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Some results on prime rings with multiplicative derivations

Gurninder Singh SANDHU[1,*], **Didem KARALARLIOĞLU CAMCI**[2]
[1]Department of Mathematics, Patel Memorial National College, Rajpura, India
[2]Department of Mathematics, Faculty of Arts and Sciences, Çanakkale Onsekiz Mart University, Çanakkale, Turkey

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Abstract: Let R be a prime ring with center $Z(R)$ and an automorphism α. A mapping $\delta : R \to R$ is called multiplicative skew derivation if $\delta(xy) = \delta(x)y + \alpha(x)\delta(y)$ for all $x, y \in R$ and a mapping $F : R \to R$ is said to be multiplicative (generalized)-skew derivation if there exists a unique multiplicative skew derivation δ such that $F(xy) = F(x)y + \alpha(x)\delta(y)$ for all $x, y \in R$. In this paper, our intent is to examine the commutativity of R involving multiplicative (generalized)-skew derivations that satisfy the following conditions: (i) $F(x^2) + x\delta(x) = \delta(x^2) + xF(x)$, (ii) $F(x \circ y) = \delta(x \circ y) \pm x \circ y$, (iii) $F([x, y]) = \delta([x, y]) \pm [x, y]$, (iv) $F(x^2) = \delta(x^2)$, (v) $F([x, y]) = \pm x^k[x, \delta(y)]x^m$, (vi) $F(x \circ y) = \pm x^k(x \circ \delta(y))x^m$, (vii) $F([x, y]) = \pm x^k[\delta(x), y]x^m$, (viii) $F(x \circ y) = \pm x(\delta(x) \circ y)x^m$ for all $x, y \in R$.

Key words: Prime ring, multiplicative generalized derivation, multiplicative (generalized)-skew derivation, multiplicative left centralizer.

1. Introduction and preliminaries

Let R be an associative ring and $Z(R)$ denotes the center of R. By a prime ring, we mean a ring R in which for every $a, b \in R$, $aRb = (0)$ implies $a = 0$ or $b = 0$. Moreover, if $aRa = (0)$ implies $a = 0$, then R is called a semiprime ring. An additive mapping $d : R \to R$ is said to be a derivation of R if $d(x + y) = d(x) + d(y)$ and $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $F : R \to R$ be a mapping associated with a derivation d such that $F(x + y) = F(x) + F(y)$ and $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. Then F is said to be a generalized derivation of R, which was introduced by Brešar [1]. In the same paper Brešar observed that if R has the property that $Rx = (0)$ implies $x = 0$; $h : R \to R$ is any function and $\mu : R \to R$ is an additive mapping satisfying $\mu(xy) = \mu(x)y + xh(y)$ for all $x, y \in R$, then μ is uniquely determined by h and moreover h must be a derivation of R. A generalized derivation F associated with a zero derivation is said to be a left multiplier of R. Thus, every derivation and every left multiplier is a generalized derivation. Daif [4] introduced a mapping $\Delta : R \to R$ satisfying $\Delta(xy) = \Delta(x)y + x\Delta(y)$ for all $x, y \in R$, which is called multiplicative derivation of R. Of course, these mappings are not necessarily additive. Furthermore, the complete description of these mappings was given by Goldmann and Šemrl [9]. Let $R = \mathfrak{C}[0, 1]$ be the ring of all continuous real-valued functions

*Correspondence: gurninder_rs@pbi.ac.in
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defined on $[0, 1]$ and a mapping $\Delta : R \to R$ such that

$$\Delta(f)(x) = \left\{ \begin{array}{ll} f(x) \log |f(x)| & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0 \end{array} \right\}.$$

Then, it is straight forward to check that Δ is not additive but satisfies $\Delta(f_1 f_2) = \Delta(f_1) f_2 + f_1 \Delta(f_2)$ for all $f_1, f_2 \in R$. Daif and Tammam-El-Sayiad [5] gave a generalization of the notion of multiplicative derivation, which is known as multiplicative generalized derivation, namely a function $F : R \to R$ associated with a derivation Δ of R is said to be a multiplicative generalized derivation if $F(xy) = F(x)y + x\Delta(y)$ for all $x, y \in R$. In addition, Dhara and Ali [7] extended the notion of multiplicative derivation to its full generality by introducing multiplicative (generalized)-derivation. Accordingly, a function $F : R \to R$ is said to be a multiplicative (generalized)-derivation of R if there exists a function $\Delta : R \to R$ such that $F(xy) = F(x)y + x\Delta(y)$ for all $x, y \in R$. Of course, F and Δ are not necessarily additive. In a recent paper [3], second author and Aydin obtained that in semiprime rings the associated function of a multiplicative (generalized)-derivation must be a multiplicative derivation. For an up-to-date discussion of these mappings one may see [3, 7, 11, 16] and references therein.

Very recently, Rehman and Khan [15] introduced the notion of multiplicative (generalized)-skew derivation as; for any automorphism α of a ring R, a function $F : R \to R$ is said to be a multiplicative (generalized)-skew derivation of R if there exists a function $\delta : R \to R$ such that $F(xy) = F(x)y + \alpha(x)\delta(y)$ for all $x, y \in R$. In case α is the trivial automorphism, F is just a multiplicative (generalized)-derivation of R. Therefore, multiplicative (generalized)-skew derivation covers the concepts of multiplicative (generalized)-derivation, multiplicative derivation and multiplicative left centralizer. Moreover, it has also been proved that in semiprime rings, the associated function δ of a multiplicative (generalized)-skew derivation F is defined as $\delta(xy) = \delta(x)y + \alpha(x)\delta(y)$ for all $x, y \in R$, which is called multiplicative skew derivation of R (see [15, Lemma 2.1]). In case δ is additive, it is called a skew derivation of R. A mapping $F : R \to R$ is called a multiplicative generalized skew derivation if it is uniquely determined by a skew derivation δ such that $F(xy) = F(x)y + \alpha(x)\delta(y)$ for all $x, y \in R$.

In 1992, Bell and Daif [6] proved that if R is a semiprime ring, I is a nonzero ideal of R and d is a nonzero derivation of R such that $d([x, y]) = [x, y]$ for all $x, y \in I$, then $I \subseteq Z(R)$. In 2003, Quadri et al. [14] proved the following result: *Let R be a prime ring and I is a nonzero ideal of R. If R admits a generalized derivation F associated with a nonzero derivation d such that $F([x, y]) = [x, y]$ for all $x, y \in I$, then R is commutative.* Moreover, Shang [17] extended these results by characterizing the conditions $F([x, y]) = x^k[x, y]x^l$ and $F([x, y]) = -x^k[x, y]x^l$ where k, l are fixed positive integers. Inspired by Shang [17], Koç and Gölbaşi [11] proved that a semiprime ring R that admits a multiplicative generalized derivation F contains a nonzero central ideal if any one of the following conditions hold:

1. $F([x, y]) = x^m[x, y]x^n$
2. $F([x, y]) = -x^m[x, y]x^n$.

In this view, it is natural to think of some more general situations: $F([x, y]) = x^m[\delta(x), y]x^n$, $F([x, y]) = -x^m[\delta(x), y]x^n$, $F([x, y]) = x^m[x, \delta(y)]x^n$ and $F([x, y]) = -x^m[x, \delta(y)]x^n$, where F is a multiplicative (generalized)-skew derivation of R associated with a multiplicative skew derivation δ. In the present paper, we examine these identities and discuss the commutative structure of the rings.

Let us recall some well-known results of this subject that will be used in the sequel.

Lemma 1.1 ([2], Lemma) *Let R be a prime ring. If functions $f : R \to R$ and $g : R \to R$ such that $f(x)yg(z) = g(x)yf(z)$ for all $x, y, z \in R$, and $0 \neq f$, then there exists λ in the extended centroid of R such that $g(x) = \lambda f(x)$ for all $x \in R$.*

Corollary 1.2 ([8], Corollary 7) *Let R be a prime ring containing a nonzero idempotent. Then any left centralizer $\varphi : R \to Q_{ml}$ is additive, where Q_{ml} denotes the maximal right ring of quotients of R.*

Corollary 1.3 ([10], Corollary pg no. 8) *Let R be a prime ring and suppose that $0 \neq a \in R$ satisfies $a[u, x] = 0$ for all $x \in R$. Then $u \in Z(R)$.*

Theorem 1.4 ([13], Theorem) *Let R be a prime ring that admits a nontrivial automorphism σ such that $[\sigma(x), x] = 0$ for all $x \in R$. Then R is commutative.*

Theorem 1.5 ([12], Theorem 1) *Let R be a noncommutative prime ring with derivations d and δ such that $[d(x^m)x^n - x^p\delta(x^q), x^r]_k = 0$ for all $x \in R$, where m, n, p, q, r, k are fixed positive integers. Then $d = 0$ and $\delta = 0$.*

2. The Results

Lemma 2.1 *Let R be a ring and α be an endomorphism of R. If R admits a multiplicative (generalized)-skew derivation F together with a multiplicative skew derivation δ, then $F - \delta$ is a multiplicative left centralizer of R.*

Proof Let us set $H = F - \delta$. Then for any $x, y \in R$, we see that $H(xy) = (F - \delta)(xy) = F(xy) - \delta(xy) = F(x)y - \delta(x)y = H(x)y$. Therefore, H is a multiplicative left centralizer of R. □

Lemma 2.2 *Let R be a ring and δ be a multiplicative derivation of R. Then δ preserves the center of R.*

Proof Let $a \in Z(R)$ be any element. Then for each $x \in R$, we have

$$\delta(xa) = \delta(x)a + x\delta(a)$$

and on the other hand

$$\delta(ax) = \delta(a)x + a\delta(x).$$

Combining these both expressions, we get $[\delta(a), x] = 0$ for all $x \in R$. Hence $\delta(a) \in Z(R)$. □

Theorem 2.3 *Let R be a prime ring with extended centroid C and $F : R \to R$ be a multiplicative (generalized)-skew derivation associated with a multiplicative skew derivation δ of R.*

1. *If $F(x \circ y) = \delta(x \circ y) \pm x \circ y$ for all $x, y \in R$, then either R is commutative or there exists $\lambda \in C$ such that $F(x) = \lambda x + \delta(x)$ for all $x \in R$, where $\lambda = \pm 1$.*

2. *If $F([x, y]) = \delta([x, y]) \pm [x, y]$ for all $x, y \in R$, then either R is commutative or there exists $\lambda \in C$ such that $F(x) = \lambda x + \delta(x)$ for all $x \in R$, where $\lambda = \pm 1$.*

Proof

1. Let us consider $F(x \circ y) = \delta(x \circ y) \pm (x \circ y)$ for all $x, y \in R$. Which is equivalent to $H(x \circ y) = \pm(x \circ y)$ for all $x, y \in R$, where $H = F - \delta$ and by Lemma 2.1, H is a multiplicative left centralizer of R. Replace x by xz, we have

$$H(x \circ y)z + H(x)[z, y] = \pm(x \circ y)z \pm x[z, y],\ \forall\ x, y, z \in R.$$

It implies that

$$H(x)[z, y] = \pm x[z, y],\ \forall\ x, y, z \in R. \tag{2.1}$$

Replace x by px in (2.1), we find that

$$H(p)x[z, y] = \pm px[z, y]. \tag{2.2}$$

Left multiply (2.1) by p and subtract from (2.2), we get

$$(H(p)x - pH(x))[z, y] = 0,\ \forall\ x, y, z, p \in R.$$

Replace x by qx, we get $(H(p)q - pH(q))R[z, y] = (0)$ for all $y, z, p, q \in R$. It implies that either $H(p)q = pH(q)$ for all $p, q \in R$ or R is commutative. In the first case, we have $H(p)q = pH(q)$ for all $p, q \in R$. Replace p by pr, where $r \in R$, we have $H(p)r1_R(q) = 1_R(p)rH(q)$, where 1_R denotes the identity mapping of R. By Fact 1.1, there exists some $\lambda \in C$ such that $H(x) = \lambda x$ for all $x \in R$. Hence $F(x) = \lambda x + \delta(x)$ for all $x \in R$. In this view, the given hypothesis yields $\lambda = \pm 1$. In the latter case we have R is commutative.

2. Let us consider $F([x, y]) = \delta([x, y]) \pm ([x, y])$ for all $x, y \in R$. Which is equivalent to $H([x, y]) = \pm([x, y])$ for all $x, y \in R$ and H is a multiplicative left centralizer of R. Replace x by xz, we have

$$H([x, y])z + H(x)[z, y] = \pm([x, y])z \pm x[z, y],\ \forall\ x, y, z \in R.$$

It implies that $H(x)[z, y] = \pm x[z, y]$ for all $x, y, z \in R$. This expression is same as (2.1), in the same way we get the conclusions.

□

Theorem 2.4 *Let R be a prime ring with extended centroid C. Let R contain a nontrivial idempotent element and $F : R \to R$ be a multiplicative (generalized)-skew derivation associated with a multiplicative skew derivation δ.*

1. *If $F(x^2) + x\delta(x) = \delta(x^2) + xF(x)$ for all $x \in R$, then either R is commutative or there exists $\lambda \in C$ such that $F(x) = \lambda x + \delta(x)$ for all $x \in R$.*

2. *If $F(x^2) = \delta(x^2)$ for all $x, y \in R$, then either R is commutative or $F = \delta$.*

Proof Let $H = F - \delta$. By Lemma 2.1, H is a multiplicative left centralizer of R. Since R is prime and containing nonzero idempotents, in view of Fact 1.2, H is a left centralizer of R.

1. Let us consider $F(x^2) + x\delta(x) = \delta(x^2) + xF(x)$ for all $x \in R$. It can be seen as $(F - \delta)(x^2) = x(F - \delta)(x)$ for all $x \in R$. That is $H(x^2) = xH(x)$ for all $x \in R$. Since H is a left centralizer of R, we may infer that

$$[H(x), x] = 0,\ \forall\ x \in R. \tag{2.3}$$

Linearizing on x and using (2.3), we get $[H(x), y] + [H(y), x] = 0$ for all $x, y \in R$. Replacing y by yz in the last relation, we get

$$y[H(x), z] = H(y)[x, z], \ \forall \ x, y, z \in R. \tag{2.4}$$

Substituting tx for x in (2.4), we obtain

$$y[H(t), z]x + yH(t)[x, z] = H(y)t[x, z] + H(y)[t, z]x, \ \forall \ x, y, z, t \in R.$$

Eq. (2.4) reduces it to $yH(t)[x, z] = H(y)t[x, z]$ for all $x, y, z, t \in R$. That is

$$(yH(t) - H(y)t)[x, z] = 0, \ \forall \ x, y, z, t \in R.$$

Replace t by tr, where $r \in R$, we find $(yH(t) - H(y)t)R[x, z] = (0)$ for all $x, y, z, t \in R$. Therefore either $yH(t) = H(y)t$ for all $y, t \in R$ or R is commutative. Let us consider $yH(t) = H(y)t$ for all $y, t \in R$. Replace y by yz, we get $1_R(y)zH(t) = H(y)z1_R(t)$ for all $y, z, t \in R$. By Fact 1.1, we find that there exists some $\lambda \in C$ such that $H(x) = \lambda x$ for all $x \in R$. Thus $F(x) = \lambda x + \delta(x)$ for all $x \in R$.

2. Let $F(x^2) = \delta(x^2)$ for all $x \in R$. That is, $H(x^2) = 0$ for all $x \in R$. Linearizing, we find $H(x \circ y) = 0$ for all $x, y \in R$. Replacing y by yz, we get $H(y)[x, z] = 0$ for all $x, y \in R$. We are done by Fact 1.3.

□

Theorem 2.5 *Let R be a $2-$torsion free prime ring and $F : R \to R$ be a multiplicative (generalized)-skew derivation associated with a multiplicative skew derivation δ and a nontrivial automorphism α of R such that $F([x, y]) = \pm x^k[x, \delta(y)]x^m$ for all $x, y \in R$ and fixed positive integers k, m. Then $\delta(x)[\alpha(x), x] = 0 = [\alpha(x), x]\delta(x)$ for all $x \in R$. Moreover, if δ is additive and $\delta(Z(R)) \neq (0)$, then R is commutative.*

Proof Suppose that

$$F([x, y]) = \pm x^k[x, \delta(y)]x^m, \ \forall \ x, y \in R.$$

Substitute yx for y in this expression, we get

$$\begin{aligned} F([x, y]x) &= \pm x^k[x, \delta(y)x + \alpha(y)\delta(x)]x^m \\ F([x, y])x + \alpha([x, y])\delta(x) &= \pm x^k[x, \delta(y)]x^{m+1} \pm x^k\alpha(y)[x, \delta(x)]x^m \\ &\quad \pm x^k[x, \alpha(y)]\delta(x)x^m. \end{aligned}$$

By the given hypothesis, we have

$$\alpha([x, y])\delta(x) = \pm x^k\alpha(y)[x, \delta(x)]x^m \pm x^k[x, \alpha(y)]\delta(x)x^m. \tag{2.5}$$

Replace y by $\alpha^{-1}(x)y$ in (2.5), we obtain

$$x\alpha([x, y])\delta(x) + [\alpha(x), x]\alpha(y)\delta(x) = \pm x^{k+1}\alpha(y)[x, \delta(x)]x^m \pm x^{k+1}[x, \alpha(y)]\delta(x)x^m.$$

Employing (2.5), we get

$$[\alpha(x), x]\alpha(y)\delta(x) = 0, \ \forall \ x, y \in R.$$

It implies that for each $x \in R$, either $[\alpha(x), x] = 0$ or $\delta(x) = 0$. These both cases imply $\delta(x)[\alpha(x), x] = 0$ and $[\alpha(x), x]\delta(x) = 0$ for all $x \in R$.

Suppose that δ is additive. Then from the relation $\delta(x)[\alpha(x), x] = 0$ for all $x \in R$, we obtain

$$\begin{aligned} \delta(x)[\alpha(x), y] + \delta(x)[\alpha(y), x] + \delta(x)[\alpha(y), y] + \delta(y)[\alpha(x), x] \\ +\delta(y)[\alpha(x), y] + \delta(y)[\alpha(y), x] = 0, \ \forall\ x, y \in R. \end{aligned} \tag{2.6}$$

Replace x by $-x$ in (2.6), we get

$$\begin{aligned} \delta(x)[\alpha(x), y] + \delta(x)[\alpha(y), x] - \delta(x)[\alpha(y), y] + \delta(y)[\alpha(x), x] \\ -\delta(y)[\alpha(x), y] - \delta(y)[\alpha(y), x] = 0, \ \forall\ x, y \in R. \end{aligned} \tag{2.7}$$

Combining (2.6) and (2.7), and using 2-torsion free condition of R, we get

$$\delta(x)[\alpha(x), y] + \delta(x)[\alpha(y), x] + \delta(y)[\alpha(x), x] = 0. \tag{2.8}$$

Left multiply (2.8) by $[\alpha(x), x]$, we get

$$[\alpha(x), x]\delta(y)[\alpha(x), x] = 0, \ \forall\ x, y \in R. \tag{2.9}$$

Choose $c \in Z(R)$ such that $0 \neq \delta(c)$ and we replace y by cy in (2.9) to get

$$[\alpha(x), x]\delta(c)y[\alpha(x), x] + \alpha(c)[\alpha(x), x]\delta(y)[\alpha(x), x] = 0.$$

Using (2.9), we get

$$[\alpha(x), x]\delta(c)y[\alpha(x), x]\delta(c) = 0, \ \forall\ x, y \in R.$$

Since α is an automorphism and R is prime, we have

$$[\alpha(x), x]\delta(c) = 0, \ \forall\ x \in R. \tag{2.10}$$

Linearizing on x, we get

$$[\alpha(x), y]\delta(c) = [x, \alpha(y)]\delta(c), \ \forall\ x, y \in R.$$

One may notice that by taking yc for y in (2.9), we have $\delta(c)[\alpha(x), x] = 0$ for all $x \in R$ and hence

$$\delta(c)[\alpha(x), y] = \delta(c)[x, \alpha(y)], \ \forall\ x, y \in R.$$

Using (2.10), we find

$$[\alpha(x), yx]\delta(c) = [\alpha(x), y]x\delta(c), \ \forall\ x, y \in R \tag{2.11}$$

and on the other side, we have

$$\begin{aligned} [x, \alpha(yx)]\delta(c) &= \alpha(y)[x, \alpha(x)]\delta(c) + [x, \alpha(y)]\alpha(x)\delta(c) \\ &= [x, \alpha(y)]\alpha(x)\delta(c). \end{aligned} \tag{2.12}$$

Combining (2.11) and (2.12), we find

$$[\alpha(x), y]x\delta(c) = [x, \alpha(y)]\alpha(x)\delta(c), \ \forall\ x, y \in R.$$

Left multiplying with $\delta(c)$, we get

$$\delta(c)[\alpha(x), y]x\delta(c) = \delta(c)[x, \alpha(y)]\alpha(x)\delta(c), \ \forall \ x, y \in R.$$

It implies that

$$\delta(c)[\alpha(x), y](x - \alpha(x))\delta(c) = 0, \ \forall \ x, y \in R. \tag{2.13}$$

Replace y by $\delta(c)y$ in (2.13), we find

$$\delta(c)^2[\alpha(x), y](x - \alpha(x))\delta(c) + \delta(c)[\alpha(x), \delta(c)]y(x - \alpha(x))\delta(c) = 0, \ \forall \ x, y \in R.$$

Using (2.13), we find that $\delta(c)[\alpha(x), \delta(c)]y(x - \alpha(x))\delta(c) = 0$ for all $x, y \in R$. It implies that for each $x \in R$ either $\delta(c)[\alpha(x), \delta(c)] = 0$ or $(x-\alpha(x))\delta(c) = 0$. Applying Brauer's trick, we find that either $(\alpha(x)-x)\delta(c) = 0$ for all $x \in R$ or $\delta(c)[\alpha(x), \delta(c)] = 0$ for all $x \in R$. The former case is not possible as it implies $\alpha(x) = x$ for all $x \in R$, and the latter case implies that

$$\delta(c) \in Z(R), \ \forall \ c \in Z(R). \tag{2.14}$$

Using (2.14) in (2.10), we find that $[\alpha(x), x] = 0$ for all $x \in R$. In view of Fact 1.4, R is commutative. □

Theorem 2.6 *Let R be a $2-$torsion free prime ring and $F : R \to R$ be a multiplicative (generalized)-skew derivation associated with a multiplicative skew derivation δ and a nontrivial automorphism α of R such that $F(x \circ y) = \pm x^k(x \circ \delta(y))x^m$ for all $x, y \in R$ and fixed positive integers k, m. Then $\delta(x)[\alpha(x), x] = 0 = [\alpha(x), x]\delta(x)$ for all $x \in R$. Moreover, if δ is additive and $\delta(Z(R)) \neq (0)$, then R is commutative.*

Proof Suppose that

$$F(x \circ y) = \pm x^k(x \circ \delta(y))x^m, \ \forall \ x, y \in R.$$

Replace y by yx, we find that

$$\begin{aligned} F((x \circ y)x) &= \pm x^k(x \circ (\delta(y)x + \alpha(y)\delta(x)))x^m \\ F(x \circ y)x + \alpha(x \circ y)\delta(x) &= \pm x^k(x \circ \delta(y))x^{m+1} \pm x^k(x \circ \alpha(y))\delta(x)x^m \\ &\quad \mp x^k\alpha(y)[x, \delta(x)]x^m. \end{aligned}$$

Using the given hypothesis, we have

$$\alpha(x \circ y)\delta(x) = \pm x^k(x \circ \alpha(y))\delta(x)x^m \mp x^k\alpha(y)[x, \delta(x)]x^m. \tag{2.15}$$

Replace y by $\alpha^{-1}(x)y$ in (2.15), we get

$$[\alpha(x), x]\alpha(y)\delta(x) = 0, \ \forall \ x, y \in R.$$

By repeating the same arguments as in Theorem 2.5, we get the desired conclusion. □

Theorem 2.7 *Let R be a prime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with a multiplicative derivation δ of R such that $F([x, y]) = \pm x^k[\delta(x), y]x^m$ for all $x, y \in R$ and fixed positive* *integers k, m. Then $x^k\delta(x) \in Z(R)$ for all $x \in R$. Moreover, if δ is additive, then either R is commutative or $\delta = 0$.*

Proof Suppose that

$$F([x, y]) = x^k[\delta(x), y]x^m, \ \forall\ x, y \in R. \tag{2.16}$$

Taking yx instead of y in (2.16), we get

$$F([x, y])x + [x, y]\delta(x) = x^k[\delta(x), y]x^{m+1} + x^k y[\delta(x), x]x^m, \ \forall\ x, y \in R.$$

By the given hypothesis, we have

$$[x, y]\delta(x) = x^k y[\delta(x), x]x^m, \ \forall\ x, y \in R. \tag{2.17}$$

Substituting ty for y in (2.17), we find

$$t[x, y]\delta(x) + [x, t]y\delta(x) = x^k ty[\delta(x), x]x^m, \ \forall\ x, y, t \in R.$$

Using (2.17), we obtain

$$[x, t]y\delta(x) = [x^k, t]y[\delta(x), x]x^m, \ \forall\ x, y, t \in R. \tag{2.18}$$

Replacing y by yx^k in (2.18), we have

$$[x, t]yx^k\delta(x) = [x^k, t]y(x^k[\delta(x), x]x^m), \ \forall\ x, y, t \in R. \tag{2.19}$$

In particular, taking $x = y$ in (2.16), we find that $0 = x^k[\delta(x), x]x^m$ for all $x \in R$. Thus from (2.19), we have

$$[x, t]yx^k\delta(x) = 0, \ \forall\ x, y \in R.$$

That is $[x, t]Rx^k\delta(x) = (0)$ for all $x, t \in R$. Since R is a prime ring, for each $x \in R$ either $[x, R] = (0)$ or $x^k\delta(x) = 0$. The first case implies that $x \in Z(R)$ and hence $x^k\delta(x) \in Z(R)$ by Lemma 2.2. Thus in each case we have $x^k\delta(x) \in Z(R)$ for all $x \in R$. We now assume that δ is additive. For some fixed positive integer j, we have $[x^k\delta(x), x^j] = 0$ for all $x \in R$. By Fact 1.5, either R is commutative or $\delta = 0$.

By repeating the same arguments with necessary variations, we can get the same conclusion for the identity $F([x, y]) = -x^k([\delta(x), y])x^m$ for all $x, y \in R$. □

Theorem 2.8 *Let R be a prime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with a multiplicative derivation δ of R such that $F(x \circ y) = \pm x(\delta(x) \circ y)x^m$ for all $x, y \in R$ and fixed positive integers m. Then $x[\delta(x), x] = 0$ for all $x \in R$. Moreover, if δ is additive, then either R is commutative or $\delta = 0$.*

Proof Suppose that

$$F(x \circ y) = -x(\delta(x) \circ y)x^m, \ \forall\ x, y \in R. \tag{2.20}$$

Replacing y by yx in (2.20), we obtain

$$F(x \circ y)x + (x \circ y)\delta(x) = -x(\delta(x) \circ y)x^{m+1} + xy[\delta(x), x]x^m, \ \forall\ x, y \in R.$$

Using (2.20), we conclude that

$$(x \circ y)\delta(x) = xy[\delta(x), x]x^m, \ \forall\ x, y \in R. \tag{2.21}$$

Taking ty in place of y in (2.21), we find

$$[x, t]y\delta(x) = [x, t]y[\delta(x), x]x^m, \ \forall\ x, y, t \in R. \tag{2.22}$$

Replacing y by xy in (2.22) and using (2.21), we get

$$\begin{aligned} [x,t]xy\delta(x) &= [x,t]xy[\delta(x),x]x^m \\ &= [x,t](x \circ y)\delta(x) \\ xtxy\delta(x) - tx^2y\delta(x) &= xtxy\delta(x) - tx^2y\delta(x) + xtyx\delta(x) - txyx\delta(x) \\ 0 &= [x,t]yx\delta(x). \end{aligned} \tag{2.23}$$

Substituting y by yx in (2.23), we find

$$[x, t]yx^2\delta(x) = 0, \ \forall\ x, y, t \in R. \tag{2.24}$$

Right multiplying (2.23) by x and combine with (2.24), we obtain $[x, t]y[x, x\delta(x)] = 0$ for all $x, y, t \in R$. In particular, we obtain $[x, x\delta(x)]y[x, x\delta(x)] = 0$ for all $x, y \in R$. Hence we obtain $x[x, \delta(x)] = 0$ for all $x \in R$. In case, δ is additive, Fact 1.5 yields that either R is commutative or $\delta = 0$.

By repeating the same arguments with necessary variations, we can get the same conclusion for the identity $F(x \circ y) = x(\delta(x) \circ y)x^m$ for all $x, y \in R$. □

In this sequel, it is natural to think of the identity $F(x \circ y) = \pm x^k(\delta(x) \circ y)x^m$ for all $x, y \in R$. At this moment we are not able to solve it, therefore we pose it as an open problem.

Conjecture 2.9 *Let R be a prime ring and $F : R \to R$ be a multiplicative (generalized)-derivation associated with a multiplicative derivation δ of R such that $F(x \circ y) = \pm x^k(\delta(x) \circ y)x^m$ for all $x, y \in R$ and fixed positive integers k, m. If δ is additive, then either R is commutative or $\delta = 0$.*

3. Examples

We conclude with some examples showing that the assumption of primeness on R is not redundant in our theorems.

Example 3.1 *Let S be a ring. Consider $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}$. Note that R is not a prime ring. Define maps $F, \delta, \alpha : R \to R$ by $F\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ac \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\delta\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\alpha\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a & -b \\ 0 & 0 & -c \\ 0 & 0 & 0 \end{pmatrix}$. Then it is verified that F is a multiplicative generalized skew* *derivation associated with a skew derivation* δ. *It is easy to see that* $F([x,y]) = \pm x^k[x,\delta(y)]x^m$ *and* $F(x \circ y) = \pm x^k(x \circ \delta(y))x^m$ *for all* $x, y \in R$. *Also* $\delta(Z(R)) \neq (0)$. *But neither* R *is commutative nor* $\delta = 0$. *Hence* R *to be prime in the hypothesis of Theorem 2.5 and 2.6 is essential.*

Example 3.2 *Let* S *be a ring. Consider* $R = \left\{ \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \mid a, b, c \in S \right\}$. *Note that* R *is not a semiprime ring. Define maps* $F, \delta : R \to R$ *by* $F \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ba \\ 0 & 0 & c^2 \\ 0 & 0 & 0 \end{pmatrix}$ *and* $\delta \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *Then it is verified that* F *is a multiplicative generalized derivation associated with a derivation* δ. *It is easy to see that* $F([x,y]) = \pm x^k[\delta(x), y]x^m$ *and* $F(x \circ y) = \pm x(\delta(x) \circ y)x^m$ *for all* $x, y \in R$. *But neither* R *is commutative nor* $\delta = 0$. *Hence* R *to be prime in the hypothesis of Theorem 2.7 and 2.8 is crucial.*

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