Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1376 – 1387

doi:10.3906/mat-1908-90

Remarks on the one-dimensional sloshing problem involving the p-Laplacian operator

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Received: 26.08.2019 • **Accepted/Published Online:** 28.05.2020 • **Final Version:** 08.07.2020

Abstract: In this paper, we study the inverse nodal problem and the eigenvalue gap for the one-dimensional sloshing problem with the p-Laplacian operator. By applying the Prüfer substitution, we first derive the reconstruction formula of the depth function by using the information of the nodal data. Furthermore, we employ the Tikhonov regularization method to consider how to reconstruct the depth function using only zeros of one eigenfunction. Finally, we investigate the eigenvalue gap under the restriction of symmetric single-well depth functions. We show the gap attains its minimum when the depth function is constant.

Key words: Inverse nodal problem, Tikhonov regularization method, eigenvalue gap, p-Laplacian

1. Introduction

Liquid sloshing problems appear in many circumstances, such as the storage and transportation of fuel tanks, the design of ship hulls, and the motion of liquid fuel in aircraft and spacecraft (see [4, 16, 25–28] and the references therein). The planar fluid motion in a shallow container is described by the following differential system:

$$\begin{array}{ll} \nabla^2\varphi = 0 & \text{in the interior of the container } V, \\ \frac{\partial\varphi}{\partial n} = 0 & \text{on the container wall,} \\ \frac{\partial\varphi}{\partial n} = \lambda\varphi & \text{on the horizontal free surface } S, \end{array}$$

where φ is the velocity potential, $\lambda = \omega^2/g$ is the eigenvalue, ω is the circular frequency, and g is the acceleration due to gravity. If we consider the fluid sloshing motion in the half-container, the corresponding eigenvalue problem can be rewritten as

$$\begin{cases} (h\,(x)\,u')' + \lambda u = 0, & x \in (0,1), \\ h(0)u'(0) = 0, \quad h(1)u'(1) = 0, \end{cases}$$

where $h(x) \in C[0,1]$ with $h(x) > 0$ is the container depth. The p-Laplacian operator $\triangle_p u = div(|\nabla u|^{p-2}\nabla u)$ has attracted considerable attention and arises in various fields, such as non-Newtonian fluids and nonlinear diffusion problems. The quantity p is a characteristic of the fluid medium. Media with $p > 2$ are called dilatant fluids and those with $p < 2$ are called pseudoplastics. If $p = 2$, they are Newtonian fluids. For more on these topics, the reader can refer to [12, 13, 17, 18, 21, 24] and their bibliographies.

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2010 *AMS Mathematics Subject Classification:* 34L30, 34B15.

When considering the two topics related to fluid dynamics mentioned above, it is intuitive to generalize the situation. In this paper we consider the one-dimensional liquid sloshing problem with the p-Laplacian:

$$\begin{cases} (h(x)|u'|^{p-2}u')' + (p-1)\lambda|u|^{p-2}u = 0, & x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $p > 1$. In 1979, Elbert [14] showed that the function $S_p(x)$ defined by the integral

$$x = \int_0^{S_p} \frac{dt}{(1-t^p)^{1/p}}$$

is a solution of the initial value problem

$$(|u'|^{p-2}u')' + (p-1)|u|^{p-2}u = 0, \qquad u(0) = 0, \quad u'(0) = 1.$$

He also showed that $\pi_p \equiv 2\int_0^{S_p}(1-t^p)^{-1/p}\,dt = 2\pi/(p\sin(\pi/p))$ is the first zero of the function S_p on $(0,\infty)$ and that the function S_p has the following properties:

(a) $|S_p(x)|^p + |S_p'(x)|^p = 1$ for $x \in \mathbb{R}$.

(b) $|S_p'(x)|^{p-2}S_p''(x) = -|S_p(x)|^{p-2}S_p(x)$ for $x \in \mathbb{R}$.

(c) Let $T_p(x) = S_p(x)/S_p'(x)$. Then $T_p'(x) = 1 + |T_p(x)|^p$ for $x \in \mathbb{R}$.

Note that the function S_p is called the *generalized sine function* and it is crucial to our analysis. In this paper, we study the inverse nodal problem and the eigenvalue gaps in the sloshing problem (1.1). The inverse nodal problem is the problem of determining the unknown function in the system from the information contained in the zeros of solutions. One can refer to [6, 9, 15, 20, 23, 29, 30] for the classical Sturm-Liouville equation and [5, 10] for the Sturm-Liouville equation with the p-Laplacian operator. Applying the Prüfer substitution (see [1, 3, 7]), it can be shown that the sloshing problem (1.1) has countably infinitely many eigenpairs $(u_n, \lambda_n)_{n\geq 0}$, and u_n has n zeros $\{x_k^{(n)} : 1 \leq k \leq n\}$ in $(0,1)$. In particular, $\lambda_0 = 0$, the corresponding eigenfunction u_0 is constant, and the nodal set $\{x_k^{(n)} : 1 \leq k \leq n; n = 1, 2, ...\}$ is dense in $(0,1)$.

Our first issue is related to the inverse nodal problems in the sloshing problem (1.1). In Theorem 1.1, we reconstruct the depth function $h(x)$ from the information of the nodal data. The convergence is almost everywhere (a.e.) for $x \in (0,1)$ and in the $L^1(0,1)$ sense. This also implies that one set of nodal points can uniquely determine the depth function $h(x)$. This is parallel to the previous consideration; see, for example, [9, 10, 15, 20, 23, 29, 30]. Note that more recently Chen et al.[5, 6] employed an original and ingenious idea towards solving a related problem. They applied the Tikhonov regularization method to reconstruct the potential function using only zeros of one eigenfunction and provided a process to find the optimal function coinciding with the finitely given nodal points. By virtue of their idea, we intend to utilize this method on (1.1) (cf. Theorem 1.2).

Now say we have a set of $n-1$ nodal points of the n-th eigenfunction and denote

$$\mathbf{X}(n) := \left\{(x_1, x_2, ..., x_{n-1}) \in \mathbb{R}^{n-1} | 0 < x_1 < \cdots < x_{n-1} < 1\right\} \text{ and}$$

$$\mathbf{Q} := \left\{ h \in W^{1,r}\left((0,1)\right) \mid \int_0^1 h(x)dx = 1 \right\}, \quad r \in (1,\infty).$$

For $\mathbf{x} \in \mathbf{X}(n)$, $\varepsilon > 0$ and $r > 1$, we define the Tikhonov functional E by

$$E(n,\varepsilon,\mathbf{x};h) := \|\mathbf{x} - \mathbf{z}(n,h)\|_r^r + \varepsilon \int_0^1 |h'(x)|^r \, dx \quad \text{ for } h \in \mathbf{Q}, \tag{1.2}$$

where $\mathbf{z}(n,h)$ is the nth nodal set associated with the depth function h and $\|\mathbf{x} - \mathbf{z}(n,h)\|_r$ is the ℓ^r-norm in $\mathbb{R}^{n-1}$. We apply the Tikhonov regularization method on (1.1) to show the existence of the optimal depth function. In particular, we calculate the first variation of $E(n,\varepsilon,\mathbf{x};h)$ in h to derive Euler-Lagrange equations so that we can find an optimal depth function $h(x)$ for finite nodal data by solving the Euler-Lagrange equations. The following result is related to the inverse nodal problem.

Theorem 1.1 *Let $\{x_k^{(n)}\}$ be a nodal set of the one-dimensional sloshing problem (1.1). For $x \in (0,1)$, define $j_n \equiv j_n(x) = \max\left\{k : x \leq x_k^{(n)}\right\}$. Then the depth $h(x)$ can be reconstructed from the nodal set by*

$$h^{\frac{1}{1-p}}(x) = \lim_{n\to\infty} \frac{1}{(p-1)} \left\{ \frac{1 + (p-1)\int_0^1 h^{\frac{1}{1-p}}(x)\,dx}{(n-1)\,\ell_{j_n}^{(n)}} - 1 \right\}. \tag{1.3}$$

The convergence is a.e. in $x \in (0,1)$ and in the $L^1(0,1)$ sense.

Theorem 1.2 *Let $\varepsilon > 0$, $n \in \mathbb{N}$, $r > 1$ and $\mathbf{x} \in X(n)$ be given. If h_ε is a minimizer of $E(n,\varepsilon,\mathbf{x};h)$, then $(\mathbf{z}, u, h_\varepsilon)$ is a solution of the following system*

$$\begin{cases} (h_\varepsilon(x)\,|u'|^{p-2}\,u')' + (p-1)\,\lambda\,|u|^{p-2}\,u = 0 \\ u(0) = 1, u'(0) = 0, \qquad \{z_1, z_2, ..., z_{n-1}\} = \{x \in [0,1] : u(x) = 0\}, \\ \left(|h_\varepsilon'|^{r-2}\,h_\varepsilon'\right)' = |u'|^p \sum_{i=0}^{n-1} a_k \chi_{[z_i,z_{i+1}]}, \; z_0 = 0, \; z_n = 1 \\ h_\varepsilon'(0) = h_\varepsilon'(1) = 0, \end{cases} \tag{1.4}$$

where (λ, u) is the nth eigenpair of (1.1), $\mathbf{z}$ is the nodal set of u, and

$$a_0 = \frac{1}{\varepsilon}\sum_{i=1}^{n-1} \frac{|z_i - x_i|^{r-2}\,(z_i - x_i)}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p} \frac{\int_{z_i}^1 |u|^p}{\int_0^1 |u|^p}, \quad a_k = a_0 + \frac{1}{\varepsilon}\sum_{i=1}^{k} \frac{|z_i - x_i|^{r-2}\,(z_i - x_i)}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p} \text{ for } k = 1, 2, ..., n-1.$$

In the Figure, we give a numerical simulation for the reconstruction formulas (1.3) and (1.4) from the true nodal data with $n = 21$ and $p = 3$. By the definition in (1.3), the reconstruction is constant in each nodal interval $(x_k^{(n)}, x_{k+1}^{(n)})$ and hence, the reconstruction obtained from (1.3) is a step function. On the other hand, (1.4) can be solved numerically by a finite difference method, and one can reasonably assume that the solution is quite smooth. Thus, the reconstruction obtained by solving (1.4) is presented as a smooth curve.

On the other hand, the eigenvalue gap/ratio is more of a concern. Consider

$$-(|y'|^{p-2}\,y')' = (p-1)\,(\lambda\rho(x) - q(x))\,|y|^{p-2}\,y \quad \text{ on } (0, \pi_p). \tag{1.5}$$

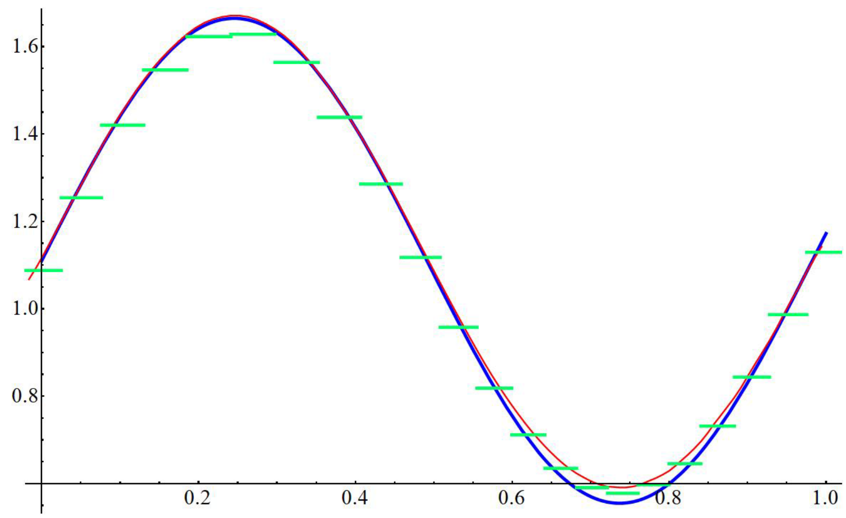

Figure 1. Comparison of the reconstruction with $n = 21$ and $p = 3$: (i) the blue curve is the true depth function h; (ii) the step function is a reconstruction using (1.3) from the nodal set $z(n,h)$; (iii) the red curve is the numerical approximation obtained using an iterative technique for finding the minimizer h_ε in (1.4) with $\varepsilon = n^{-3}$.

We say a function f is "single-well" with transition point a if f is decreasing on $(0,a)$ and increasing on (a,π_p); f is "single-barrier" if $-f$ is single-well. Based on the generalized Prüfer transformation and comparison theorem, researchers have investigated the eigenvalue ratios of (1.5). In 2010, Bognár and Dosly [2] showed that the Dirichlet eigenvalues for (1.5) with $\rho \equiv 1$ and nonnegative single-well $q(x)$ satisfy $\mu_n/\mu_m \le n^p/m^p$. In 2013, Chen, Law, Lian and Wang [8] showed the Dirichlet eigenvalues for (1.5) with $\rho \equiv 1$ and nonnegative continuous $q(x)$ satisfy $\mu_n/\mu_1 \le n^p$. In 2015, Li [19] studied the problem

$$-(|h(x)y'|^{p-2}\, h(x)y')' = (p-1)\,(\lambda\rho(x) - q(x))|y|^{p-2}y$$

coupled with the separated boundary conditions

$$\begin{cases} y(0)S_p'(\gamma) + h(0)y'(0)S_p(\gamma) = 0, \\ y(1)S_p'(\delta) - h(1)y'(1)S_p(\delta) = 0, \end{cases}$$

where $0 \le \gamma, \delta < \pi_p$ and found the upper bound and lower bound of the eigenvalue ratio λ_m/λ_n when $0 < k \le h\rho \le K$ and $q \ge 0$. On the other hand, by investigating the properties of the generalized trigonometric functions, the authors in [11] studied the first two Dirichlet eigenvalues for (1.5) and showed that (i) $\mu_2 - \mu_1 \ge 2^p - 1$ if $\rho \equiv 1$ and $q(x)$ is single-well with transition point at $\pi_p/2$; (ii) $\mu_2/\mu_1 \ge 2^p$ if $q(x) \equiv 0$ and $\rho(x)$ is single-barrier with transition point at $\pi_p/2$.

In the final theorem, we study the eigenvalue gap $\lambda_2 - \lambda_1$ for the symmetric single-well h. We give an optimal lower bound and show that the minimum is attained when $h(x) \equiv 1$

Theorem 1.3 *Consider the one-dimensional sloshing problem (1.1) with symmetric single-well depth $h(x) \ge 1$. Then the eigenvalue gap $\lambda_2 - \lambda_1 \ge (2^p - 1)\,\pi_p^p$. The equality holds only for $h(x) \equiv 1$.*

The paper is organized as follows. In Section 2, we apply the modified Prüfer substitution to derive the asymptotic expansion for eigenvalues and nodal points. Then using the information of the nodal data, we give a reconstruction formula of the depth function $h(x)$. In section 3, we show the minimizer of E exists. Furthermore, we derive Euler-Lagrange equations for the Tikhonov functional and prove Theorem 1.2. In section 4, we use the homotopy method to prove that $\lambda_2-\lambda_1$ attains its minimum when h is a constant function, and then give a proof of Theorem 1.3.

2. Reconstruction from a set of nodal data

Consider the Prüfer substitution

$$u(x)=r(x)S_p\left(\lambda^{\frac{1}{p}}\theta(x)\right), \qquad h^{\frac{1}{p-1}}(x)u'(x)=r(x)\lambda^{\frac{1}{p}}S_p'\left(\lambda^{\frac{1}{p}}\theta(x)\right).$$

We can find

$$\theta'(x)=h^{\frac{1}{1-p}}(x)\left|S_p'\left(\lambda^{\frac{1}{p}}\theta(x)\right)\right|^p+\left|S_p\left(\lambda^{\frac{1}{p}}\theta(x)\right)\right|^p. \tag{2.1}$$

Applying some standard arguments (see [1, 3, 7]), it can be shown that (1.1) has countably many eigenpairs $\{u_n,\lambda_n\}_{n\geq 0}$ and u_n has n zeros $\left\{x_k^{(n)}\right\}$ in $(0,1)$. Note that $\theta(0,\lambda_n)=\frac{1}{2}\pi_p\lambda_n^{-\frac{1}{p}}$, $\theta(1,\lambda_n)=(n+1/2)\,\pi_p\lambda_n^{-\frac{1}{p}}$ and $\theta(x_k^{(n)},\lambda_n)=k\pi_p\lambda_n^{-\frac{1}{p}}$ for $k=0,1,2,...,n$. Moreover, by integrating (2.1) from 0 to 1, we find that

$$\begin{aligned} n\pi_p\lambda_n^{-\frac{1}{p}} &= \int_0^1\theta_n'(x)dx \\ &= \frac{1}{p}+\frac{p-1}{p}\int_0^1 h^{\frac{1}{1-p}}(x)\,dx+\int_0^1\left(1-h^{\frac{1}{1-p}}(x)\right)\left(\left|S_p'\left(\lambda_n^{\frac{1}{p}}\theta(x)\right)\right|^p-\frac{1}{p}\right)dx. \end{aligned}$$

This implies that $\lambda_n^{1/p}=O(n)$. Hence, by $\lambda_n^{1/p}=O(n)$, (2.1) and the general Riemann-Lebesgue Lemma (see [10, Lemma 3.1]) , we can obtain

$$\lambda_n^{\frac{1}{p}}=\frac{p\,(n-1)\,\pi_p}{1+(p-1)\int_0^1 h^{\frac{1}{1-p}}(x)\,dx}+o(n). \tag{2.2}$$

Similarly, by integrating (2.1) from $x_k^{(n)}$ to $x_{k+1}^{(n)}$ and letting $\ell_k^{(n)}=x_{k+1}^{(n)}-x_k^{(n)}$, we find that

$$\begin{aligned} \pi_p\lambda_n^{-\frac{1}{p}} &= \int_{x_k^{(n)}}^{x_{k+1}^{(n)}}\theta_n'(x)dx \\ &= \frac{\ell_k^{(n)}}{p}+\frac{p-1}{p}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} h^{\frac{1}{1-p}}(x)\,dx+\int_{x_k^{(n)}}^{x_{k+1}^{(n)}}\left(1-h^{\frac{1}{1-p}}(x)\right)\left(\left|S_p'\left(\lambda_n^{\frac{1}{p}}\theta(x)\right)\right|^p-\frac{1}{p}\right)dx. \end{aligned}$$

Hence,

$$\ell_k^{(n)}=p\pi_p\lambda_n^{-\frac{1}{p}}-(p-1)\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} h^{\frac{1}{1-p}}(x)\,dx+o(\frac{1}{n}).$$

Now, we are prepared to prove Theorem 1.1.

Proof [Proof of Theorem 1.1]

For $x \in (0,1)$, let $j_n \equiv j_n(x) = \max\left\{k : x \le x_k^{(n)}\right\}$ for convenience . Since the nodal set is dense in $[0,1]$, we find that the sequence of intervals $\left\{\left[x_{j_n}^{(n)}, x_{j_n+1}^{(n)}\right) : n \text{ is sufficiently large}\right\}$ shrinks to x nicely (cf. Rudin [22, p.140]). Hence, we can obtain

$$\frac{1}{\ell_{j_n}^{(n)}} \int_{x_{j_n}^{(n)}}^{x_{j_n+1}^{(n)}} h^{\frac{1}{1-p}}(t)\, dt \to h^{\frac{1}{1-p}}(x) \quad \text{a.e. for } x \in (0,1).$$

In particular, by the Lebesgue dominated convergence theorem, we find

$$\frac{1}{\ell_{j_n}^{(n)}} \int_{x_{j_n}^{(n)}}^{x_{j_n+1}^{(n)}} h^{\frac{1}{1-p}}(t)\, dt \to h^{\frac{1}{1-p}}(x) \quad \text{in } L^1(0,1).$$

Hence, we can find

$$h^{\frac{1}{1-p}}(x) = \lim_{n\to\infty} \frac{1}{(p-1)} \left(\frac{p\pi_p}{\lambda_n^{\frac{1}{p}} \ell_{j_n}^{(n)}} - 1 \right)$$

a.e. for $x \in (0,1)$ and in $L^1(0,1)$. Finally, by the eigenvalue asymptotic expansion (2.2), we can conclude that

$$h^{\frac{1}{1-p}}(x) = \lim_{n\to\infty} \frac{1}{(p-1)} \left\{ \frac{1 + (p-1)\int_0^1 h^{\frac{1}{1-p}}(t)\, dt}{(n-1)\, \ell_{j_n}^{(n)}} - 1 \right\}$$

a.e. for $x \in (0,1)$ and in $L^1(0,1)$. $\square$

3. Tikhonov regularization for the inverse nodal problem

In this section, we apply the Tikhonov regularization method to study the minimization problem for the sloshing problem (1.1). We will reconstruct an optimal depth function by using finite nodal data. To do this, we first show that $E(n, \varepsilon, \mathbf{x}; \cdot)$ has a minimizer in $\mathbf{Q}$.

Theorem 3.1 *Given $n \in N$, $\varepsilon > 0$ and $\mathbf{x} \in \mathbf{X}(n)$, $E(n, \varepsilon, \mathbf{x}; \cdot)$ has at least one minimizer in $\mathbf{Q}$.*

Proof For $r > 1$, since $E(n, \varepsilon, \mathbf{x}; \cdot)$ is a nonnegative well-defined functional on $\mathbf{Q}$, we find that there is a sequence $\{h_i\}_{i=1}^{\infty} \subset \mathbf{Q}$ such that

$$\lim_{i\to\infty} E(n, \varepsilon, \mathbf{x}; h_i) = \inf_{h\in\mathbf{Q}} E(n, \varepsilon, \mathbf{x}; h).$$

This implies that $\{h_i\}$ is a bounded family in $W^{1,r}([0,1])$. Hence, by applying the weak compactness argument on $\mathbf{Q}$, there exists a subsequence of $\{h_i\}$, also called $\{h_i\}$, such that

$$h_i \to h_\varepsilon \in \mathbf{Q} \quad \text{weakly in } W^{1,r}([0,1]) \quad \text{and} \quad \text{uniformly in } C([0,1]).$$

Furthermore, by the weak lower semicontinuity of the functional $\int_0^1 |h'|^r$, we have

$$\int_0^1 |h_\varepsilon'(x)|^r\, dx \le \liminf_{i\to\infty} \int_0^1 |h_i'(x)|^r\, dx.$$

In particular, we also have

$$E(n,\varepsilon,\mathbf{x};h_\varepsilon) \le \lim\inf_{i\to\infty} E(n,\varepsilon,\mathbf{x};h_i) = \inf_{h\in\mathbf{Q}} E(n,\varepsilon,\mathbf{x};h).$$

This concludes that h_ε is a minimizer of $E(n,\varepsilon,\mathbf{x};\cdot)$ in $\mathbf{Q}$.

□

In the following, we derive the Euler-Lagrange equation for the sloshing problem (1.1). We will show the minimization problem can be formulated as shown in Theorem 1.2. To achieve our goal, we calculate the first variation of $E(n,\varepsilon,\mathbf{x};h)$ in h, i.e. solve

$$\lim_{t\to 0} \frac{E(n,\varepsilon,\mathbf{x};h_\varepsilon+t\nu)-E(n,\varepsilon,\mathbf{x};h_\varepsilon)}{t} = 0.$$

For $t\in\mathbb{R}$, let $\varphi(t) = h_\varepsilon + t\nu$, $E_0(t) = E(n,\varepsilon,\mathbf{x};\varphi(t))$, $Z(t) = (Z_1(t),...,Z_{n-1}(t)) = \mathbf{z}(n,\varphi(t))$, $\Lambda(t) = \lambda(n,\varphi(t))$ and $U(x,t) = u(x;n,\varphi(t))$. Denote $' = \frac{d}{dx}$ and $\dot{} = \frac{d}{dt}$. Let $V(x,t) = \dot{U}(x,t)$. Then $Z(0) = \mathbf{z}(n,h_\varepsilon)$, $\Lambda(0) = \lambda$ and $U(x,0) = u(x)$. Moreover, we have

$$\dot{E}_0(t) = r\,\|Z-\mathbf{x}\|_r^{r-1}\cdot\dot{Z} + \varepsilon r\int_0^1 (|h_\varepsilon' + t\nu'|^{r-2}(h_\varepsilon'+t\nu')\nu'.$$

On the other hand, since $U(Z_i(t),t) = 0$, we have

$$\dot{Z}_i(t) = -\frac{V(Z_i(t),t)}{U'(Z_i(t),t)}.$$

Now, differentiating the equation

$$(\varphi(t)\,|U'|^{p-2}\,U')' + (p-1)\,\Lambda\,|U|^{p-2}\,U = 0 \tag{3.1}$$

with respect to t, we find $V'(0,t) = V'(1,t) = 0$ and

$$(\nu\,|U'|^{p-2}\,U')' + (p-1)(\varphi(t)\,|U'|^{p-2}\,V')' + (p-1)\,\dot{\Lambda}\,|U|^{p-2}\,U + (p-1)^2\,\Lambda\,|U|^{p-2}\,V = 0 \tag{3.2}$$

By (3.1) and (3.2), we have

$$(\varphi(t)\,|U'|^{p-2}\,U')'V - (\varphi(t)\,|U'|^{p-2}\,V')'U = \frac{1}{p-1}(\nu\,|U'|^{p-2}\,U')'U + \dot{\Lambda}\,|U|^p \tag{3.3}$$

Since $U'(0,t) = U'(1,t) = V'(0,t) = V'(1,t) = 0$, we can integrate (3.3) from 0 to x and apply integration by parts to obtain

$$\varphi(t)\left(|U'|^{p-2}\,U'V - |U'|^{p-2}\,V'U\right) = \frac{1}{p-1}\nu\,|U'|^{p-2}\,U'U - \frac{1}{p-1}\int_0^x \nu\,|U'|^p + \dot{\Lambda}\int_0^x |U|^p. \tag{3.4}$$

We can take $x=1$ in (3.4) to obtain

$$\dot{\Lambda}(t) = \frac{\int_0^1 \nu\,|U'|^p}{(p-1)\int_0^1 |U|^p}.$$

On the other hand, since $z_i = Z_i(0)$, $u(x) = U(x,0)$ and $u(z_i) = 0$, we can take $x = z_i$ and $t = 0$ in (3.4) to obtain

$$h_\varepsilon(z_i)\,|u'(z_i)|^{p-2}\,u'(z_i)V(z_i,0) = -\frac{1}{p-1}\int_0^{z_i} \nu\,|u'|^p + \dot{\Lambda}(0)\int_0^{z_i} |u|^p\,.$$

Hence

$$\begin{aligned}
\dot{Z}_i(0) &= -\frac{V(z_i,0)}{u'(z_i)} \\
&= \frac{-1}{h_\varepsilon(z_i)\,|u'(z_i)|^p}\left[-\frac{1}{p-1}\int_0^{z_i} \nu\,|u'|^p + \dot{\Lambda}(0)\int_0^{z_i} |u|^p\right] \\
&= \frac{-1}{h_\varepsilon(z_i)\,|u'(z_i)|^p}\left[-\frac{1}{p-1}\int_0^{z_i} \nu\,|u'|^p + \frac{\int_0^1 \nu\,|u'|^p}{(p-1)\int_0^1 |u|^p}\int_0^{z_i} |u|^p\right] \\
&= \frac{1}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p}\left[\int_0^{z_i} \nu\,|u'|^p - \frac{\int_0^1 \nu\,|u'|^p \int_0^{z_i} |u|^p}{\int_0^1 |u|^p}\right] \\
&= \frac{1}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p}\int_0^1 \nu\,|u'|^p\left[\chi_{[0,z_i]} - \frac{\int_0^{z_i} |u|^p}{\int_0^1 |u|^p}\right] \\
&= \frac{1}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p}\int_0^1 \nu\,|u'|^p\left[\frac{\int_{z_i}^1 |u|^p}{\int_0^1 |u|^p} - \chi_{[z_i,1]}\right].
\end{aligned}$$

This implies that the first variation of $E(n,\varepsilon,\mathbf{x};h)$ in h is

$$\begin{aligned}
0 &= \dot{E}_0(0) \\
&= r\,\|Z(0)-\mathbf{x}\|_r^{r-1}\cdot \dot{Z}(0) + \varepsilon r\int_0^1 |h_\varepsilon'|^{r-2}\,h_\varepsilon'\nu' \\
&= r\sum_{i=1}^{n-1}\frac{|z_i-x_i|^{r-2}\,(z_i-x_i)}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p}\int_0^1 \nu\,|u'|^p\left[\frac{\int_{z_i}^1 |u|^p}{\int_0^1 |u|^p} - \chi_{[z_i,1]}\right] + \varepsilon r\int_0^1 |h_\varepsilon'|^{r-2}\,h_\varepsilon'\nu' \\
&= r\varepsilon\int_0^1\left\{|h_\varepsilon'|^{r-2}\,h_\varepsilon'\nu' + \frac{1}{\varepsilon}\nu\,|u'|^p\sum_{i=1}^{n-1}\frac{|z_i-x_i|^{r-2}\,(z_i-x_i)}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p}\left(\frac{\int_{z_i}^1 |u|^p}{\int_0^1 |u|^p} - \chi_{[z_i,1]}\right)\right\}
\end{aligned}$$

Set $\mathbf{a} = (a_0, a_1, ..., a_{n-1})$ where

$$a_0 = \frac{1}{\varepsilon}\sum_{i=1}^{n-1}\frac{|z_i-x_i|^{r-2}\,(z_i-x_i)}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p}\frac{\int_{z_i}^1 |u|^p}{\int_0^1 |u|^p} \quad \text{and} \quad a_k = a_0 + \frac{1}{\varepsilon}\sum_{i=1}^{k}\frac{|z_i-x_i|^{r-2}\,(x_i-z_i)}{(p-1)\,h_\varepsilon(z_i)\,|u'(z_i)|^p} \text{ for } k = 1,2,...,n-1.$$

We obtain the following equations about the minimizer:

$$\begin{aligned}
\left(|h_\varepsilon'|^{r-2}\,h_\varepsilon'\right)' &= |u'|^p\sum_{i=0}^{n-1} a_k\chi_{[z_i,z_{i+1}]}, z_0 = 0, z_n = 1 \\
h_\varepsilon'(0) &= h_\varepsilon'(1) = 0.
\end{aligned}$$

This completes the proof of Theorem 1.2.

4. Eigenvalue gap for symmetric single-well depth

In this section, we study the eigenvalue gap for $\lambda_2 - \lambda_1$ where $\{\lambda_n\}_{n\geq 0}$ is the eigenvalue set of (1.1). We first observe that, if $h(x) \equiv m > 0$, then the normalized eigenpairs are $u_n(x) = p^{1/p} S_p\left(n\pi_p x + \pi_p/2\right)$ and $\lambda_n = m\left(n\pi_p\right)^p$. Hence, $\lim_{m\to 0^+}\lambda_n = 0$ and $\lim_{m\to\infty}\lambda_n = \infty$ for all $n \in \mathbb{N}$. In particular, $\lambda_2 - \lambda_1$ is increasing in m .

Now, let $h(x,t)$ be a one-parameter family of functions such that $\frac{\partial}{\partial t}h(x,t)$ exists and let $(\lambda_n(t), u_n(x,t))_{n\geq 0}$ be the nth normalized eigenpair of (1.1). We have the following lemma.

Lemma 4.1 *Consider (1.1) with the function $h(x,t)$. Then we have*

$$\dot{\lambda}_n(t) = \int_0^1 \dot{h}(x,t)\left|u_n'(x,t)\right|^p dx. \tag{4.1}$$

Proof First, replacing $h(x)$ by $h(x,t)$, (1.1) is reduced to

$$\left(h(x,t)\left|u_n'(x,t)\right|^{p-2}u_n'(x,t)\right)' + \lambda(t)\left|u_n(x,t)\right|^{p-2}u_n(x,t) = 0, \quad 0\leq x\leq 1. \tag{4.2}$$

Denote $\dot{y}(x,t) \equiv \frac{\partial}{\partial t}y(x,t)$. Then, differentiating (4.2) with respect to t and combining with (4.2), we obtain

$$\begin{aligned}\dot{\lambda}(t)\left|u_n(x,t)\right|^p &= (p-1)\left(h(x,t)\left|u_n'(x,t)\right|^{p-2}u_n'(x,t)\right)'\dot{u}_n(x,t)\\ &\quad -\left(\dot{h}(x,t)\left|u_n'(x,t)\right|^{p-2}u_n'(x,t) + (p-1)\,h(x,t)\left|u_n'(x,t)\right|^{p-2}\dot{u}_n'(x,t)\right)'u_n(x,t)\end{aligned}$$

Integrating the above equation from 0 to 1 with respect to x, using integration by parts, we obtain

$$\dot{\lambda}_n(t) = \int_0^1 \dot{h}(x,t)\left|u_n'(x,t)\right|^p dx.$$

□

Next, we prove that $|u_1'|$ and $|u_2'|$ have at most two intersections in $(0,1)$ for the eigenfunctions u_1 and u_2.

Lemma 4.2 *For the normalized eigenfunctions u_1 and u_2 of (1.1), the equation $|u_1'(x)| = |u_2'(x)|$ has at most two solutions in $(0,1)$.*

Proof Let $\alpha_0 \in (0,1)$ be a local minimum of $u_2(x)$. Then $u_2'(\alpha_0) = 0$. We will first show that $u_1'(x) < 0$ on $(0,1)$ while $u_2'(x) < 0$ on $(0,\alpha_0)$ and $u_2'(x) > 0$ on $(\alpha_0, 1)$. Let $x_1^{(1)}$ be the zero of $u_1(x)$. Then $u_1(x) > 0$ on $(0, x_1^{(1)})$ and $u_1(x) < 0$ on $(x_1^{(1)}, 1)$. In particular, we have, for $x \in (0, x_1^{(1)})$,

$$h(x)\left|u_1'(x)\right|^{p-2}u_1'(x) = \int_0^x \left(h(z)\left|u_1'(z)\right|^{p-2}u_1'(z)\right)'dz = -\lambda_1\int_0^x \left|u_1(z)\right|^{p-2}u_1(z)dz < 0,$$

and for $x \in (x_1^{(1)}, 1)$,

$$-h(x)\left|u_1'(x)\right|^{p-2}u_1'(x) = \int_x^1 \left(h(z)\left|u_1'(z)\right|^{p-2}u_1'(z)\right)'dt = -\lambda_1\int_x^1 \left|u_1(z)\right|^{p-2}u_1(z)dt > 0.$$

Hence, $u_1'(x)<0$ on $(0,1)$. Similarly, we can show that $u_2'(x)<0$ on $(0,\alpha_0)$ and $u_2'(x)>0$ on $(\alpha_0,1)$. This implies that $u_2(x)$ has only one local minimum in $(0,1)$.

In order to compare the behaviors of $u_1'(x)$ and $u_2'(x)$, we introduce a Prüfer substitution

$$u_n(x)=r(x)S_p\left(\varphi_n(x)\right), \qquad h^{\frac{1}{p-1}}(x)u_n'(x)=r(x)S_p'\left(\varphi_n(x)\right).$$

It can be shown that $\varphi_n(0)=\pi_p/2$,

$$\varphi_n'(x)=h^{\frac{1}{p-1}}(x)\left|S_p'\left(\varphi_n(x)\right)\right|^p+\lambda_n\left|S_p\left(\varphi_n(x)\right)\right|^p$$

and, by the comparison theorem [3], we have $\varphi_2(x)>\varphi_1(x)$ on $(0,\alpha_0)$. Now, we calculate that

$$\begin{aligned}
&\left(\frac{|u_2'(x)|^{p-2}u_2'(x)}{|u_1'(x)|^{p-2}u_1'(x)}\right)'\\
=\;&\left(\frac{h(x)|u_2'(x)|^{p-2}u_2'(x)}{h(x)|u_1'(x)|^{p-2}u_1'(x)}\right)'\\
=\;&\frac{\left(h(x)|u_2'(x)|^{p-2}u_2'(x)\right)'h(x)|u_1'(x)|^{p-2}u_1'(x)-h(x)|u_2'(x)|^{p-2}u_2'(x)\left(h(x)|u_1'(x)|^{p-2}u_1'(x)\right)'}{\left(h(x)|u_1'(x)|^{p-2}u_1'(x)\right)^2}\\
=\;&\frac{-\left(\lambda_2|u_2(x)|^{p-2}u_2(x)\right)h(x)|u_1'(x)|^{p-2}u_1'(x)+h(x)|u_2'(x)|^{p-2}u_2'(x)\left(\lambda_1|u_1(x)|^{p-2}u_1(x)\right)}{\left(h(x)|u_1'(x)|^{p-2}u_1'(x)\right)^2}\\
=\;&\frac{\lambda_1h(x)|u_2'(x)|^{p-2}u_2'(x)|u_1(x)|^{p-2}u_1(x)-\lambda_2h(x)|u_1'(x)|^{p-2}u_1'(x)|u_2(x)|^{p-2}u_2(x)}{\left(h(x)|u_1'(x)|^{p-2}u_1'(x)\right)^2}.
\end{aligned}$$

Denote $\varphi(x)=\lambda_1h(x)|u_2'(x)|^{p-2}u_2'(x)|u_1(x)|^{p-2}u_1(x)-\lambda_2h(x)|u_1'(x)|^{p-2}u_1'(x)|u_2(x)|^{p-2}u_2(x)$. Then we find $\varphi(0)=0$, $\varphi(\alpha_0)<0$ and

$$\begin{aligned}
&\varphi(x)-\frac{1}{(p-1)}\varphi'(x)\\
=\;&\lambda_2h(x)|u_1'(x)|^{p-2}u_1'(x)|u_2(x)|^{p-2}u_2'(x)u_1(x)-\lambda_2h(x)|u_1'(x)|^{p-2}u_1'(x)|u_2(x)|^{p-2}u_2(x)u_1'(x)\\
=\;&\lambda_2h^{\frac{p}{p-1}}(x)|u_1'(x)|^{p-2}u_1'(x)|u_2(x)|^{p-2}u_1'(x)u_2'(x)\left[\frac{u_1(x)}{h^{\frac{1}{p-1}}(x)u_1'(x)}-\frac{u_2(x)}{h^{\frac{1}{p-1}}(x)u_2'(x)}\right]\\
=\;&\lambda_2h^{\frac{p}{p-1}}(x)|u_1'(x)|^{p-2}u_1'(x)|u_2(x)|^{p-2}u_1'(x)u_2'(x)\left[T_p\left(\varphi_1(x)\right)-T_p\left(\varphi_2(x)\right)\right]\\
>\;&0\ \text{ on } (0,\alpha_0),
\end{aligned}$$

where the last inequality comes from the monotonicity of T_p on $(\pi_p/2,3\pi_p/2)$. This implies that $\varphi'(x)<0$ whenever $\varphi(x)=0$. Hence, we find that $\varphi(x)<0$ on $(0,\alpha_0)$. Moreover, since $\frac{u_2'(\alpha_0)}{u_1'(\alpha_0)}=0$, we find that $\frac{u_2'(x)}{u_1'(x)}$ decreases to 0 on $(0,\alpha_0)$ and hence $|u_2'(x)|=|u_1'(x)|$ has at most one solution on $(0,\alpha_0)$. Similarly, it can be shown that $|u_2'(x)|=|u_1'(x)|$ has at most one solution on $(\alpha_0,1)$. □

Now, we are prepared to prove Theorem 1.3.

Proof [Proof of Theorem 1.3]

Let $h(x)$ be a symmetric single-well function with $h(x) \geq 1$ on $[0,1]$, and assume $(\lambda_2 - \lambda_1)[h]$ attains its minimum at some symmetric single-well function h_0. Define by (λ_n, u_n) the n-th normalized eigenpair of (1.1) corresponding to h_0. It is clear that u_1 is antisymmetric and u_2 is symmetric. By Lemma 4.2, there exists $0 \leq \hat{x} < 1/2$ such that

$$|u_2'(x)|^p - |u_1'(x)|^p \begin{cases} > 0 & \text{for } (0, \hat{x}), \\ < 0 & \text{for } (\hat{x}, 1-\hat{x}), \\ > 0 & \text{for } (1-\hat{x}, 1). \end{cases}$$

Now, consider the one-parameter family of functions $h(x,t) \equiv t h_0(\hat{x}) + (1-t) h_0(x)$ where $0 < t < 1$ and denote by $(\lambda_n(t), u_n(x,t))$ the n-th normalized eigenpair of (1.1) corresponding to $h(x,t)$. It is clear that $\lambda_n(0) = \lambda_n$ and $u_n(x,0) = u_n(x)$. By (4.1) and the optimality of h_0, we find that

$$0 \leq \frac{d}{dt}(\lambda_2(t) - \lambda_1(t))\bigg|_{t=0} = \int_0^1 [h_0(\hat{x}) - h_0(x)] \left[|u_2'(x,0)|^p - |u_1'(x,0)|^p\right] dx \leq 0.$$

This forces that $h_0(x) = h_0(\hat{x})$ for all $x \in (0,1)$. In particular, $(\lambda_2 - \lambda_1)(h)$ attains its minimum if h is constant. Finally, for $h(x) \equiv m \geq 1$, $(\lambda_2 - \lambda_1)[h]$ is increasing in m. So we find the minimum of $\lambda_2 - \lambda_1$ occurs at $m = 1$ with $(\lambda_2 - \lambda_1)[1] = (2\pi_p)^p - \pi_p^p = (2^p - 1)\pi_p^p$. □

Acknowledgment

The second author, Yan-Hsiou Cheng, was supported in part by the Ministry of Science and Technology of the Republic of China under the grant number MOST 105-2115-M-152-002.

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