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Positive periodic solutions for a class of second-order differential equations with state-dependent delays

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Abstract: In this paper, we consider a class of second order differential equations with iterative source term. The main results are obtained by virtue of a Krasnoselskii fixed point theorem and some useful properties of a Green's function which allows us to prove the existence of positive periodic solutions. Finally, an example is included to illustrate the correctness of our results.

Key words: Positive periodic solutions, nonlinear differential equation, fixed point theorem, Green's function

1. Introduction

Iterative functional differential equation which relates a function to its derivatives and its iterates may be regarded as a special type of the differential equations with complex delays depending on time and state.

This type of equations often arises in modeling of several natural phenomena and has numerous applications in various fields of science such as electrodynamics, epidemiology, biology, etc.

The origin of this type of equations goes back to the early nineteenth century and to the best of our knowledge their study started with a simple example but very important treated by Babage [2] in 1815 and after Babage equation the theory was slowly evolved (see for example [1, 8, 9, 12]).

In recent years, many authors have paid close attention to the study of first order iterative differential equations and many works have been done in this direction. Some mathematicians used Picard's successive approximation, other used the nonexpansive operators technique and some of them applied fixed point theory (see [4, 16, 17] and references therein).

Although these equations may be hard to handle and even though the mathematicians avoid them when possible, some efforts have also been made to investigate higher order iterative differential equations. Among these few works, we mention the works of H. Y. Zhao and J. Liu [18] for studying the following equation

$$c_0x''(t) + c_1x'(t) + c_2x(t) = x(p(t) + bx(t)) + h(t),$$

and E. R. Kaufmann [10] for investigating the below second order iterative functional differential equation

$$x''(t) = f(t, x(t), x(x(t))),$$

where the authors used Schauder's fixed point theorem for proving some results about the existence, uniqueness and stability of periodic solutions. In 2018, we used the same fixed point theorem combined with some useful

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properties of a Green's function to establish sufficient conditions which ensure the existence of periodic solutions for the following second order iterative differential equation (see [5]):

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)$$
$$=\frac{d}{dt}g\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right),$$

where $x(t)=t,\ x^{[1]}(t)=x(t),\ x^{[2]}(t)=x(x(t)),...,x^{[n]}(t)=x^{[n-1]}(x(t))$.

The motivation for our study comes from these previous works and that the positivity of a solution is very important because it is required in the modeling of many phenomena where the state is a density, number of individuals, concentration, electric charge, etc.

In this paper, based on some recent work on positivity of solotions [6, 7, 15] and by virtue of some properties of a Green's function, some recent results such as those obtained in [3, 5, 11, 14, 18] and a Krasnoselskii fixed point theorem, we investigate the existence and positivity of periodic solutions for the following class of second order iterative differential equations:

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)$$
$$=\frac{d}{dt}f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+\sum_{i=1}^{n}c_i(t)x^{[i]}(t), \tag{1.1}$$

where p and q are positive continuous real-valued functions and the functions $f:\mathbb{R}^{n+1}\to\mathbb{R}$ is continuous with respect to its arguments.

The remainder of the paper is as follows: We will give some preliminary results in the next section and in the third and the last section, we establish the existence of positive periodic solutions for our equation and we give an example to illustrate the obtained results.

2. Preliminaries

For the purpose of making the iterative terms $x^{[2]}(t),...,x^{[n]}(t)$ well-defined and applying the Krasnoselskii's fixed point theorem for a sum of 2 mappings, we choose an appropriate Banach space, a closed convex and bounded subset of it and moreover, we assume a set of assumptions.

For $T>0,\ L_1,M\geq 0$ and $L_2>0$, let

$$P_T=\{x\in\mathcal{C}(\mathbb{R},\mathbb{R}),\ x(t+T)=x(t)\},$$

equipped with the norm

$$\|x\|=\sup_{t\in\mathbb{R}}|x(t)|=\sup_{t\in[0,T]}|x(t)|,$$

and

$$P_T(M,L_1,L_2)=\{x\in P_T: L_1\leq x\leq L_2,\ |x(t_2)-x(t_1)|\leq M|t_2-t_1|,\ \forall t_1,t_2\in\mathbb{R}\}.$$

Then $(P_T,\|.\|)$ is a Banach space and $P_T(M,L_1,L_2)$ is a closed convex and bounded subset of P_T.

We assume that p,q and $c_i,\ i=\overline{1,n}$ are continuous real-valued functions such that

$$p(t+T)=p(t),\ q(t+T)=q(t),\ c_i(t+T)=c_i(t), \tag{2.1}$$

and

$$\int_0^T p(s)\,ds > 0, \quad \int_0^T q(s)\,ds > 0. \tag{2.2}$$

The function $f(t, x_1, x_2, ..., x_n)$ is supposed periodic in t with period T and globally Lipschitz in $x_1, ..., x_n$, i.e.

$$f(t+T, x_1, ..., x_n) = f(t, x_1, ..., x_n), \tag{2.3}$$

and there exist n positive constants $k_1,\ k_2, ..., k_n$ such that

$$|f(t, x_1, ..., x_n) - f(t, y_1, ..., y_2)| \leq \sum_{i=1}^{n} k_i \|x_i - y_i\|. \tag{2.4}$$

Lemma 2.1 *([11]) Suppose that (2.1) and (2.2) hold and*

$$\frac{R_1 \left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \geq 1, \tag{2.5}$$

where

$$R_1 = \max_{t\in[0,T]} \left| \int_t^{t+T} \frac{\exp\left(\int_t^s p(u)\,du\right)}{\exp\left(\int_0^T p(u)\,du\right) - 1} q(s)ds \right|,$$

and

$$Q_1 = \left(1 + \exp\left(\int_0^T p(u)\,du\right)\right)^2 R_1^2.$$

Then, there are continuous and T-periodic functions a and b such that

$$b(t) > 0, \quad \int_0^T a(u)\,du > 0, \ a(t) + b(t) = p(t),$$

and

$$\frac{d}{dt} b(t) + a(t)b(t) = q(t),$$

for all $t \in \mathbb{R}$.

Lemma 2.2 *([14]) Suppose the conditions of Lemma 2.1 hold and $\varphi \in P_T$. Then the equation*

$$\frac{d^2}{dt^2} x(t) + p(t)\frac{d}{dt} x(t) + q(t)x(t) = \varphi(t),$$

has a $T-$periodic solution. Moreover, the periodic solution can be expressed as

$$x(t) = \int_t^{t+T} G(t,s)\varphi(s)ds,$$

where

$$G(t,s)=\frac{\int_t^s \exp\left[\int_t^u b(v)\,dv+\int_u^s a(v)\,dv\right]du}{\left[\exp(\int_0^T a(u)\,du)-1\right]\left[\exp(\int_0^T b(u)\,du)-1\right]}+\frac{\int_s^{t+T} \exp\left[\int_t^u b(v)\,dv+\int_u^{s+T} a(v)\,dv\right]du}{\left[\exp(\int_0^T a(u)\,du)-1\right]\left[\exp(\int_0^T b(u)\,du)-1\right]}. \tag{2.6}$$

Corollary 2.3 *([14]) Green's function G defined by (2.6) satisfies the following properties*

$$G(t,t+T)=G(t,t),\quad G(t+T,s+T)=G(t,s),$$

$$\frac{\partial}{\partial s}G(t,s)=a(s)G(t,s)-\frac{\exp(\int_t^s b(v)\,dv)}{\exp(\int_0^T b(v)\,dv)-1},\ \frac{\partial}{\partial t}G(t,s)=-b(t)G(t,s)-\frac{\exp(\int_t^s a(v)\,dv)}{\exp(\int_0^T a(v)\,dv)-1}. \tag{2.7}$$

Theorem 2.4 *([13])(Krasnoselskii) Let $\mathbb{B}$ be a closed convex nonempty subset of a Banach space $(\mathbb{X},\|.\|)$. Suppose that $\mathcal{A}$ and $\mathcal{B}$ map $\mathbb{B}$ into $\mathbb{X}$ such that*
(i) $x,y\in\mathbb{B}$, implies $\mathcal{A}x+\mathcal{B}y\in\mathbb{B}$,
(ii) $\mathcal{A}$ is compact and continuous,
(iii) $\mathcal{B}$ is a contraction mapping.
Then there exists $z\in\mathbb{B}$ with $z=\mathcal{A}z+\mathcal{B}z$.

Lemma 2.5 *Suppose (2.1)-(2.3) and (2.5) hold. Then $x\in P_T(M,L_1,L_2)\cap C^2(\mathbb{R},\mathbb{R})$ is a solution of (1.1) if and only if $x\in P_T(M,L_1,L_2)$ is a solution of*

$$x(t)=\int_t^{t+T}\left[E(t,s)-a(s)G(t,s)\right]f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds+\sum_{i=1}^{n}\int_t^{t+T}G(t,s)c_i(s)x^{[i]}(s)\,ds,$$

where

$$E(t,s)=\frac{\exp(\int_t^s b(v)\,dv)}{\exp(\int_0^T b(v)\,dv)-1}. \tag{2.8}$$

Proof Suppose that $x\in P_T(M,L_1,L_2)\cap C^2(\mathbb{R},\mathbb{R})$ is a solution of (1.1). From Lemma 2.2, we have

$$x(t)=\int_t^{t+T}G(t,s)\left[\frac{d}{ds}f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]ds+\sum_{i=1}^{n}\int_t^{t+T}G(t,s)c_i(s)x^{[i]}(s)\,ds.$$

Integration by parts gives

$$\int_t^{t+T}G(t,s)\left[\frac{d}{ds}f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]ds=\left[G(t,s)f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\right]_t^{t+T}-\int_t^{t+T}\left(\frac{d}{ds}G(t,s)\right)f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)ds.$$

It follows from Corollary 2.3 that

$$\left[G(t,s)f\left(s,x\left(s\right),x^{[2]}\left(s\right),...,x^{[n]}\left(s\right)\right)\right]_{t}^{t+T}=0,$$

and

$$\begin{aligned}&\int_{t}^{t+T}G(t,s)\left[\frac{d}{dt}f\left(s,x\left(s\right),x^{[2]}\left(s\right),...,x^{[n]}\left(s\right)\right)\right]ds\\&=\int_{t}^{t+T}f\left(s,x\left(s\right),x^{[2]}\left(s\right),...,x^{[n]}\left(s\right)\right)\left[\frac{\exp(\int_{t}^{s}b\left(v\right)dv)}{\exp(\int_{0}^{T}b\left(v\right)dv)-1}-a\left(s\right)G(t,s)\right]ds.\end{aligned}$$

Consequently

$$\begin{aligned}x\left(t\right)=&\int_{t}^{t+T}\left[E\left(t,s\right)-a(s)G(t,s)\right]f\left(s,x\left(s\right),x^{[2]}\left(s\right),...,x^{[n]}\left(s\right)\right)ds\\&+\sum_{i=1}^{n}\int_{t}^{t+T}G(t,s)c_{i}\left(s\right)x^{[i]}\left(s\right)ds,\end{aligned}$$

and the solutions of (1.1) are solutions of this last integral equation.

Now, we show that the solutions of the integral equation are solutions of (1.1). In view of Lemma 2.1 and Corollary 2.3 , $G(t,s)$ is a differentiable function and for $x\in P_{T}\left(M,L_{1},L_{2}\right)$ we have

$$\begin{aligned}\frac{d}{dt}x\left(t\right)=&f\left(t,x\left(t\right),...,x^{[n]}\left(t\right)\right)+\int_{t}^{t+T}\left[-b\left(t\right)E\left(t,s\right)+\left(a(s)b(t)G(t,s)-a(s)V\left(t,s\right)\right)\right]f\left(s,x\left(s\right),...,x^{[n]}\left(s\right)\right)ds\\&+\sum_{i=1}^{n}\int_{t}^{t+T}\left(V\left(t,s\right)-b(t)G(t,s)\right)c_{i}\left(s\right)x^{[i]}\left(s\right)ds,\end{aligned}$$

where

$$V\left(t,s\right)=\frac{\exp(\int_{t}^{s}a\left(v\right)dv)}{\exp(\int_{0}^{T}a\left(v\right)dv)-1}.$$

Thus

$$\begin{aligned}\frac{d^{2}}{dt^{2}}x\left(t\right)=&\frac{d}{dt}f\left(t,x\left(t\right),...,x^{[n]}\left(t\right)\right)-p\left(t\right)f\left(t,x\left(t\right),x^{[2]}\left(t\right),...,x^{[n]}\left(t\right)\right)\\&+\int_{t}^{t+T}\left\{\left(b^{2}\left(t\right)-b'\left(t\right)\right)E\left(t,s\right)+a(s)\left(b'\left(t\right)-b^{2}(t)\right)G(t,s)+a(s)p(t)V\left(t,s\right)\right\}f\left(s,x\left(s\right),...,x^{[n]}\left(s\right)\right)ds\\&+\sum_{i=1}^{n}c_{i}\left(t\right)x^{[i]}\left(t\right)+\sum_{i=1}^{n}\int_{t}^{t+T}\left\{-p\left(t\right)V\left(t,s\right)+\left(b^{2}\left(t\right)-b'(t)\right)G(t,s)-b(t)\right\}c_{i}\left(s\right)x^{[i]}\left(s\right)ds.\end{aligned}$$

Consequently

$$\begin{aligned}\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t) &= \frac{d}{dt}f\left(t,x(t),...,x^{[n]}(t)\right)\\ &+\int_t^{t+T}\left(b^2(t)+q(t)-b'(t)-p(t)b(t)\right)E(t,s)f\left(s,x(s),...,x^{[n]}(s)\right)ds\\ &+\int_t^{t+T}a(s)\left(b'(t)+p(t)b(t)-b^2(t)-q(t)\right)G(t,s)f\left(s,x(s),...,x^{[n]}(s)\right)ds\\ &+\sum_{i=1}^{n}c_i(t)x^{[i]}(t)+\sum_{i=1}^{n}\int_t^{t+T}\left(b^2(t)+q(t)-b'(t)-p(t)b(t)\right)G(t,s)c_i(s)x^{[i]}(s)ds.\end{aligned}$$

It follows from Lemma 2.1 that

$$b'(t)+p(t)b(t)=b^2(t)+q(t),$$

which gives

$$\frac{d^2}{dt^2}x(t)+p(t)\frac{d}{dt}x(t)+q(t)x(t)=\frac{d}{dt}f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)+\sum_{i=1}^{n}c_i(t)x^{[i]}(t).$$

This completes the proof. □

Lemma 2.6 *([14]) Let* $A=\int_0^T p(u)\,du$ *and* $B=T^2\exp\left(\frac{1}{T}\int_0^T \ln(q(u))\,du\right)$. *If*

$$A^2\geq 4B, \tag{2.9}$$

then

$$\min\left\{\int_0^T a(u)\,du,\int_0^T b(u)\,du\right\}\geq\frac{1}{2}\left(A-\sqrt{A^2-4B}\right):=l,$$

and

$$\max\left\{\int_0^T a(u)\,du,\int_0^T b(u)\,du\right\}\leq\frac{1}{2}\left(A+\sqrt{A^2-4B}\right):=m.$$

Corollary 2.7 *([14]) Functions* G *and* E *satisfy*

$$\alpha_1\leq G(t,s)\leq\alpha_2 \text{ and } |E(t,s)|\leq\beta, \tag{2.10}$$

where

$$\alpha_1=\frac{T}{(e^m-1)^2},\ \alpha_2=\frac{T\exp(\int_0^T p(u)\,du)}{(e^l-1)^2} \text{ and } \beta=\frac{e^m}{e^l-1}.$$

Lemma 2.8 *([17]) For any* $\varphi,\ \psi\in P_T(M,L_1,L_2)$,

$$\left\|\varphi^{[m]}-\psi^{[m]}\right\|\leq\sum_{j=0}^{m-1}M^j\|\varphi-\psi\|,\ m=1,2,...$$

Lemma 2.9 *([16]) It holds*

$$P_T(M, L_1, L_2) = \{x \in P_T,\ L_1 \le x \le L_2,\ |x(t_2) - x(t_1)| \le M|t_2 - t_1|,\ \forall t_1, t_2 \in [0, T]\}.$$

Lemma 2.10 *([5]) For any $t_1, t_2 \in \mathbb{R}$*

$$\int_{t_1}^{t_1+T} |G_2(t_2, s) - G_2(t_1, s)|\, ds \le \mu |t_2 - t_1|,$$

where

$$\mu = Te^{2m}\delta\left[T\lambda\gamma\left(2e^{2m} + 1\right) + e^m + 1\right],\ \delta = \frac{1}{\left[\exp\left(\int_0^T a(v)\, dv\right) - 1\right]\left[\exp\left(\int_0^T b(v)\, dv\right) - 1\right]},$$

and

$$\lambda = \max_{t\in[0,T]} |b(t)|,\ \gamma = \exp\left(-\int_0^T b(v)\, dv\right).$$

3. Existence of positive periodic solutions

In this section we will use the Krasnoselskii's fixed point theorem for a sum of 2 mappings to prove the existence of positive periodic solutions of the equation (1.1). So, we need to define a Banach space $\mathbb{X}$, a closed convex subset $\mathbb{B}$ of $\mathbb{X}$ and to construct 2 mappings, one of which is compact and the other is a contraction. Let $\mathbb{X} = P_T$ and $\mathbb{B} = P_T(M, L_1, L_2)$. By virtue of Lemma 2.5 we define 2 operators $\mathcal{A},\ \mathcal{B} : P_T(M, L_1, L_2) \longrightarrow P_T$ by

$$\begin{aligned}(\mathcal{A}\varphi)(t) = \sum_{i=1}^{n} \int_t^{t+T} &\left[c_i(s)\varphi^{[i]}(s)\right] G(t,s)ds \\ &- \int_t^{t+T} a(s) f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t,s)ds, \end{aligned} \tag{3.1}$$

and

$$(\mathcal{B}\varphi)(t) = \int_t^{t+T} E(t,s) f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) ds. \tag{3.2}$$

Clearly, $(\mathcal{A}\varphi)(t+T) = (\mathcal{A}\varphi)(t)$ and $(\mathcal{B}\varphi)(t+T) = (\mathcal{B}\varphi)(t)$. Furthermore, fixed points of the operator $\mathcal{A} + \mathcal{B} = \mathcal{H}$ are solutions of equation (1.1) and vice versa.

To prove the existence of positive periodic solutions of equation (1.1) we distinguish 2 cases:

$f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \ge 0$, and $f\left(t, x(t), x^{[2]}(t), ..., x^{[n]}(t)\right) \le 0$, for all $t \in \mathbb{R}$ and $x \in P_T(M, L_1, L_2)$.

Case 1: Now, we start by the case $f(t, x_1, x_2, ..., x_2) \ge 0$ and we will prove set of preparatory Lemmas in order to use them in the proof of the main existence results. In this reason, we assume that there exist a nonnegative constant ξ and a positive constants ν and σ such that

$$E(t,s) > \sigma,\ \text{for all } t, s \in [0,T] \times [0,T], \tag{3.3}$$

$$\nu x \le f\left(t, x, x^{[2]}, ..., x^{[n]}\right) \le \xi x,\ \text{for all } t \in [0,T],\ x \in P_T(M, L_1, L_2), \tag{3.4}$$

$$T\beta\xi < 1, \tag{3.5}$$

and for all $s \in [0,T]$, $x \in P_T(M, L_1, L_2)$,

$$L_1 \frac{1 - T\sigma\nu}{T\alpha_1} \leq \sum_{i=1}^{n} c_k(s)\, x^{[k]}(s) - a(s)\, f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \leq L_2 \frac{1 - T\beta\xi}{T\alpha_2}. \tag{3.6}$$

Lemma 3.1 *Suppose that* $c_i \in P_T\left(M_{c_i}, L_{1_{c_i}}, L_{2_{c_i}}\right)$, $i = \overline{1,n}$. *If conditions (2.4) and (2.9) hold, then the operator* $\mathcal{A}$ *defined by (3.1) is continuous and compact on* $P_T(M, L_1, L_2)$.

Proof Obviously $P_T(M, L_1, L_2)$ is an uniformly bounded and equicontinuous subset of the space of continuous functions on $[0,T]$, so it follows from the Ascoli-Arzela theorem that $P_T(M, L_1, L_2)$ is a compact subset from this space.

We know that any continuous operator maps compact sets into compact sets, then to show that $\mathcal{A}$ is a compact operator it suffices to prove that it is continuous.

Let $\varphi, \theta \in P_T(M, L_1, L_2)$ and $c_i \in P_T\left(M_{c_i}, L_{1_{c_i}}, L_{2_{c_i}}\right)$, $i = \overline{1,n}$. We have

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\theta)(t)\right| \\
&\leq \sum_{i=1}^{n} \int_t^{t+T} |G(t,s)|\, |c_k(s)| \left|\varphi^{[k]}(s) - \theta^{[k]}(s)\right| ds \\
&+ \int_t^{t+T} |G(t,s)|\, |a(s)| \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right)\right| ds.
\end{aligned}$$

By using (2.10), we obtain

$$\begin{aligned}
&\left|(\mathcal{A}\varphi)(t) - (\mathcal{A}\theta)(t)\right| \leq \alpha_2 \sum_{i=1}^{n} \int_t^{t+T} L_{2_{c_i}} \left|\varphi^{[k]}(s) - \theta^{[k]}(s)\right| ds \\
&+ \alpha_2 \|a\| \int_t^{t+T} \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right)\right| ds.
\end{aligned}$$

Due to Lemma 2.8, we get

$$\alpha_2 \sum_{i=1}^{n} \int_t^{t+T} L_{2_{c_i}} \left|\varphi^{[k]}(s) - \theta^{[k]}(s)\right| ds \leq T\alpha_2 \sum_{i=1}^{n} L_{2_{c_i}} \sum_{j=0}^{i-1} M^j \|\varphi - \theta\|.$$

It follows from (2.4) and Lemma 2.8, that

$$\begin{aligned}
&\alpha_2 \|a\| \int_t^{t+T} \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) - f\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right)\right| ds \\
&\leq T\alpha_2 \|a\| \sum_{i=1}^{n} k_i \left\|\varphi^{[i]} - \theta^{[i]}\right\| \\
&\leq T\alpha_2 \|a\| \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j \|\varphi - \theta\|.
\end{aligned}$$

Consequently

$$|(\mathcal{A}\varphi)(t) - (\mathcal{A}\theta)(t)| \leq T\alpha_2 \sum_{i=1}^{n} L_{2_{c_i}} \sum_{j=0}^{i-1} M^j \|\varphi - \theta\| + T\alpha_2 \|a\| \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j \|\varphi - \theta\|$$

$$= T\alpha_2 \sum_{i=1}^{n} \left(L_{2_{c_i}} + \|a\| k_i\right) \sum_{j=0}^{i-1} M^j \|\varphi - \theta\|,$$

which shows that the operator $\mathcal{A}$ is continuous. Therefore, the compactness of $\mathcal{A}$ follows from its continuity. □

Lemma 3.2 *If condition (2.4) and (2.9) hold and*

$$T\beta \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j < 1, \tag{3.7}$$

then the operator given by (3.2) is a contraction.

Proof Let φ and $\theta \in P_T(M, L_1, L_2)$. By using (2.10), we obtain

$$|(\mathcal{B}\varphi)(t) - (\mathcal{B}\theta)(t)| \leq \int_t^{t+T} |E(t,s)| \left| f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right.$$

$$\left. - f\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right) \right| ds$$

$$\leq T\beta \sum_{i=1}^{n} k_i \left\| \varphi^{[i]} - \theta^{[i]} \right\|.$$

Due to Lemma 2.8, we get

$$\|(\mathcal{B}\varphi) - (\mathcal{B}\theta)\| = \sup_{t\in[0,T]} |(\mathcal{B}\varphi)(t) - (\mathcal{B}\theta)(t)| \leq \Gamma \|\varphi - \theta\|,$$

where

$$\Gamma = T\beta \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j.$$

Using (3.7), we conclude that $\mathcal{B}$ is a contraction. □

Lemma 3.3 *Suppose (2.3)-(2.5) and (3.3)-(3.6) hold. If*

$$L_2 \left[(\mu + 2\alpha_2) \frac{1 - T\beta\xi}{T\alpha_2} + \left(2\beta + \frac{T\|a\| \exp\left(\int_0^T b(v)\, dv\right)}{\exp(\int_0^T b(v)\, dv) - 1} \right) \xi \right] \leq M, \tag{3.8}$$

then

$$(\mathcal{A}\varphi) + (\mathcal{B}\theta) \in P_T(M, L_1, L_2),$$

for all $\varphi, \theta \in P_T(M, L_1, L_2)$.

Proof Let φ and $\theta \in P_T(M, L_1, L_2)$, we have

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\theta)(t) = \int_t^{t+T} \left[\left(\sum_{i=1}^{n} c_k(s)\varphi^{[k]}(s) \right) - a(s) f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) \right] G(t,s)ds$$
$$+ \int_t^{t+T} E(t,s) f\left(s, \theta(s), \theta^{[2]}(s), ..., \theta^{[n]}(s)\right) ds.$$

It follows from (2.10), (3.3), (3.4) and (3.6), that

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\theta)(t) \leq T\beta L_2\xi + \alpha_2 T L_2 \frac{1 - T\beta\xi}{T\alpha_2}$$
$$= L_2,$$

and

$$(\mathcal{A}\varphi)(t) + (\mathcal{B}\theta)(t) \geq T\sigma L_1\nu + \alpha_1 T L_1 \frac{1 - T\sigma\nu}{T\alpha_1}$$
$$= L_1.$$

which implies that

$$L_1 \leq (\mathcal{A}\varphi)(t) + (\mathcal{B}\theta)(t) \leq L_2. \tag{3.9}$$

Now, let $t_1, \ t_2 \in [0, T]$ with $t_1 < t_2$. We have

$$|((\mathcal{A}\varphi) + (\mathcal{B}\theta))(t_2) - ((\mathcal{A}\varphi) + (\mathcal{B}\theta))(t_1)| = |((\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)) + (\mathcal{B}\theta)(t_2) - (\mathcal{B}\theta)(t_1)|$$
$$\leq |(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| + |(\mathcal{B}\theta)(t_2) - (\mathcal{B}\theta)(t_1)|,$$

where

$$|(\mathcal{A}\varphi)(t_2) - (\mathcal{A}\varphi)(t_1)| = \left| \sum_{i=1}^{n} \int_{t_2}^{t_2+T} G(t_2,s) c_k(s) \varphi^{[k]}(s) ds \right.$$
$$- \sum_{i=1}^{n} \int_{t_1}^{t_1+T} G(t_1,s) c_k(s) \varphi^{[k]}(s) ds - \int_{t_2}^{t_2+T} a(s) f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_2,s)$$
$$\left. + \int_{t_1}^{t_1+T} a(s) f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) G(t_1,s) ds \right|,$$

and

$$|(\mathcal{B}\theta)(t_2) - (\mathcal{B}\theta)(t_1)| = \left| \int_{t_2}^{t_2+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) E(t_2,s) ds \right.$$
$$\left. - \int_{t_1}^{t_1+T} f\left(s, \varphi(s), \varphi^{[2]}(s), ..., \varphi^{[n]}(s)\right) E(t_1,s) ds \right|.$$

We have

$$
\begin{aligned}
&\left|\left(\mathcal{A}\varphi\right)(t_2)-\left(\mathcal{A}\varphi\right)(t_1)\right| \\
&\leq \int_{t_2}^{t_1}\left|\left(\sum_{i=1}^{n}c_k(s)\varphi^{[k]}(s)\right)-a(s)f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|G(t_2,s)\,ds \\
&+\int_{t_1+T}^{t_2+T}\left|\left(\sum_{i=1}^{n}c_k(s)\varphi^{[k]}(s)\right)-a(s)f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|G(t_2,s)\,ds \\
&+\int_{t_1}^{t_1+T}\left|\left(\sum_{i=1}^{n}c_k(s)\varphi^{[k]}(s)\right)-a(s)f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\right|\left|G(t_2,s)-G(t_1,s)\right|ds.
\end{aligned}
$$

It follows from (2.10), (3.6) and Lemma 2.9, that

$$
\begin{aligned}
\left|\left(\mathcal{A}\varphi\right)(t_2)-\left(\mathcal{A}\varphi\right)(t_1)\right| &\leq 2\alpha_2L_2\frac{1-T\beta\xi}{T\alpha_2}\left|t_2-t_1\right|+\mu L_2\frac{1-T\beta\xi}{T\alpha_2}\left|t_2-t_1\right| \\
&=\left(\mu+2\alpha_2\right)L_2\frac{1-T\beta\xi}{T\alpha_2}\left|t_2-t_1\right|. \qquad (3.10)
\end{aligned}
$$

In the other hand,

$$
\begin{aligned}
\left|\left(\mathcal{B}\theta\right)(t_2)-\left(\mathcal{B}\theta\right)(t_1)\right| \leq &\int_{t_2}^{t_1}f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)E(t_2,s)\,ds \\
&+\int_{t_1+T}^{t_2+T}f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)E(t_2,s)\,ds \\
&+\int_{t_1}^{t_1+T}f\left(s,\varphi(s),\varphi^{[2]}(s),...,\varphi^{[n]}(s)\right)\left|E(t_2,s)-E(t_1,s)\right|ds.
\end{aligned}
$$

But

$$
\begin{aligned}
&\int_{t_1}^{t_1+T}\left(E(t_2,s)-E(t_1,s)\right)ds \\
&=\frac{1}{\exp(\int_0^T b(v)\,dv)-1}\int_{t_1}^{t_1+T}\left(\exp(\int_{t_2}^{s}b(v)\,dv)-\exp(\int_{t_1}^{s}b(v)\,dv)\right)ds \\
&=\frac{1}{\exp(\int_0^T b(v)\,dv)-1}\int_{t_1}^{t_1+T}\exp\left(\int_{t_2}^{s}b(v)\,dv\right)\left|1-\exp\left(\int_{t_1}^{t_2}b(v)\,dv\right)\right|ds,
\end{aligned}
$$

It follows from the mean value theorem, that

$$
\int_{t_1}^{t_1+T}\left(E(t_2,s)-E(t_1,s)\right)ds \leq T\left\|a\right\|\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp(\int_0^T b(v)\,dv)-1}\left|t_2-t_1\right|.
$$

So,

$$\begin{aligned}\left|(\mathcal{B}\theta)(t_2)-(\mathcal{B}\theta)(t_1)\right| &\leq 2\beta L_2\xi\left|t_2-t_1\right|+T\|a\| L_2\xi\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp(\int_0^T b(v)\,dv)-1}\left|t_2-t_1\right| \\ &=\left(2\beta+T\|a\|\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp(\int_0^T b(v)\,dv)-1}\right)L_2\xi\left|t_2-t_1\right|. \end{aligned} \tag{3.11}$$

From (3.10), (3.11) and Lemma 2.8, we get

$$\begin{aligned}\left|((\mathcal{A}\varphi)+(\mathcal{B}\theta))(t_2)-((\mathcal{A}\varphi)+(\mathcal{B}\theta))(t_1)\right| &\leq(\mu+2\alpha_2)L_2\frac{1-T\beta\xi}{T\alpha_2}\left|t_2-t_1\right| \\ &+\left(2\beta+T\|a\|\frac{\exp\left(\int_0^T b(v)\,dv\right)}{\exp(\int_0^T b(v)\,dv)-1}\right)L_2\xi\left|t_2-t_1\right| \\ &=\Lambda\left|t_2-t_1\right|,\ \text{for all } t_1,t_2\in\mathbb{R}, \end{aligned} \tag{3.12}$$

where

$$\Lambda=L_2\left[(\mu+2\alpha_2)\frac{1-T\beta\xi}{T\alpha_2}+\left(2\beta+\frac{T\|a\|\exp\left(\int_0^T b(v)\,dv\right)}{\exp(\int_0^T b(v)\,dv)-1}\right)\xi\right].$$

It follows from (3.8)

$$\left|((\mathcal{A}\varphi)+(\mathcal{B}\theta))(t_2)-((\mathcal{A}\varphi)+(\mathcal{B}\theta))(t_1)\right|\leq M\left|t_2-t_1\right|.$$

According to (3.9) and (3.12), $(\mathcal{A}\varphi)+(\mathcal{B}\theta)\in P_T(M,L_1,L_2)$, for all φ and $\theta\in P_T(M,L_1,L_2)$ and the proof is complete. □

Theorem 3.4 *Suppose that $c_i\in P_T\left(M_{c_i},L_{1_{c_i}},L_{2_{c_i}}\right)$, $i=\overline{1,n}$. If conditions (2.3)-(2.5), (2.9) and (3.3)-(3.7) hold, then equation (1.1) has at least a solution $x\in P_T(M,L_1,L_2)$.*

Proof As consequences of Lemmas 3.1, 3.2, 3.3, all the conditions of Theorem 2.4 are satisfied. So, $\mathcal{H}=\mathcal{A}+\mathcal{B}$ has at least a fixed point on $P_T(M,L_1,L_2)$. Finally, via Lemma 2.5, equation (1.1) has at least a solution on $P_T(M,L_1,L_2)$. □

Case 2: To treat the case $f\left(t,x(t),x^{[2]}(t),...,x^{[n]}(t)\right)\leq 0$, we replace conditions (3.4)-(3.6) by the following conditions respectively. We assume that there exist a negative constant $\xi^{'}$ and a nonpositive constants $\nu^{'}$ such that

$$\xi^{'}x\leq f\left(t,x,x^{[2]},...,x^{[n]}\right)\leq\nu^{'}x,\ \text{for all } t\in[0,T],\ x\in P_T(M,L_1,L_2), \tag{3.13}$$

$$-T\beta\xi^{'}<1, \tag{3.14}$$

and for all $s\in[0,T]$, $x\in P_T(M,L_1,L_2)$,

$$\frac{L_1-T\beta L_2\xi^{'}}{T\alpha_1}\leq\sum_{i=1}^{n}c_k(s)\,\varphi^{[k]}(s)-a(s)\,f\left(s,x(s),x^{[2]}(s),...,x^{[n]}(s)\right)\leq\frac{L_2-\sigma TL_1\nu^{'}}{T\alpha_2}. \tag{3.15}$$

and we will follow the same steps as in the previous theorem to prove the following theorem:

Theorem 3.5 *Suppose that* $c_i \in P_T\left(M_{c_i}, L_{1_{c_i}}, L_{2_{c_i}}\right),\ i = \overline{1,n}$. *If conditions (2.3)-(2.5), (2.9) and (3.3), (3.7) and (3.13)-(3.15) hold, then equation (1.1) has at least a solution* $x \in P_T\left(M, L_1, L_2\right)$.

4. Example

In this section, an example of an application is given as a test case.

We consider the following equation:

$$\begin{aligned} & x^{''} + (1.48)\, x^{'}(t) + q(t)(0.52)\, x(t) = \frac{0.000\,5}{2} x(t) + \frac{0.0005}{2} x^{[2]}(t) \\ & + \left(0.014\,5 + 0.00013 \sin^2(16\pi x)\right) x(t) + \left(0.014\,5 + 0.00013 \sin^2(16\pi x)\right) x^{[2]}(t), \end{aligned} \tag{4.1}$$

where

$$p(t) = 1.48,\ q(t) = 0.52,\ f(t,x,y) = \frac{0.000\,5}{2} x + \frac{0.0005}{2} y,\ c_1(t) = c_2(t) = 0.014\,5 + 0.00013 \sin^2(16\pi x).$$

Let $M = 12,\ L_1 = 0.01,\ L_2 = 0.1$ and $T = \frac{1}{8}$, i.e. $P_T(M, L_1, L_2) = P_{\frac{1}{8}}(12, 0.01, 0.1)$.
If $x_1, y_1, x_2, y_2 \in P_{\frac{1}{8}}(12, 0.01, 0.1)$, then

$$|f(t, x_1, y_1) - f(t, x_2, y_2)| \le k_1 \|x_1 - x_2\| + k_2 \|y_1 - y_2\|,$$

where

$$k_1 = 0.000\,25,\ k_2 = 0.003\,25.$$

By direct calculation we obtain

$$\begin{aligned} & R_1 \backsimeq 0.351\,35,\ Q_1 \backsimeq 0.173\,35,\ \frac{R_1\left[\exp\left(\int_0^T p(u)\,du\right) - 1\right]}{Q_1 T} \backsimeq 3.\,295\,1 \ge 1, \\ & A \backsimeq 0.185,\ B \backsimeq 0.008125,\ A^2 \backsimeq 0.03422\,5 \ge 4B \backsimeq 0.032\,5, \\ & l \backsimeq 0.07173\,3,\ m \backsimeq 0.113\,27,\ \alpha_1 \backsimeq 8.\,690\,1,\ \alpha_2 \backsimeq 27.\,194, \\ & \beta \backsimeq 15.\,059,\ a(t) = 0.573\,87,\ b(t) = 0.906\,13,\ \delta \backsimeq 112.\,12, \\ & \lambda = 0.906\,13,\ \gamma \backsimeq 0.892\,91,\ \mu \backsimeq 43.\,502. \end{aligned}$$

For $\sigma = 7.445\,3,\ \xi = 0.005$ and $\nu = 0.00005$, we have

$$\nu x \le 0.000\,05 \le f\left(t, x, x^{[2]}, ..., x^{[n]}\right) \le 0.0005 \le \xi x,\ \forall t \in \left[0, \frac{1}{8}\right],\ \forall x \in P_{\frac{1}{8}}(12, 0.1, 1),$$

$$T\beta\xi \backsimeq 0.0094119 < 1,\ L_1 \frac{1 - T\sigma\nu}{T\alpha_1} \backsimeq 0.0092054,\ L_2 \frac{1 - T\beta\xi}{T\alpha_2} \backsimeq 0.02914\,1,$$

$$T\beta \sum_{i=1}^{n} k_i \sum_{j=0}^{i-1} M^j \backsimeq 0.080001 < 1,\ \Lambda \backsimeq 2.868 < M = 12,$$

and

$$0.028687 \leq \sum_{i=1}^{n} c_k(s)\, x^{[k]}(s) - a(s)\, f\left(s, x(s), x^{[2]}(s), ..., x^{[n]}(s)\right) \leq 0.2899\,7.$$

Therefore by the existence Theorem 3.4, there exists at least a positive solution to the equation (4.1) in $P_{\frac{1}{8}}(12, 0.01, 0.1)$.

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