\,|\tau|\,\gamma}{(p+2\lambda)}.$$

Remark 2.8 *In Corollary 2.7,*

1. *if we take $p = 1$ and $\tau = 1$, then we obtain the results of Frasin and Aouf [23],*
2. *if we take $\tau = 1$, then we have the results which were given by Caglar et al. [17].*

If we set

$$\varphi(z) = \frac{1+(1-2\upsilon)z}{1-z} = 1 + 2(1-\upsilon)z + 2(1-\upsilon)z^2 + ...(0 \leq \upsilon < 1, z \in U)$$

in Definition 2.2 of the bi-univalent function class $B_\sigma(p,\lambda,\tau,\varphi)$, we obtain a new class $B_\sigma^\upsilon(p,\lambda,\tau)$ given by Definition 2.9 below.

Definition 2.9 *Let $0 \leq \upsilon < 1$. A function $f \in \sigma$ given by (1.1) is said to be in the class $B_{\sigma}^{\upsilon}(p, \lambda, \tau)$, if the following conditions hold true:*

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{f(z)}{z})^{p} + \lambda \frac{zf^{'}(z)}{f(z)}(\frac{f(z)}{z})^{p} - 1] \prec \frac{1+(1-2\upsilon)z}{1-z} \quad (z \in U)$$

and

$$1 + \frac{1}{\tau}[(1-\lambda)(\frac{g(w)}{w})^{p} + \lambda \frac{wg^{'}(w)}{g(w)}(\frac{g(w)}{w})^{p} - 1] \prec \frac{1+(1-2\upsilon)\omega}{1-\omega} \quad (\omega \in U),$$

where $g = f^{-1}$.

Using the parameter setting of Definition 2.9 in Theorems 2.3 and 2.5, respectively, we get the following corollaries.

Corollary 2.10 *Let the function $f \in B_{\sigma}^{\upsilon}(p, \lambda, \tau)$, be given by (1.1). If $a_k = 0$ for $2 \leq k \leq n-1$, then*

$$|a_n| \leq \frac{2\,|\tau|\,(1-\upsilon)}{[p + \lambda(n-1)]}, \qquad n \geq 3.$$

Remark 2.11 *In Corollary 2.10, if we set $\lambda = \tau = 1$, then we obtain the results of Jahangiri and Hamidi [28].*

Corollary 2.12 *For $0 \leq \upsilon < 1$, let the function $f \in B_{\sigma}^{\upsilon}(p, \lambda, \tau)$ be given by (1.1). Then*

$$|a_2| \leq \sqrt{\frac{4\,|\tau|\,(1-\upsilon)}{(p+2\lambda)(p+1)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{2\,|\tau|\,(1-\upsilon)}{(p+2\lambda)}.$$

Remark 2.13 *In Corollary 2.12,*

1. *if we take $p = 1$ and $\tau = 1$, then we obtain the results of Frasin and Aouf [23],*
2. *if we take $\tau = 1$, then we have the results which were given by Caglar et al. [17],*
3. *if we set $\lambda = \tau = 1$, then we have the results which were given by Jahangiri and Hamidi [28].*

3. Coefficient estimates for the class $B_{\sigma}(\alpha, \beta, \varphi)$

Definition 3.1 *A function $f \in \sigma$ given by (1.1) is said to be in the class $B_{\sigma}(\alpha, \beta, \varphi)(\alpha, \beta \geq 0, \alpha + \beta \leq 1)$ if the following conditions are satisfied:*

$$\alpha \frac{f(z)}{z} + \beta f^{'}(z) + (1-\alpha-\beta)\frac{zf^{'}(z)}{f(z)} \prec \varphi(z) \quad (z \in U)$$

and

$$\alpha\frac{g(w)}{w}+\beta g^{'}(w)+(1-\alpha-\beta)\frac{wg^{'}(w)}{g(w)}\prec\varphi(w),\quad(\omega\in U),$$

where $g=f^{-1}$.

Unless otherwise mentioned, we shall assume in the remainder of this section that $\alpha,\beta\geq 0$ and $\alpha+\beta\leq 1$.

Theorem 3.2 *Let the function* $f\in\sigma$ *given by (1.1) be in the class* $B_\sigma(\alpha,\beta,\varphi)$. *If* $a_k=0$ *for* $2\leq k\leq n-1$, *then*

$$|a_n|\leq\frac{B_1}{[\alpha+\beta+(1-\alpha)(n-1)]},\qquad n\geq 3.$$

Proof Let $f\in B_\sigma(\alpha,\beta,\varphi)$. Then there are analytic functions $u,v:U\to U$ given by (2.2) such that

$$\alpha\frac{f(z)}{z}+\beta f^{'}(z)+(1-\alpha-\beta)\frac{zf^{'}(z)}{f(z)}=\varphi(u(z))\tag{3.1}$$

and

$$\alpha\frac{g(w)}{w}+\beta g^{'}(w)+(1-\alpha-\beta)\frac{wg^{'}(w)}{g(w)}=\varphi(v(\omega)).\tag{3.2}$$

Now, from (1.4), we get that

$$\begin{aligned}&\alpha\frac{f(z)}{z}+\beta f^{'}(z)+(1-\alpha-\beta)\frac{zf^{'}(z)}{f(z)}\\=&\ 1+\sum_{j=2}^{\infty}[(\alpha+\beta j)\,a_j-(1-\alpha-\beta)F_{j-1}(a_2,a_3,...,a_j)]z^{j-1},\end{aligned}$$

and

$$\begin{aligned}&\alpha\frac{g(w)}{w}+\beta g^{'}(w)+(1-\alpha-\beta)\frac{wg^{'}(w)}{g(w)}\\=&\ 1+\sum_{j=2}^{\infty}[(\alpha+\beta j)\,d_j-(1-\alpha-\beta)F_{j-1}(d_2,d_3,...,d_j)]w^{j-1},\end{aligned}$$

where $d_n=\frac{1}{n}K_{n-1}^{-n}(a_2,a_3,...,a_n)$. It follows from (2.4), (2.5), (3.1), and (3.2) that

$$\begin{aligned}&(\alpha+\beta n)\,a_n-(1-\alpha-\beta)F_{n-1}(a_2,a_3,...,a_n)\\=&\ -B_1K_{n-1}^{-1}(b_1,b_2,...,b_{n-1},B_1,B_1,B_2,B_3,...,B_{n-1})\end{aligned}\tag{3.3}$$

and

$$\begin{aligned}&(\alpha+\beta n)\,d_n-(1-\alpha-\beta)F_{n-1}(d_2,d_3,...,d_n)\\=&\ -B_1K_{n-1}^{-1}(c_1,c_2,...,c_{n-1},B_1,B_2,B_3,...,B_{n-1}).\end{aligned}\tag{3.4}$$

Since $a_k = 0$ for $2 \leq k \leq n-1$, by using $d_n = -a_n$ and $F_{n-1} = -(n-1)a_n$, we have

$$[\alpha + \beta + (1-\alpha)(n-1)]a_n = B_1 b_{n-1} \tag{3.5}$$

and

$$-[\alpha + \beta + (1-\alpha)(n-1)]a_n = B_1 c_{n-1}. \tag{3.6}$$

By using (2.3), we conclude that

$$|a_n| \leq \frac{B_1}{[\alpha + \beta + (1-\alpha)(n-1)]}.$$

□

Remark 3.3 *In Theorem 3.2,*

1. *If we set $\alpha = \beta = 0$ and $\varphi(z) = \frac{1+Az}{1+Bz} = 1 + (A-B)z - B(A-B)z^2 + \ldots$ $(-1 \leq B < A \leq 1)$, then we obtain the results of Hamidi and Jahangiri [26].*

2. *If we take $\alpha + \beta = 1$, then we have the results which were given by Zireh et al. [51] when $\varphi(z) = 1$.*

3. *If we take $\alpha + \beta = 1$ and $\varphi(z) = \frac{1+z}{1-z}$, then we obtain the results of Altınkaya and Yalcın [5] when p=1.*

Theorem 3.4 *If the function $f \in \sigma$ given by (1.1) be in the class $B_\sigma(\alpha, \beta, \varphi)$, then*

$$|a_2| \leq \begin{cases} \frac{B_1\sqrt{B_1}}{\sqrt{(1+2\beta)B_1^2 + (1+\beta)^2(B_1+B_2)}} & (B_2 \leq 0, B_1 + B_2 \geq 0) \\ \frac{B_1\sqrt{B_1}}{\sqrt{(1+2\beta)B_1^2 + (1+\beta)^2(B_1-B_2)}} & (B_2 > 0, B_1 - B_2 \geq 0) \end{cases}, \tag{3.7}$$

and

$$\left|a_3 - a_2^2\right| \leq \begin{cases} \frac{B_1}{(2-\alpha+\beta)} & (B_1 \geq |B_2|) \\ \frac{|B_2|}{(2-\alpha+\beta)} & (B_1 < |B_2|). \end{cases} \tag{3.8}$$

Proof Letting $n = 2$ and 3 in (3.3) and (3.4), respectively, we find that

$$(1+\beta)a_2 = B_1 b_1, \tag{3.9}$$

$$\left[(2-\alpha+\beta)a_3 - (1-\alpha-\beta)a_2^2\right] = B_1 b_2 + B_2 b_1^2, \tag{3.10}$$

$$-(1+\beta)a_2 = B_1 c_1, \tag{3.11}$$

$$\left[(2-\alpha+\beta)\left(2a_2^2 - a_3\right) - (1-\alpha-\beta)a_2^2\right] = B_1 c_2 + B_2 c_1^2. \tag{3.12}$$

Eqs. (3.9) and (3.11) lead to

$$b_1 = -c_1. \tag{3.13}$$

Adding (3.10) and (3.12) yields

$$2(1+2\beta)a_2^2 = B_1(b_2 + c_2) + B_2\left(b_1^2 + c_1^2\right) \tag{3.14}$$

or

$$\left|a_2^2\right| \leq \frac{B_1}{2(1+2\beta)}\left(\left|b_2+\frac{B_2}{B_1}b_1^2\right|+\left|c_2+\frac{B_2}{B_1}c_1^2\right|\right).$$

First, let $B_2 \leq 0$ $(\rho = \frac{B_2}{B_1} \leq 0, B_1+B_2 \geq 0)$. Applying Lemma 2.4 and (3.13), we get

$$\left|a_2^2\right| \leq \frac{B_1}{(1+2\beta)}\left(1-[\frac{B_1+B_2}{B_1}]\left|b_1^2\right|\right). \tag{3.15}$$

From (3.9) and (3.15) it follows that

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+2\beta)B_1^2+(1+\beta)^2(B_1+B_2)}}. \tag{3.16}$$

Similarly, for $B_2 > 0$ $(\rho = \frac{B_2}{B_1} > 0, B_1-B_2 \geq 0)$, we have

$$|a_2| \leq \frac{B_1\sqrt{B_1}}{\sqrt{(1+2\beta)B_1^2+(1+\beta)^2(B_1-B_2)}}. \tag{3.17}$$

From (3.16) and (3.17) we obtain the desired estimate of $|a_2|$ given by (3.7). Next, in order to find the bound on $\left|a_3-a_2^2\right|$, by subtracting (3.12) from (3.10), we have

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(2-\alpha+\beta)}\left(\left|b_2+\frac{B_2}{B_1}b_1^2\right|+\left|c_2+\frac{B_2}{B_1}c_1^2\right|\right). \tag{3.18}$$

If $B_2 \leq 0$, let $\rho = \frac{B_2}{B_1} \leq 0$ in Lemma 2.4 we get

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(2-\alpha+\beta)}\left(\left[1-\frac{B_1+B_2}{B_1}\left|b_1\right|^2\right]+\left[1-\frac{B_1+B_2}{B_1}\left|c_1\right|^2\right]\right). \tag{3.19}$$

If $B_1+B_2 \geq 0$ then (3.19) gives $\left|a_3-a_2^2\right| \leq \frac{B_1}{(2-\alpha+\beta)}$. Let $B_1+B_2 < 0$; thus, (2.3) and (3.19) give

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{(2-\alpha+\beta)}\left[1-\frac{B_1+B_2}{B_1}\right] = -\frac{B_2}{(2-\alpha+\beta)}.$$

If $B_2 > 0$, let $\rho = \frac{B_2}{B_1} > 0$ in Lemma 2.4, then (3.18) gives

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{2(2-\alpha+\beta)}\left(\left[1-\frac{B_1-B_2}{B_1}\left|b_1\right|^2\right]+\left[1-\frac{B_1-B_2}{B_1}\left|c_1\right|^2\right]\right). \tag{3.20}$$

If $B_1-B_2 \geq 0$, then (3.20) gives

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{(2-\alpha+\beta)}.$$

If $B_1-B_2 < 0$, then from (2.3) and (3.20) we get

$$\left|a_3-a_2^2\right| \leq \frac{B_1}{(2-\alpha+\beta)}\left[1-\frac{B_1-B_2}{B_1}\right] = \frac{B_2}{(2-\alpha+\beta)}.$$

This completes the proof of Theorem 3.4. □

Remark 3.5 *In Theorem 3.4, if we set* $\alpha=\beta=0$ *and* $\varphi(z)=\frac{1+Az}{1+Bz}$ $(-1\le B<A\le 1)$, *then we obtain the results of Hamidi and Jahangiri [26].*

If we set $\varphi(z)=\left(\frac{1+z}{1-z}\right)^{\gamma}$ $(0<\gamma\le 1, z\in U)$ in Definition 3.1 of the bi-univalent function class $B_\sigma(\alpha,\beta,\varphi)$, we obtain a new class $B_\sigma(\alpha,\beta,\gamma)$ given by Definition 3.6 below.

Definition 3.6 *Let* $0<\gamma\le 1$. *A function* $f\in\sigma$ *given by (1.1) is said to be in the class* $B_\sigma(\alpha,\beta,\gamma)$ *if the following subordinations hold:*

$$\alpha\frac{f(z)}{z}+\beta f^{'}(z)+(1-\alpha-\beta)\frac{zf^{'}(z)}{f(z)}\prec\left(\frac{1+z}{1-z}\right)^{\gamma}\ (z\in U)$$

and

$$\alpha\frac{g(w)}{w}+\beta g^{'}(w)+\frac{wg^{'}(w)}{g(w)}\prec\left(\frac{1+\omega}{1-\omega}\right)^{\gamma}\ (\omega\in U),$$

where $g=f^{-1}$.

Using the parameter setting of Definition 3.6 in the Theorem 3.4, we get the following corollary.

Corollary 3.7 *Let* $0<\gamma\le 1$. *If the function* $f\in\sigma$ *given by (1.1) be in the class* $B_\sigma(\alpha,\beta,\gamma)$, *then*

$$|a_2|\le\frac{2\gamma}{\sqrt{2\gamma(1+2\beta)+(1+\beta)^2(1-\gamma)}}$$

and

$$\left|a_3-a_2^2\right|\le\frac{2\gamma}{(2-\alpha+\beta)}.$$

If we set $\varphi(z)=\frac{1+(1-2\upsilon)z}{1-z}$ $(0\le\upsilon<1, z\in U)$ in Definition 3.1 of the bi-univalent function class $B_\sigma(\alpha,\beta,\varphi)$, we obtain a new class $B_\sigma^{\upsilon}(\alpha,\beta)$ given by Definition 3.8 below.

Definition 3.8 *For* $0\le\upsilon<1$, *a function* $f\in\sigma$ *given by (1.1) is said to be in the class* $B_\sigma^{\upsilon}(\alpha,\beta)$, *if the following conditions are satisfied:*

$$\alpha\frac{f(z)}{z}+\beta f^{'}(z)+(1-\alpha-\beta)\frac{zf^{'}(z)}{f(z)}\prec\frac{1+(1-2\upsilon)z}{1-z}\ (z\in U)$$

and

$$\alpha\frac{g(w)}{w}+\beta g^{'}(w)+(1-\alpha-\beta)\frac{wg^{'}(w)}{g(w)}\prec\frac{1+(1-2\upsilon)\omega}{1-z}\ (\omega\in U),$$

where $g=f^{-1}$.

Using the parameter setting of Definition 3.8 in the Theorem 3.4, we get the following corollary.

Corollary 3.9 *For* $0 \leq \upsilon < 1$, *let the function* $f \in B_{\sigma}^{\upsilon}(\alpha, \beta)$ *be of the form (1.1). Then*

$$|a_2| \leq \sqrt{\frac{2(1-\upsilon)}{(1+2\beta)}}$$

and

$$\left|a_3 - a_2^2\right| \leq \frac{2(1-\upsilon)}{(2-\alpha+\beta)}.$$

Remark 3.10 *1. If we take* $\alpha + \beta = 1$ *in Corollaries 3.7 and 3.9, respectively, then we have the results which were given by Frasin and Aouf [23].*

2. If we take $\alpha + \beta = 1$ *in Corollary 3.9, we obtain that the bounds on* $\left|a_3 - a_2^2\right|$ *given by Altınkaya and Yalcın, [5].*

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References

[1] Airault H, Bouali A. Differential calculus on the Faber polynomials. Bulletin des Sciences Mathématiques 2006; 130: 179-222.

[2] Airault H, Ren J. An algebra of differential operators and generating functions on the set of univalent functions. Bulletin des Sciences Mathématiques 2002; 126: 343-367.

[3] Ali RM, Lee SK, Ravichandran V, Supramaniam S. Coefficient estimates for bi-univalent Ma–Minda starlike and convex functions. Applied Mathematics Letters 2012; 25: 344-351.

[4] Altınkaya S. Bounds for a new subclass of bi-univalent functions subordinate to the Fibonacci numbers. Turkish Journal of Mathematics 2020; 44 (2): 553-560.

[5] Altınkaya S, Yalcın S. Faber polynomial coefficient bounds for a subclass of bi-univalent functions. Comptes Rendus Mathematique 2015; 353 (12): 1075-1080.

[6] Aouf MK, El-Ashwah RM, Abd-Eltawab AM. New subclasses of bi-univalent functions involving Dziok–Srivastava operator. ISRN Mathematical Analysis 2013; Article ID 387178: 5 p.

[7] Bazilevic IE. On a case of integrability in quadratures of the Loewner-Kufarev equation. Matematicheskii Sbornik 1955; 37 (79): 471-476.

[8] Bouali A. Faber polynomials, Cayley-Hamilton equation and Newton symmetric functions. Bulletin des Sciences Mathématiques 2006; 130: 49-70.

[9] De Branges L. A proof of the Bieberbach conjecture. Acta Mathematica 1985; 154: 137-152.

[10] Brannan DA, Clunie JG (Eds). Aspects of Contemporary Complex Analysis (Proceedings of the NATO Advanced Study Institute held at the University of Durham, Durham; July 1–20, 1979), Academic Press, New York and London, 1980.

[11] Brannan DA, Clunie J, Kirwan WE. Coefficient estimates for a class of starlike functions. Canadian Journal of Mathematics 1970; 22: 476-485.

[12] Brannan DA, Taha TS. On some classes of bi-univalent functions. Studia Universitatis Babes-Bolyai Mathematica 1986; 31 (2): 70-77.

[13] Bulut S. Coefficient estimates for initial Taylor–Maclaurin coefficients for a subclass of analytic and bi-univalent functions defined by Al-Oboudi differential operator. Scientific World Journal 2013; Article ID 171039: 6 p.

[14] Bulut S. Coefficient estimates for new subclasses of analytic and bi-univalent functions defined by Al-Oboudi differential operator. Journal of Function Spaces and Applications 2013; Article ID 181932: 7 p.

[15] Bulut S. Coefficient estimates for a class of analytic and bi-univalent functions. Novi Sad Journal of Mathematics 2013; 43 (2): 59-65.

[16] Bulut S. Coefficient estimates for a new subclass of analytic and bi-univalent functions defined by Hadamard product. Journal of Complex Analysis 2014; 2014 Article ID 302019: 7 p.

[17] Caglar M, Orhan H, Yagmur N. Coefficient bounds for new subclasses of bi-univalent functions. Filomat 2013; 27 (7): 1165-1171.

[18] Darwish HE, Lashin AY, Soileh SM. On Certain Subclasses of Starlike p-valent Functions. Kyungpook Mathematical Journal 2016; 56: 867-876.

[19] Deniz E. Certain subclasses of bi-univalent functions satisfying subordinate conditions. Journal of Classical Analysis 2013; 2(1): 49-60.

[20] Deniz E, Jahangiri JM, Hamidi SG, Kina SK. Faber polynomial coefficients for generalized bi–subordinate functions of complex order. Journal of Mathematical Inequalities 2018; 12 (3): 645-653.

[21] Duren PL. Univalent Functions. Grundlehren der mathematischen Wissenschaften, Band 259. New York, NY, USA: Springer-Verlag, 1983.

[22] Faber G. Uber polynomische Entwicklungen. Mathematische Annalen 1903; 57: 385-408.

[23] Frasin BA, Aouf MK. New subclasses of bi-univalent functions. Applied Mathematics Letters 2011; 24: 1569-1573.

[24] Goyal SP, Goswami P. Estimate for initial Maclaurin coefficients of bi-univalent functions for a class defined by fractional derivatives. Journal of the Egyptian Mathematical Society 2012; 20: 179-182.

[25] Goyal SP, Kumar R. Coefficient estimates and quasi-subordination properties associated with certain subclasses of analytic and bi-univalent functions. Mathematica Slovaca 2015; 65 (3): 533-544.

[26] Hamidi SG, Jahangiri JM. Faber polynomial coefficients of bi-subordinate functions. Comptes Rendus Mathematique 2016; 354 (4): 365-370.

[27] Hayami T, Owa S. Coefficient bounds for bi-univalent functions. Pan-American Mathematical Journal 2012; 22 (4): 15-26.

[28] Jahangiri JM, Hamidi SG. Faber polynomial coefficient estimates for analytic bi-bazilevic functions. Matematicki Vesnik 2015; 67 (2): 123-129.

[29] Lashin AY. On certain subclasses of analytic and bi-univalent functions. Journal of the Egyptian Mathematical Society 2016; 24 (2): 220-225.

[30] Lashin AY. Coefficient estimates for two subclasses of analytic and bi-univalent functions. Ukrainian Mathematical Journal 2019; 70 (9): 1484-1492.

[31] Lewin M. On a coefficient problem for bi-univalent functions. Proceedings of the American Mathematical Society 1967; 18: 63-68.

[32] Li X-F, Wang A-P. Two new subclasses of bi-univalent functions. International Mathematical Forum 2012; 7: 1495-1504.

[33] Liu M. On certain subclass of p-valent functions. Soochow Journal of Mathematics 2000; 26 (2): 163-171.

[34] Magesh N, Rosy T, Varma S. Coefficient estimate problem for a new subclass of bi-univalent functions. Journal of Complex Analysis 2013; 2013 Article ID 474231: 3p.

[35] Magesh N, Yamini J. Coefficient bounds for certain subclasses of bi-univalent functions. International Mathematical Forum 2013; 8: 1337-1344.

[36] Murugusundaramoorthy G, Magesh N, Prameela V. Coefficient bounds for certain subclasses of bi-univalent function. Abstract and Applied Analysis 2013; Article ID 573017: 3 p.

[37] Peng Z-G, Han Q-Q. On the coefficients of several classes of bi-univalent functions. Acta Mathematica Sinica, English Series 2014; 34: 228-240.

[38] Ponnusamy S. Polya-Schoenberg conjecture by Caratheodory functions. Journal of the London Mathematical Society 1995; 51 (2): 93-104.

[39] Ponnusamy S, Rajasekaran S. New sufficient conditions for starlike and univalent functions. Soochow Journal of Mathematics 1995; 21 (2): 193-201.

[40] Porwal S, Darus M. On a new subclass of bi-univalent functions. Journal of the Egyptian Mathematical Society 2013; 21 (3): 190-193.

[41] Prajapat JK, Agarwal R. Some results on certain class of analytic functions based on differential subordination. Bulletin of the Korean Mathematical Society 2013; 50 (1): 1-10.

[42] Srivastava HM, Bulut S, Caglar M, Yagmur N. Coefficient estimates for a general subclass of analytic and bi-univalent functions. Filomat 2013; 27 (5): 831-842.

[43] Srivastava HM, Mishra AK, Gochhayat P. Certain subclasses of analytic and bi-univalent functions. Applied Mathematics Letters 2010; 23: 1188-1192.

[44] Srivastava HM, Murugusundaramoorthy G, Magesh N. Certain subclasses of bi-univalent functions associated with the Hohlov operator. Global Journal of Mathematical Analysis 2013; 1 (2): 67-73.

[45] Srivastava HM, Murugusundaramoorthy G, Vijaya K. Coefficient estimates for some families of bi-Bazilevic functions of the Ma–Minda type involving the Hohlov operator. Journal of Classical Analysis 2013; 2: 167-181.

[46] Taha TS. Topics in univalent function theory. PhD, University of London, London, UK, 1981.

[47] Tang H, Deng G-T, Li S-H. Coefficient estimates for new subclasses of Ma–Minda bi-univalent functions. Journal of Inequalities and Applications 2013; 2013 Article ID 317: 10 p.

[48] Xu Q-H, Gui Y-C, Srivastava HM. Coefficient estimates for a certain subclass of analytic and bi-univalent functions. Applied Mathematics Letters 2012; 25: 990-994.

[49] Xu Q-H, Xiao H-G, Srivastava HM. A certain general subclass of analytic and bi-univalent functions and associated coefficient estimate problems. Applied Mathematics and Computation 2012; 218 (23): 11461-11465.

[50] Yang D. Some multivalent starlikeness conditions for analytic functions. Bulletin of the Institute of Mathematics Academia Sinica 2005; 33 (1): 55-67.

[51] Zireh A, Adegani EA, Bidkham M. Faber polynomial coefficient estimates for subclass of bi-univalent functions defined by quasi-subordinate. Mathematica Slovaca 2018; 68 (2): 369-378.