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On Holomorphic poly-Norden Manifolds

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Abstract: In this paper, we investigated a new manifold with a poly-Norden structure, which is inspired by the positive root of the equation $x^2 - mx - 1 = 0$. We call this new manifold as holomorphic poly-Norden manifolds. We examine some properties of the Riemann curvature tensor and give an example of this manifold. Then, we define a different connection on this manifold which is named the semisymmetric metric poly F-connection and study some properties of the curvature and torsion tensor field according to this connection.

Key words: Poly-Norden structure, semisymmetric metric connection, Tachibana operator, bronze ratio

1. Introduction

The theory of differential structures on manifolds is studied with great interest. In [19], the authors have extensively investigated complex, product, contact, and f-structures. Later, a very interesting structure was defined on the manifolds, which is called the golden-structure [1]. In fact, the golden-structure is inspired by the equation $x^2 - x - 1 = 0$, whose positive root $\eta = \frac{1+\sqrt{5}}{2} = 1.61803398874989...$ is the golden ratio. If the equation $\varphi^2 - \varphi - I = 0$ is provided on manifold M, then (M, φ) is called golden manifold, where φ is the tensor field of type $(1,1)$ on the manifold.

In [9], the authors defined metallic structures which are the generalization of the golden structure. For integers p and q, the metallic ratio $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ is the root of the equation $x^2 - px - q = 0$. Also, a manifold M endowed with the tensor field J of type $(1,1)$, such that $J^2 - pJ - qI = 0$, is named metallic manifold. Many authors have made interesting studies on golden and metallic manifolds. In one of them [4], they defined a semisymmetric metric F-connection on golden manifolds and made studies on it. A semisymmetric connection $\overline{\nabla}$ is a connection whose torsion tensor checks the equation $S(U,V) = w(V)U - w(U)V$, where U, V are vector fields and w is a covector field. In addition, if this connection holds the requirements $\overline{\nabla} g = 0$ and $\overline{\nabla} F = 0$, then this connection is called semisymmetric metric F-connection. See [2, 3, 5, 11, 13, 17, 18] studies for more information.

The new bronze ratio is defined by

$$B_m = \frac{m + \sqrt{m^2 - 4}}{2},$$

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which is the positive root of the equation $x^2 - mx + 1 = 0$ [10]. In [14], by inspiring from the ratio, the author introduced a new structure on a manifold, which is called a poly-Norden structure. In his work, the author examined some geometric properties of the poly-Norden manifold and investigated certain maps between poly-Norden manifolds and other manifolds endowed with different structures.

A poly-Norden structure on a differentiable manifold M is a $(1,1)$-type tensor field (affinor) F, which satisfies the relation $F^2 = mF - I$, where I is the identity operator on the Lie algebra of vector fields on the manifold. Thus, the pair (M, F) is named an almost poly-Norden manifold. We say that a semi-Riemann metric g is pure (or self-adjoint) with respect to a poly-Norden structure F if $g(FU, V) = g(U, FV)$ for any vector fields U, V. Also, if $g(FU, FV) = mg(FU, V) - g(U, V)$, then the semi-Riemann metric g is called a F-compatible metric (see [6, 7]). So, an almost poly-Norden manifold (M, F) endowed with a semi-Riemann metric g is called an almost poly-Norden semi-Riemann manifold and is represented by (M, g, F) [14]. Also, see [12] study for more information on almost poly-Norden manifolds.

In this paper, we derive the integrability condition of the almost poly-Norden structure F on (M, g, F) with the help of a different operator whose name is φ operator (or Tachibana operator) [16]. Then, we named this manifold (M, g, F) as holomorphic poly-Norden manifold because it satisfies the condition $\varphi_F g = 0$ and by examining the curvature property, we gave an example of such manifold. After that, we introduced a connection ${}^p\nabla$ with semisymmetric torsion endowed with poly-Norden structure F on this manifold and proved that this new connection satisfies the equations ${}^p\nabla g = 0$ and ${}^p\nabla F = 0$, that is, ${}^p\nabla$ is a semisymmetric metric F-connection. Finally, by using the operator φ, we investigated the curvature and torsion properties of this connection ${}^p\nabla$.

2. Preliminaries

Let M_n be $(n = 2k)$ differentiable manifold of class C^∞. Throughout this paper, all connections and tensor fields on the manifold will be assumed to be of class C^∞. In addition, the set of tensor fields of type (p, q) will be represented by $\Im_q^p(M_n)$. For example, the set of vector and covector fields will be indicated by $\Im_0^1(M_n)$ and $\Im_1^0(M_n)$, respectively. Now, let us give some definitions that we will use in this article.

Definition 2.1 ([16]) *Let M_n be differentiable manifold. For any $K \in \Im_q^0(M_n)$, if the following condition holds, then the tensor field K is called a pure tensor field.*

$$\begin{aligned} K(JV_1, V_2, ..., V_q) &= K(V_1, JV_2, ..., V_q) \\ &= ... = K(V_1, V_2, ..., JV_q), \end{aligned}$$

where $V_1, V_2, ..., V_q \in \Im_0^1(M_n)$ and $J \in \Im_1^1(M_n)$.

Definition 2.2 ([16]) *Let M_n be differentiable manifold. If K is a pure tensor field, then the operator φ (or Tachibana operator) applied to this tensor is given by*

$$\begin{aligned} & (\varphi_J K)(X, V_1, V_2, ..., V_q) \\ = \; & (JX)(K(V_1, V_2, ..., V_q)) - X(K(JV_1, V_2, ..., V_q)) \\ & + \sum_{i=1}^{q} K(V_1, ..., (L_{V_i} J)X, ..., V_q), \end{aligned} \tag{2.1}$$

where $X \in \Im_0^1(M_n)$ *and* L_V *represents the Lie differentiation according vector field* V.

Let J be a complex structure, that is, $J^2 = -I$. In the equation (2.1), if $\varphi_J K = 0$, then the vector field K is called a holomorphic (or analytic) tensor field. The Riemann metric g on an almost complex manifold (M_n, J) is called a Norden (or anti-Hermitian) manifold if it satisfies the condition

$$g(JU, V) = g(U, JV) \ \ or \ \ g(JU, JV) = -g(U, V),$$

where $U, V \in \Im_0^1(M_n)$. It is easy to see that g is a semi-Riemannian metric [6]. Then, the triplet (M_n, g, J) is named almost Norden manifold. Besides, if $\nabla J = 0$, then the triplet (M_n, g, J) becomes a Norden (anti-Kähler) manifold, where ∇ is the Riemannian connection of g.

On almost Norden manifold (M_n, g, J), if $\varphi_J g = 0$, then g is holomorphic and this manifold is called almost holomorphic Norden manifold.

3. Holomorphic poly-Norden manifolds

In [14], the author (Propositions 3.4 and 3.5) shows that complex and poly-Norden structures will be written in terms of each other, such that

$$F_{\pm} = \frac{m}{2} I \pm \frac{\sqrt{4 - m^2}}{2} J \tag{3.1}$$

and

$$J_{\pm} = \pm \left(\frac{-m}{\sqrt{4 - m^2}} I + \frac{2}{\sqrt{4 - m^2}} F \right),$$

where $-2 < m < 2$. From the equation (2.1) and (3.1), we obtain

$$\varphi_F K = \frac{\sqrt{4 - m^2}}{2} \varphi_J K \tag{3.2}$$

and from here, we can easily say that if $\varphi_F K = 0$, then the tensor K is holomorphic. This means that we can study holomorphicity on the almost poly-Norden semi-Riemann manifold (M_n, g, F).

Theorem 3.1 *Let* (M_n, g, F) *be an almost poly-Norden semi-Riemann manifold. If* ∇ *denotes the Levi-Civita connection of the metric* g, *then* $\nabla F = 0$ *if and only if* $\varphi_F g = 0$.

Proof From the covariant derivation of the $g(FU, V) = g(U, FV)$ with respect to Riemann connection ∇, we obtain

$$g((\nabla_X F) U, V) = g(U, (\nabla_X F) V).$$

Applying φ to the Riemannian tensor g and from $L_U V = [U, V] = \nabla_U V - \nabla_V U$, we get

$$\begin{aligned} (\varphi_{FX} g)(U, V) &= (FX) g(U, V) - X g(FU, V) \\ &\quad + g((L_U F) X, V) + g(U, (L_V F) X) \\ &= -g((\nabla_X F) U, V) + g((\nabla_U F) X, V) + g(X, (\nabla_V F) U) \end{aligned}$$

and

$$(\varphi_{FV} g)(U, X) = -g((\nabla_V F) U, X) + g((\nabla_U F) V, X) + g(V, (\nabla_X F) U). \tag{3.3}$$

From the last two equations, we have

$$(\varphi_{FX}g)(U,V) + (\varphi_{FV}g)(U,X) = 2g((\nabla_U F)V,X). \tag{3.4}$$

It is clear that in the equation (3.3), if $\nabla F = 0$, then $\varphi_F g = 0$ and in the equation (3.4), if $\varphi_F g = 0$, than $\nabla F = 0$. □

Also, from the equation (3.2), we have

$$\varphi_F g = \frac{\sqrt{4-m^2}}{2}\varphi_J g.$$

Then, if $\varphi_F g = 0$ (or $\nabla F = 0$), then the triplet (M_n, g, F) is called holomorphic poly-Norden manifold.

Twin metric G of the almost poly-Norden semi-Riemann manifold (M_n, g, F) is defined by

$$G(U,V) = g(FU,V), \tag{3.5}$$

for $X, Y \in \Im_0^1(M_n)$. Then,

$$\begin{aligned} G(U,V) &= g(FU,V) \\ &= g(V,FU) \\ &= g(FY,U) = G(V,U) \end{aligned}$$

and

$$\begin{aligned} G(FU,V) &= g(F^2U,V) \\ &= g(FU,FV) = G(U,FV), \end{aligned}$$

that is, twin metric G is both symmetric and pure according to poly-Norden structure F. From the covariant derivation of the equation (3.5) with respect to Riemann connection ∇, we obtain

$$\begin{aligned} (\nabla_X G)(U,V) &= (\nabla_X g)(FU,V) + g((\nabla_X F)U,V) \\ &= g((\nabla_X F)U,V) \end{aligned}$$

and then,

Proposition 3.2 *Let (M_n, g, F) be a holomorphic poly-Norden manifold. The Riemann connection of the metric g equals to the Riemann connection of the twin metric G, i.e., ${}^G\nabla = \nabla$.*

Let ${}^gR$ and ${}^GR$ be Riemann curvature tensors of the metric g and the twin metric G, respectively. From the proposition 3.2, we can easily see that ${}^gR = {}^GR$. The Ricci identity for poly-Norden structure F on holomorphic poly-Norden manifold (M_n, g, F) is as follows:

$${}^gR(U,V,FZ) - F({}^gR(U,V,Z)) = 0. \tag{3.6}$$

Also, for the $(0,4)$-type of the curvature tensor ${}^gR$, we get ${}^gR(U,V,Z,W) = g(R(U,V,Z),W)$ and

$$\begin{aligned} {}^gR(U,V,FZ,W) &= g({}^gR(U,V,FZ),W) \\ &= g(F{}^gR(U,V,Z),W) \\ &= g({}^gR(U,V,Z),FW) \\ &= {}^gR(U,V,Z,FW), \end{aligned}$$

that is, the curvature tensor ${}^gR$ is pure according to Z and W. Besides, from the ${}^gR(U,V,Z,W) = {}^gR(Z,W,U,V)$ property of the curvature tensor ${}^gR$, we have

$${}^gR(FU,V,Z,W) = {}^gR(U,FV,Z,W).$$

Finally, for ${}^gR = {}^GR$ and (3.5), the curvature tensor ${}^GR$ of the twin metric G is as follows:

$$\begin{aligned} {}^GR(U,V,Z,W) &= G({}^GR(U,V,Z),W) \\ &= g(F({}^GR(U,V,Z)),W) \\ &= g({}^GR(U,V,Z),FW) \\ &= {}^gR(U,V,Z,FW) \end{aligned}$$

and

$${}^GR(Z,W,U,V) = {}^gR(Z,FW,U,V).$$

From the last two equations, we obtain ${}^gR(U,V,Z,FW) = {}^gR(Z,FW,U,V)$. After all, we say that the curvature tensor ${}^gR$ is pure with regard to poly-Norden structure F, i.e.

$$\begin{aligned} {}^gR(FU,V,Z,W) &= {}^gR(U,FV,Z,W) \\ &= {}^gR(U,V,FZ,W) = {}^gR(U,V,Z,FW). \end{aligned}$$

Then,

Theorem 3.3 *Let (M_n, g, F) be a holomorphic poly-Norden manifold. Then, $\varphi_F\, {}^gR = 0$, that is, the curvature tensor ${}^gR$ is a holomorphic tensor.*

Proof From the covariant derivation of the equation (3.6), we have,

$$(\nabla_X\, {}^gR)(FU_1,U_2,U_3) = F(\nabla_X\, {}^gR)(U_1,U_2,U_3). \tag{3.7}$$

If the operator φ is applied to the Riemann curvature tensor ${}^gR$, we obtain

$$\begin{aligned} (\varphi_{FX}\, {}^gR)(U_1,U_2,U_3,U_4) &= (\nabla_{FX}\, {}^gR)(U_1,U_2,U_3,U_4) \\ &\quad -(\nabla_X\, {}^gR)(FU_1,U_2,U_3,U_4). \end{aligned} \tag{3.8}$$

Substituting (3.7) in (3.8) and using the Bianchi's 2nd identity for the tensor field ${}^{g}R$, we obtain

$$\begin{aligned}(\varphi_{FX}\ {}^{g}R)(U_1,U_2,U_3,U_4) &= g((\nabla_{FX}\ {}^{g}R)(U_1,U_2,U_3)-(\nabla_X\ {}^{g}R)(FU_1,U_2,U_3),U_4)\\ &= g((\nabla_{FX}\ {}^{g}R)(U_1,U_2,U_3)-F(\nabla_X\ {}^{g}R)(U_1,U_2,U_3),U_4)\\ &= -g((\nabla_{U_2}\ {}^{g}R)(FX,U_1,U_3)+(\nabla_{U_1}\ {}^{g}R)(U_2,FX,U_3)\\ &\quad +F(\nabla_X\ {}^{g}R)(U_1,U_2,U_3),U_4)\\ &= -g(F((\nabla_{U_2}\ {}^{g}R)(X,U_1,U_3)+(\nabla_{U_1}\ {}^{g}R)(U_2,X,U_3)\\ &\quad +(\nabla_X\ {}^{g}R)(U_1,U_2,U_3),U_4)\\ &= -g(\underset{(X,U_1,U_2)}{\sigma}(\nabla_X\ {}^{g}R)(U_1,U_2,U_3),FU_4)\\ &= 0\end{aligned}$$

where σ represents the cyclic sum over X, U_1, and U_2. Finally, from the equation (3.2), we have

$$\varphi_F\ {}^{g}R=\frac{\sqrt{4-m^2}}{2}\varphi_J\ {}^{g}R,$$

namely, the curvature tensor ${}^{g}R$ is a holomorphic tensor. □

Example 3.4 *Let $\mathbb{R}_{2n}$ be a semi-Euclidean space endowed with semi-Euclidean metric g, that is,*

$$g=\begin{pmatrix}\delta_i^j & 0\\ 0 & -\delta_{\bar{i}}^{\bar{j}}\end{pmatrix}$$

where $i,j=1,...,n,\ \bar{i},\bar{j}=n+1,...,2n$. Also, let $\mathbb{C}_n$ be a complex space with $\mathbb{R}_{2n}$ such that

$$s:z\in\mathbb{C}_n\longrightarrow s(z)=Z\in\mathbb{R}_{2n},$$

where $z=(z_1,z_2,...,z_n)$, $s(z)=Z=(x_1,x_2,...,x_n;y_1,y_2,...,y_n)$ and $z_t=x_t+iy_t$, $t=1,2,...,n$. Then, the complex structure J on $\mathbb{R}_{2n}$ is given by

$$J=\begin{pmatrix}0 & \delta_i^j\\ -\delta_{\bar{i}}^{\bar{j}} & 0\end{pmatrix}.$$

From here, we easily see that $g_{im}F_j^m=g_{mj}F_i^m=$, i.e. the structure J is compatible (purity) with metric g and then $(\mathbb{R}_{2n},J,g)$ is a holomorphic Norden Euclidean space. Also, poly-Norden structures $F_{\pm}$ on $\mathbb{R}_{2n}$ obtained from complex structure J are as follows:

$$F_{\pm}=\begin{pmatrix}\frac{m}{2}\delta_i^j & \pm\frac{\sqrt{4-m^2}}{2}\delta_{\bar{i}}^{\bar{j}}\\ \mp\frac{\sqrt{4-m^2}}{2}\delta_{\bar{i}}^{\bar{j}} & \frac{m}{2}\delta_i^j\end{pmatrix}$$

and the triple $(\mathbb{R}_{2n},F,g)$ is called a holomorphic poly-Norden Euclidean space.

4. Semisymmetric metric poly F-connection

In this section, we are going to study the holomorphic poly-Norden manifold endowed with another connection rather than the metric connection.

Theorem 4.1 *Let* (M_n, g, F) *be a holomorphic poly-Norden manifold and* ${}^p\nabla$ *be a connection with torsion* ${}^pT$ *on that manifold such that*

$$ {}^pT(U,V) = \gamma(V)(U) - \gamma(U)(V) - \gamma(FV)(FU) + \gamma(FU)(FV) \tag{4.1}$$

where $U, V \in \Im_0^1(M_n)$ *and* $\gamma \in \Im_1^0(M_n)$. *If this connection satisfies* ${}^p\nabla g = 0$ *and* ${}^p\nabla F = 0$, *then*

$$\begin{aligned} {}^p\nabla_U V &= \nabla_U V + \gamma(V)(U) - g(U,V)(W) \\ &\quad -\gamma(FV)(FU) + g(FU,V)(FW), \end{aligned} \tag{4.2}$$

where ∇ *stands for the Levi-Civita connection of the metric* g *and* $g(W,Y) = \gamma(Y)$, $W \in \Im_0^1(M_n)$.

Proof It is well known that a new connection ${}^p\nabla$ can be formed with

$$ {}^p\nabla_U V = \nabla_U V + D(U,V), \tag{4.3}$$

where D is the deformation tensor field of type (1,2). Then, from ${}^pT(U,V) = {}^p\nabla_U V - {}^p\nabla_V U - [U,V]$ and the method of Hayden [8], we obtain

$$ {}^pT(U,V) = D(U,V) - D(V,U) \tag{4.4}$$

If ${}^p\nabla g = 0$, we have

$$ D(U,V,Z) + D(U,Z,V) = 0. \tag{4.5}$$

From the equations (4.4) and (4.5), we have

$$ {}^pT(U,V,Z) = D(U,V,Z) - D(V,U,Z)$$

$$ {}^pT(Z,U,V) = D(Z,U,V) - D(U,Z,V)$$

$$ {}^pT(Z,V,U) = D(Z,V,U) - D(V,Z,U)$$

and then

$$ {}^pT(U,V,Z) + {}^pT(Z,U,V) + {}^pT(Z,V,U) = 2D(U,V,Z),$$

where ${}^pT(U,V,Z) = g({}^pT(U,V),Z)$.

Substituting (4.1) in the last equation, we get

$$ D(U,V) = \gamma(V)(U) - g(U,V)(W) - \gamma(FV)(FU) + g(FU,V)(FW).$$

Also, the connection given by (4.2) satisfies the condition ${}^p\nabla F = 0$. So, this proof is complete. □

From now on, the connection ${}^p\nabla$ will be called semisymmetric metric poly F-connection.

With a simple calculation, we can see that the torsion tensor ${}^pT$ is pure according to poly-Norden structure F, i.e.

$$ {}^pT(FU,V) = {}^pT(U,FV) = F\,{}^pT(U,V).$$

Also, in [15], the author has proved that an $F-$connection is pure if and only if its torsion tensor is pure. Then, we can easily write as following equation:

$$ {}^{p}\nabla_{FU}V = {}^{p}\nabla_{U}(FV) = F^{p}\nabla_{U}V. $$

Then,

Theorem 4.2 *Let (M_n, g, F) be a holomorphic poly-Norden manifold. If the covector γ in (4.1) is holomorphic, then the torsion tensor ${}^{p}T$ is also a holomorphic tensor, i.e. $\varphi_F\gamma = 0$ and $\varphi_F\ {}^{p}T = 0$.*

Proof By applying the φ operator to the torsion tensor ${}^{p}T$, we get

$$ \begin{aligned} (\varphi_{FX}\ {}^{p}T)(U,V) &= ({}^{p}\nabla_{FX}\ {}^{p}T)(U,V) - ({}^{p}\nabla_{X}\ {}^{p}T)(FU,V) \\ &= [({}^{p}\nabla_{FX}\gamma)(V) - ({}^{p}\nabla_{X}\gamma)(FV)](U) \\ &\quad -[({}^{p}\nabla_{FX}\gamma)(U) - ({}^{p}\nabla_{X}\gamma)(FU)](V) \\ &\quad +[({}^{p}\nabla_{FX}\gamma)(FU) - m({}^{p}\nabla_{X}\gamma)(FU) \\ &\qquad +({}^{p}\nabla_{X}\gamma)(U)](FV) \\ &\quad -[({}^{p}\nabla_{FX}\gamma)(FV) - m({}^{p}\nabla_{X}\gamma)(FV) \\ &\qquad +({}^{p}\nabla_{X}\gamma)(V)](FU). \end{aligned} \tag{4.6} $$

Also, for the covector field γ in the equation (4.1), we obtain

$$ \begin{aligned} (\varphi_{FX}\gamma)(U) &= ({}^{p}\nabla_{FX}\gamma)(U) - ({}^{p}\nabla_{X}\gamma)(FU) \\ &= 0 \end{aligned} \tag{4.7} $$

Finally, from the equation (4.6) and (4.7), we get

$$ \begin{aligned} (\varphi_{FX}\ {}^{p}T)(U,V) &= (\varphi_{FX}\gamma)(V)(U) - (\varphi_{FX}\gamma)(U)(V) \\ &\quad +(\varphi_{FX}\gamma)(FV)(FU) - (\varphi_{FX}\gamma)(FU)(FV) \\ &= 0 \end{aligned} $$

□

Then, we write the following corollary:

Remark 4.3 *1. From the equation (3.2), it is obvious that*

$$ \varphi_F\gamma = \frac{\sqrt{4-m^2}}{2}\varphi_J\gamma, $$

and

$$ \varphi_F\ {}^{p}T = \frac{\sqrt{4-m^2}}{2}\varphi_J\ {}^{p}T, $$

2. If $\varphi_F\ {}^{p}T = 0$, from $(\varphi_{FX}{}^{p}T)(U,V) = ({}^{p}\nabla_{FX}{}^{p}T)(U,V) - ({}^{p}\nabla_{X}{}^{p}T)(FU,V)$, we can write

$$ \begin{aligned} ({}^{p}\nabla_{FX}{}^{p}T)(U,V) &= ({}^{p}\nabla_{X}{}^{p}T)(FU,V) \\ &= ({}^{p}\nabla_{X}{}^{p}T)(U,FV) = F({}^{p}\nabla_{X}{}^{p}T)(U,V), \end{aligned} $$

that is, the covariant derivation of the torsion tensor ${}^pT$ *according to* ${}^p\nabla$ *is pure according to poly-Norden structure* F*.*

3. The last theorem can also be proved for Riemann connection ∇*, that is, we write* $(\varphi_{FX}{}^pT)(U,V) = (\nabla_{FX}{}^pT)(U,V) - (\nabla_X{}^pT)(FU,V)$ *and*

$$\begin{aligned}(\nabla_{FX}{}^pT)(U,V) &= (\nabla_X{}^pT)(FU,V) \\ &= (\nabla_X{}^pT)(U,FV) = F(\nabla_X{}^pT)(U,V).\end{aligned}$$

Throughout this article, we will assume that $\varphi_F\gamma = 0$, that is,

$$({}^p\nabla_{FX}\gamma)(U) - ({}^p\nabla_X\gamma)(FU) = 0.$$

5. The curvature tensor of semisymmetric metric poly F-connection

It is well-known that the curvature tensor of any linear connection $\overline{\nabla}$ for all vector fields is as follows:

$$\overline{R}(U,V,Z) = (\overline{\nabla}_U\overline{\nabla}_V - \overline{\nabla}_V\overline{\nabla}_U - \overline{\nabla}_{[U,V]})Z.$$

Then, $(0,4)$-type of the curvature tensor for the connection (4.2) has the following form:

$$\begin{aligned}{}^pR(U,V,Z,W) &= {}^gR(U,V,Z,W) \\ &+\varsigma(U,Z)g(V,W) - \varsigma(V,Z)g(U,W) \\ &+\varsigma(V,W)g(U,Z) - \varsigma(U,W)g(V,Z) \\ &+\varsigma(V,FZ)g(FU,W) - \varsigma(U,FZ)g(FV,W) \\ &+\varsigma(U,FW)g(FV,Z) - \varsigma(V,FW)g(FU,Z),\end{aligned} \tag{5.1}$$

where

$$\begin{aligned}\varsigma(U,V) &= (\nabla_U\gamma)(V) - \gamma(U)\gamma(V) + \frac{1}{2}\gamma(W)g(U,V) \\ &+\gamma(FU)\gamma(FV) - \frac{1}{2}\omega(FW)g(FU,V).\end{aligned} \tag{5.2}$$

It is said that the curvature tensor ${}^pR$ is hold:

$${}^pR(U,V,W,Z) = -\,{}^pR(U,V,Z,W) = -\,{}^pR(V,U,Z,W),$$

that is, ${}^pR$ is antisymmetric according to the first and last two components. Also,

$$\varsigma(U,V) - \varsigma(V,U) = (\nabla_U\gamma)(V) - (\nabla_V\gamma)(U) \tag{5.3}$$

and for the exterior differential operator d applied to the covector field ω, we get

$$\begin{aligned}2(d\gamma)(U,V) &= U\gamma(V) - V\gamma(U) - \gamma([U,V]) \\ &= (\nabla_U\gamma)V + \gamma(\nabla_UV) - (\nabla_V\gamma)U - \gamma(\nabla_VU) - \gamma([U,V]) \\ &= (\nabla_U\gamma)V - (\nabla_V\gamma)U + \gamma(\nabla_UV - \nabla_VU) - \gamma([U,V]) \\ &= (\nabla_U\gamma)(V) - (\nabla_V\gamma)(U).\end{aligned} \tag{5.4}$$

From the equations (5.3) and (5.4), we obtain

$$\begin{aligned} \varsigma(U,V) - \varsigma(V,U) &= (\nabla_U \gamma)(V) - (\nabla_V \gamma)(U) \\ &= 2(d\gamma)(U,V). \end{aligned} \tag{5.5}$$

Then, we write the following corollary.

Corollary 5.1 *1. The covector field γ is closed if and only if the tensor field ς is symmetric.*

2. If the covector field γ is a gradient, that is $\gamma = \partial f$, then the tensor ς is symmetric.

For the tensor ς given by (5.2) is pure with regard to poly-Norden structure F, that is,

$$\begin{aligned} \varsigma(U,FV) - \varsigma(FU,V) &= [(\nabla_U \gamma)(FV) - (\nabla_{FU} \gamma)(V)] \\ &= (\varphi_{FU} \gamma)(V) \\ &= 0 \end{aligned}$$

and from the equation (5.5), we get

$$\begin{aligned} &{}^pR(U,V,Z,W) - {}^pR(Z,W,U,V) \\ = \;& 2(d\gamma)(U,Z)g(V,W) - 2(d\gamma)(V,Z)g(U,W) \\ & +2(d\gamma)(V,W)g(U,Z) - 2(d\gamma)(U,W)g(V,Z) \\ & +2(d\gamma)(FV,Z)g(FU,W) - 2(d\gamma)(FU,Z)g(FV,W) \\ & +2(d\gamma)(FU,W)g(FV,Z) - 2(d\gamma)(FV,W)g(FU,Z), \end{aligned}$$

then,

Proposition 5.2 *For the curvature tensor ${}^pR$, if the covector field γ is closed ($d\gamma = 0$), then ${}^pR(U,V,Z,W) - {}^pR(Z,W,U,V) = 0$.*

By applying the φ operator to the tensor ς, we get

$$\begin{aligned} (\varphi_{FX}\varsigma)(U,V) &= ({}^p\nabla_{FX}\varsigma)(U,V) - ({}^p\nabla_X\varsigma)(FU,V) \\ &= (\nabla_{FX}\varsigma)(U,V) - (\nabla_X\varsigma)(FU,V) \end{aligned} \tag{5.6}$$

From the equation (5.2), we have

$$(\varphi_{FX}\varsigma)(U,V) = (\nabla_{FX}\ \nabla_U\gamma)(V) - (\nabla_X\ \nabla_{FU}\gamma)(V). \tag{5.7}$$

In the last equation, if we apply the Ricci identity to the 1-form γ, we obtain

$$(\nabla_{FX}\ \nabla_U\gamma)(V) = (\nabla_U\ \nabla_{FX}\gamma)(V) - \frac{1}{2}\gamma({}^gR(FX,U,V))$$

and

$$\begin{aligned}(\nabla_X \nabla_{FU}\gamma)(V) &= (\nabla_X \nabla_U\gamma)(FV)\\ &= (\nabla_U \nabla_X\gamma)(FV) - \frac{1}{2}\gamma({}^gR(X,U,FV))\\ &= (\nabla_U \nabla_{FX}\gamma)(V) - \frac{1}{2}\gamma({}^gR(X,FU,V))\end{aligned}$$

Substituting (5.2) in the equation (5.6), we get

$$\begin{aligned}(\varphi_{FX}\varsigma)(U,V) &= -\frac{1}{2}\gamma[{}^gR(FX,U,V) - {}^gR(X,U,FV)]\\ &= 0.\end{aligned}$$

Then,

Proposition 5.3 *Let (M_n, g, F) be a holomorphic poly-Norden manifold. The tensor ς given by the equation (5.2) is a holomorphic tensor, that is, $\varphi_F\ \varsigma = \frac{\sqrt{4-m^2}}{2}\varphi_J\ \varsigma$ and*

$$({}^p\nabla_{FX}\varsigma)(U,V) = ({}^p\nabla_X\varsigma)(FU,V) = ({}^p\nabla_X\varsigma)(U,FV). \tag{5.8}$$

Because of the purity of the tensor ς, we say that the curvature tensor of the semisymmetric metric poly F-connection is a pure tensor, namely,

$$\begin{aligned}{}^pR(FU,V,Z,W) &= {}^pR(U,FV,Z,W)\\ &= {}^pR(U,V,FZ,W) = {}^pR(U,V,Z,FW)\end{aligned}$$

and from (2.1) and (5.1), we obtain

$$\begin{aligned}(\varphi_{FX}\ {}^pR)(U_1,U_2,U_3,U_4) &= ({}^p\nabla_{FX}\ {}^pR)(U_1,U_2,U_3,U_4)\\ &\quad -({}^p\nabla_X\ {}^pR)(FU_1,U_2,U_3,U_4).\end{aligned} \tag{5.9}$$

Substituting (5.1) in the last equation, we have

$$\begin{aligned}&(\varphi_{FX}{}^pR)(U_1,U_2,U_3,U_4)\\ =\ &({}^p\nabla_{FX}{}^gR)(U_1,U_2,U_3,U_4) - ({}^p\nabla_X{}^gR)(FU_1,U_2,U_3,U_4)\\ &+(\varphi_{FX}\varsigma)(U_1,U_3)g(U_2,U_4) - (\varphi_{FX}\varsigma)(U_2,U_3)g(U_1,U_4)\\ &+(\varphi_{FX}\varsigma)(U_2,U_4)g(Y_1,U_3) - (\varphi_{FX}\varsigma)(Y_1,Y_4)g(U_2,U_3)\\ &+(\varphi_{FX}\varsigma)(FU_2,U_3)g(FU_1,U_4) - (\varphi_{FX}\varsigma)(FU_1,U_3)g(FU_2,U_4)\\ &+(\varphi_{FX}\varsigma)(FU_1,U_4)g(FU_2,U_3) - (\varphi_{FX}\varsigma)(FU_2,U_4)g(FU_1,U_3)\end{aligned}$$

From the proposition 5.3 and theorem 3.3, we obtain

$$\begin{aligned}(\varphi_{FX}{}^pR)(U_1,U_2,U_3,U_4) &= ({}^p\nabla_{FX}{}^gR)(U_1,U_2,U_3,U_4)\\ &\quad -({}^p\nabla_X{}^gR)(FU_1,U_2,U_3,U_4)\\ &= (\varphi_{FX}{}^gR)(U_1,U_2,U_3,U_4)\\ &= 0.\end{aligned}$$

Finally,

Theorem 5.4 *Let (M_n, g, F) be a holomorphic poly-Norden manifold. The curvature tensor ${}^pR$ of the semi-symmetric metric poly F-connection is a holomorphic tensor, i.e.* $\varphi_F\ {}^pR = \frac{\sqrt{4-m^2}}{2}\varphi_J\ {}^pR$ *and*

$$\begin{aligned}({}^p\nabla_{FX}\ {}^pR)(U_1,U_2,U_3,U_4) &= ({}^p\nabla_X\ {}^pR)(FU_1,U_2,U_3,U_4)\\ &= ({}^p\nabla_X\ {}^pR)(U_1,FU_2,U_3,U_4)\\ &= ({}^p\nabla_X\ {}^pR)(U_1,U_2,FU_3,U_4)\\ &= ({}^p\nabla_X\ {}^pR)(U_1,U_2,U_3,FU_4)\\ &= F({}^p\nabla_X\ {}^pR)(U_1,U_2,U_3,U_4).\end{aligned}$$

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