

T -soft equality relation

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Abstract: The desire of generalizing some set-theoretic properties to the soft set theory motivated many researchers to define various types of soft operators. For example, they redefined the complement of a soft set, and soft union and intersection between two soft sets in a way that satisfies De Morgan's laws. In this paper, we introduce and study the concepts of T -soft subset and T -soft equality relations. Then, we utilize them to define the concepts of T -soft union and T -soft intersection for arbitrary family of soft sets. By T -soft union, we successfully keep some classical properties via soft set theory. We conclude this work by giving and investigating new types of soft linear equations with respect to some soft equality relations. Illustrative examples are provided to elucidate main obtained results.

Key words: T -soft subset, T -soft equality, T -soft union, T -soft intersection, soft linear equations

1. Introduction

Many mathematical tools such as probability theory, fuzzy set theory [33], and rough set theory [25] were introduced by several mathematicians in order to solve and model complicated problems which contain uncertainty, vagueness, and ambiguity in different fields like economics, engineering, medical sciences, social sciences. Molodtsov [24] demonstrated that these theories have their inherent difficulties which are attributed to examine the existence of mean by performing a large number of trials in the case of probability theory, and to the possibility of determining a membership function in the case of fuzzy set theory. Consequently, Molodtsov [24] originated a new mathematical approach, namely soft set. He investigated its applications in various disciplines such as game theory, operations research, and probability theory. Immediately afterwards, Maji et al. [21] gave an application of soft sets in the decision-making problems.

In the literature, many types of operators of soft sets were defined to exploit in the theoretical and applied studies of soft set theory. In 2003, Maji et al. [22] were the first ones who formulated some of these operators such as union and intersection of two soft sets and the complement of a soft set. Also, they defined the null and absolute soft sets as a soft version of the empty and universal crisp sets. Aktas and Cagman [3], in 2007, proved that a soft set may be considered a rough set (fuzzy set). In 2008, Yang [31] showed one of the errors of [22], by a counterexample. Feng et al. [13] addressed a biintersection operator and generalized a soft union operator for arbitrary family of soft sets. To make the soft set theory easier and more convenient for applications, Ali et al. [4] modified a notion of the complement of a soft set and established some soft operators such as restricted union, restricted intersection, extended intersection, and restricted difference of two soft sets.

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They also provided various examples to point out that some results obtained in [22] are erroneous. Feng and Li [14] investigated ontology-based soft sets and soft inclusion relation. Jiang et al. [17] presented the extended soft set theory and studied some soft operators of extended soft sets.

In 2010, Qin and Hong [26] made an interesting theoretical study concerning two new types of soft equality relations. Feng et al. [15] improved the definition of soft subset and presented the concepts of an intersection complete soft set, a full soft set, and a covering soft set. The authors of [16] established a new type of soft sets, namely a bijective soft set and successfully applied it in decision-making problems. Sezgin and Atagün [27], in 2011, defined a concept of restricted symmetric differences of two soft sets and discussed its main features. Jun and Yang [18] initiated the notions of soft generalized soft subset and generalized soft equality relations and they illustrated their relationships with the soft subset and soft equality relations introduced in [4]. Liu et al. [20], in 2012, formulated the definition of a soft L -equality and proved that the associative laws of soft product operations are satisfied in the sense of a soft L -equal relation. The authors of [32] defined the concepts of anti-reflexive kernel, symmetric kernel, reflexive closure, and symmetric closure of a soft set relation. Min [23] introduced the notion of similarity between soft sets and investigated modified operations of soft set theory in terms of ordered parameters. Zhu and Wen [34], in 2013, redefined some notions of the soft set theory such as the codomain of soft set and null soft set.

In order to preserve some classical set-theoretic laws true for soft sets, Abbas et al. [1] gave the notions of g -null soft set, g -absolute soft set, g -soft subset and g -soft equality with illustrative examples. In addition, they compared g -soft equality with lower and upper soft equality relations introduced in [26] and probed algebraic structure with respect to g -soft equality relation. Çağman [7] carried out a comparative study on some soft operators given in [4]. Finally, Abbas et al. [2], in 2017, proposed the concepts of gf -soft subset and gf -soft equality in order to avoid a shortcoming of nonexistence of a g -soft union of two soft sets (see, Example 2.15 of [2]). They defined gf -soft union and gf -soft intersection of two soft sets and discussed main properties. Al-shami et al. [6] defined monotone soft sets and investigated main properties.

In the literature, many sorts of soft operators between soft sets were initiated and studied. The diversity of these operators is attributed to the nature of soft sets which are completely different from crisp sets. This work aims to establish some concepts such as T -soft union and T -soft intersection for arbitrary family of soft sets. These two concepts are inspired by the relations of T -soft subset and T -soft equality which are defined in this work. We also remove the shortcoming related to gf -soft union for arbitrary family of soft sets by using the concept of T -soft union. Ultimately, we construct new types of soft linear equations with respect to some soft equality relations and present some examples and counterexamples to show our main results.

2. Preliminaries

We start with some definitions related to soft set theory which we need in the sequel. In this study, the notions X and K denote a universe set and a set of parameters, respectively. Also, we consider that 2^X is the power set of X and $A, B, C, D \subseteq K$.

Definition 2.1 [24] *A pair (G, K) is said to be a soft set over X provided that G is a map of K into 2^X .*

A soft set is identified as a set of ordered pairs: $(G, K) = \{(k, G(k)) : k \in K \text{ and } G(k) \in 2^X\}$. The set of all soft sets, over X under a set of parameters K , is denoted by $S(X_K)$.

Shabir and Naz [30] defined belong and nonbelong relations which are denoted by \in and \notin , respectively.

El-Shafei et al. [11] introduced partial belong and total nonbelong relations which are denoted by \Subset and $\not\Subset$, respectively. They showed the significant role of them in studying soft separation axioms, see, for example [5, 12].

Definition 2.2 [11, 30] For a soft set (G, K) over X and $x \in X$, we say that:

- (i) $x \Subset (G, K)$ if $x \in G(k)$ for some $k \in K$; and $x \not\Subset (G, K)$ if $x \notin G(k)$ for each $k \in K$.
- (ii) $x \in (G, K)$ if $x \in G(k)$ for each $k \in K$; and $x \notin (G, K)$ if $x \notin G(k)$ for some $k \in K$.

Definition 2.3 [22] A soft set (G, K) over X is said to be:

- (i) A null soft set, denoted by $\tilde{\emptyset}$, if $G(k) = \emptyset$ for each $k \in K$.
- (ii) An absolute soft set, denoted by \tilde{X} , if $G(k) = X$ for each $k \in K$.

Zhu and Wen [34] made a slight difference on the original definition of soft set in [24] by restricting the codomain of soft set on all nonempty subsets of X . Then they constructed a new definition of null soft set to keep the property which reads as follows: a null soft set is a subset of any soft set. However, according to their definition, many set-theoretic properties of null soft set is invalid via soft set theory. Abbas et al. [1] pointed out the shortcoming that arise from the two definitions of null soft set introduced in [22] and [34], by a counterexample.

Definition 2.4 [22] A soft set (G, A) is called an M -soft subset of a soft set (F, B) , denoted by $(G, A) \subseteq_M (F, B)$ if $A \subseteq B$ and the two approximations $G(a)$, $F(a)$ are identical for all $a \in A$.

The soft sets (G, A) and (F, B) are called M -soft equal, denoted by $(G, A) =_M (F, B)$, if each one of them is an M -soft subset of the other.

The definition above was improved in [15] to be as follows.

Definition 2.5 [15] A soft set (G, A) is called an F -soft subset of a soft set (F, B) , denoted by $(G, A) \subseteq_F (F, B)$, if $A \subseteq B$ and $G(a) \subseteq F(a)$ for all $a \in A$.

The soft sets (G, A) and (F, B) are called F -soft equal, denoted by $(G, A) =_F (F, B)$, if each of them is an F -soft subset of the other.

Definition 2.6 [15] A soft set (G, A) over X is said to be:

- (i) A full soft set provided that $\bigcup_{a \in A} G(a) = X$.
- (ii) A partition soft set provided that $\{G(a) : a \in A\}$ constitutes a partition of X .

We observe that Gong et al. [16] introduced and studied a notion of a partition soft set under the name of a bijective soft set. They presented an example of applying bijective soft sets in the decision-making problems.

Definition 2.7 [22] The union of two soft sets (G, A) and (F, B) over X , denoted by $(G, A) \widetilde{\cup} (F, B)$, is the soft set (V, D) , where $D = A \cup B$ and a map $V : D \rightarrow 2^X$ is given as follows:

$$V(d) = \begin{cases} G(d) & : d \in A \setminus B \\ F(d) & : d \in B \setminus A \\ G(d) \cup F(d) & : d \in A \cap B \end{cases}$$

Definition 2.8 [4] The restricted union of two soft sets (G, A) and (F, B) over X such that $A \cap B \neq \emptyset$, is the soft set (V, D) , where $D = A \cap B$, and a map $V : D \rightarrow 2^X$ is given by $V(d) = G(d) \cup F(d)$. It is written as $(G, A) \cup_{\mathfrak{R}} (F, B)$.

Definition 2.9 [22] The intersection of two soft sets (G, A) and (F, B) over X , denoted by $(G, A) \widetilde{\cap} (F, B)$, is the soft set (V, D) , where $D = A \cap B$, and a map $V : D \rightarrow 2^X$ is given by $V(d) = G(d)$ or $F(d)$.

With the help of some illustrative examples, Ali et al. [4] showed that Definition (2.9) suffers from many problems related to its existence. To overcome these problems, they adopted two new definitions of soft intersection given as follows.

Definition 2.10 [4] The extended intersection of two soft sets (G, A) and (F, B) over X , denoted by $(G, A) \square_{\varepsilon} (F, B)$, is the soft set (V, D) , where $D = A \cup B$ and a map $V : D \rightarrow 2^X$ is given as follows:

$$V(d) = \begin{cases} G(d) & : d \in A \setminus B \\ F(d) & : d \in B \setminus A \\ G(d) \cap F(d) & : d \in A \cap B \end{cases}$$

Definition 2.11 [4] The restricted intersection of two soft sets (G, A) and (F, B) over X such that $A \cap B \neq \emptyset$, is the soft set (V, D) , where $D = A \cap B$, and a map $V : D \rightarrow 2^X$ is given by $V(d) = G(d) \cap F(d)$. It is written as $(G, A) \cap (F, B)$.

Definition 2.12 [13] The biintersection of two soft sets (G, A) and (F, B) over X , denoted by $(G, A) \widetilde{\cap} (F, B)$, is the soft set (V, D) , where $D = A \cap B$, and a map $V : D \rightarrow 2^X$ is given by $V(d) = G(d) \cap F(d)$.

It must be imposed that $A \cap B \neq \emptyset$ in the definition of any operator whose domain is $A \cap B$. Otherwise, we obtain a contradiction with the original definition of the soft set given in [24] if $A \cap B = \emptyset$. Hence, Definition (2.11) is more accurate than Definition (2.12).

Remark 2.13 It is worth to note that:

- (i) The union of an arbitrary family of soft sets was given in [13].
- (ii) The restricted union (extended intersection, restricted intersection) of an arbitrary family of soft sets was given in [28].
- (iii) In [4] and [28], the authors pointed out that Definition (2.9) is an ill-defined definition and leads to some incorrect claims. It turns out that the errors existing in [3, 22] are attributed to the utilization of Definition (2.9).

Definition 2.14 [22] *The complement of a soft set (G, K) is a soft set (G^c, K) , where $G^c : K \rightarrow 2^X$ is the mapping defined by $G^c(k) = X \setminus G(k)$ for each $k \in K$.*

Ali et al. [4] redefined the complement of a soft set to satisfy some classical set-theoretic laws such as the union (restricted union) of soft set and its complement is an absolute soft set and the extended intersection (restricted intersection) of soft set and its complement is a null soft set.

Definition 2.15 [4] *The relative complement of a soft set (G, K) is a soft set (G^c, K) , where $G^c : K \rightarrow 2^X$ is the mapping defined by $G^c(k) = X \setminus G(k)$ for each $k \in K$.*

Qin and Hong [26] introduced two soft equalities \approx_s, \approx^s . Herein, we keep their names as proposed by [1], lower soft equality \approx_s and upper soft equality \approx^s . They concluded many results related to the lower and upper soft equality relations and investigated soft lattice structures in terms of lower and upper soft equality relations.

Definition 2.16 [26] *Let (G, A) and (F, B) be two soft sets over X . We say that:*

- (i) *(G, A) is called lower soft equal to (F, B) , denoted by $(G, A) \approx_s (F, B)$, provided that $G(k) = \emptyset$ for each $k \in A \setminus B$, $F(k) = \emptyset$ for each $k \in B \setminus A$ and $G(k) = F(k)$ for each $k \in A \cap B$.*
- (ii) *(G, A) is called upper soft equal to (F, B) , denoted by $(G, A) \approx^s (F, B)$, provided that $G(k) = X$ for each $k \in A \setminus B$, $F(k) = X$ for each $k \in B \setminus A$ and $G(k) = F(k)$ for each $k \in A \cap B$.*

In the following, we recall some kinds of soft subset and soft equality relations.

Definition 2.17 [18] *A soft set (G, A) is called a generalized soft subset of a soft set (F, B) , denoted by $(G, A) \prec (F, B)$, if for each $a \in A$, there exists $b \in B$ such that $G(a) \subseteq F(b)$.*

The soft sets (G, A) and (F, B) are called generalized soft equal, denoted by $(G, A) \doteq (F, B)$, if each one of them is a generalized soft subset of the other.

Some authors (see, for example, [14, 20]) prefer to utilize the notations \subseteq_j and $=_j$ in the places of \prec and \doteq , respectively.

Definition 2.18 [20] *A soft set (G, A) is called an L-soft subset of a soft set (F, B) , denoted by $(G, A) \subseteq_L (F, B)$, if for each $a \in A$, there exists $b \in B$ such that $G(a) = F(b)$.*

The soft sets (G, A) and (F, B) are called L-soft equal, denoted by $(G, A) =_L (F, B)$, if each one of them is an L-soft subset of the other.

Definition 2.19 [1] *A soft set (G, A) is called a g-soft subset of (F, B) , denoted by $(G, A) \subseteq_g (F, B)$, if $A = \emptyset$ or for each $a \in A$, there is $b \in B$ such that $G(a) \subseteq F(b)$.*

The soft sets (G, A) and (F, B) are called g-soft equal, denoted by $(G, A) \cong_g (F, B)$, if each one of them is a g-soft subset of the other.

The above definition contradicts the original definition of soft set if a set of parameters A is empty. Therefore, the authors of [2] utilized generalized soft subset under the name of g-soft subset. They formulated

new soft operators of two soft sets, called g -soft union and g -soft intersection by using g -soft subset and g -soft equality relations.

Definition 2.20 [2] A g -soft union of two soft sets $(F, A), (G, B) \in S(X_K)$, denoted by $(F, A) \sqcup_g (G, B)$, as the set consisting of all soft sets (H, C) , where $C \subseteq K$, satisfying the following two conditions:

(i) $(F, A) \sqsubseteq_g (H, C)$ and $(G, B) \sqsubseteq_g (H, C)$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(F, A) \sqsubseteq_g (J, D)$ and $(G, B) \sqsubseteq_g (J, D)$, then $(H, C) \sqsubseteq_g (J, D)$.

That is, (H, C) is a minimal g -soft superset of $(F, A), (G, B)$ in the sense that if there exists another soft set (J, D) satisfying (i), then (H, C) is g -soft subset of (J, D) .

Definition 2.21 [2] A g -soft intersection of two soft sets $(F, A), (G, B) \in S(X_K)$, denoted by $(F, A) \sqcap_g (G, B)$, as the set consisting of all soft sets (H, C) , where $C \subseteq K$, satisfying the following two conditions:

(i) $(H, C) \sqsubseteq_g (F, A)$ and $(H, C) \sqsubseteq_g (G, B)$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(J, D) \sqsubseteq_g (F, A)$ and $(J, D) \sqsubseteq_g (G, B)$, then $(J, D) \sqsubseteq_g (H, C)$.

That is, (H, C) is a maximal g -soft subset of $(F, A), (G, B)$ in the sense that if there exists another soft set (J, D) satisfying (i), then (J, D) is g -soft subset of (H, C) .

To handle the problem which arises from the nonexistence of g -soft union of two soft sets (see, Example 2.15 of [2]), Abbas et al. [2] formulated the following definition.

Definition 2.22 [2] A soft set (G, A) is called a gf -soft subset of (F, B) , denoted by $(G, A) \sqsubseteq_{gf} (F, B)$, if for each $a \in A$, there is a finite set $E \subseteq B$ such that $G(a) \subseteq \bigcup_{e \in E} F(e)$.

The soft sets (G, A) and (F, B) are called gf -soft equal, denoted by $(G, A) \cong_{gf} (F, B)$, if each one of them is a gf -soft subset of the other.

By replacing g -soft subset by gf -soft subset in Definitions (2.20) and (2.21), the authors of [2] formulated the concepts of gf -soft union and gf -soft intersection of two soft sets.

Das and Samanta [9, 10] introduced and investigated soft real numbers and soft complex numbers in 2012 and 2013, respectively.

Definition 2.23 A soft set (F, K) over X is said to be a singleton soft set if for each $k \in K$, there is $x \in X$ such that $F(k) = \{x\}$.

Definition 2.24 [9, 10] Let \mathcal{R} and \mathcal{C} be the set of real numbers and the set of complex numbers, respectively. Then a singleton soft set (F, K) is said to be a soft real number if $F(k) \in \mathcal{R}$, and it is said to be a soft complex number if $F(k) \in \mathcal{C}$.

3. T-soft equality relation

It is known in the set theory that if $Y_i \subseteq X$ for each $i \in I$, then $\bigcup_{i \in I} Y_i \subseteq X$. However, the following example shows that this property does not hold in general with respect to gf -soft subset. In other words, if $(F_i, A_i) \sqsubseteq_{gf} (H, C)$ for each $i \in I$, unfortunately, $\bigcup_{i \in I} (F_i, A_i) \sqsubseteq_{gf} (H, C)$ need not be true.

Example 3.1 Let X be the set of natural numbers \mathcal{N} and $A = \{k_1, k_2\}$, $B = \mathcal{N}$ be two sets of parameters. Consider the soft sets (G, B) and (F_n, A) for each $n \in \mathcal{N}$ defined as follows:

$$(G, B) = \{(n, \{n\}) : n \in \mathcal{N}\}$$

$$(F_n, A) = \{(k_1, \{4n - 3, 4n - 1\}), (k_2, \{4n - 2, 4n\})\}.$$

It can be noted that $(F_n, A) \sqsubseteq_{gf} (G, B)$ for each $n \in \mathcal{N}$.

However, $\sqcup_{gf} \{(F_n, A) : n \in \mathcal{N}\} = \{(k_1, \{1, 3, 5, \dots\}), (k_2, \{2, 4, 6, \dots\})\} \not\sqsubseteq_{gf} (G, B)$.

Thus, the above example demonstrates that the property which reads as follows: "if the soft sets (F_i, A) is gf -soft subsets of a soft set (G, K) for each $i \in I$, then $\sqcup_{gf} \{(F_i, A) : i \in I\}$ is a gf -soft subset of a soft set (G, K) " does not inherit in soft set theory. In order to remove this shortcoming, we shall modify in this section the notions of gf -soft subset and gf -soft union by introducing the concepts of T -soft subset and T -soft union.

Definition 3.2 Let (F, A) and (G, B) be two soft sets over X . We say that:

- (i) (F, A) is called a T -soft subset of (G, B) if for each $k \in A$, there exists a subset B' (finite or infinite) of B such that $F(k) \subseteq \bigcup_{b \in B'} G(b)$. We denote it as $(F, A) \sqsubseteq_T (G, B)$.
- (ii) (F, A) and (G, B) are called T -soft equal if $(F, A) \sqsubseteq_T (G, B)$ and $(G, B) \sqsubseteq_T (F, A)$. We denote it as $(F, A) \cong_T (G, B)$.

The proofs of the following two results are easy; thus, they are omitted.

Proposition 3.3 Let (F, A) and (G, B) be two soft sets over X . Then:

- (i) If $(F, A) \sqsubseteq_{gf} (G, B)$, then $(F, A) \sqsubseteq_T (G, B)$.
- (ii) If $(F, A) \cong_{gf} (G, B)$, then $(F, A) \cong_T (G, B)$.

Proposition 3.4 Let (F, A) and (G, B) be two soft sets over X . Then:

- (i) If there exists a subset B' of B such that $\bigcup_{b \in B'} F(b) = X$, then $(F, A) \sqsubseteq_T (G, B)$.
- (ii) If there exist subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigcup_{a \in A'} F(a) = \bigcup_{b \in B'} G(b) = X$, then $(F, A) \cong_T (G, B)$.

We construct the following example to point out that Definition (3.2) is real generalization of Definition (2.22).

Example 3.5 Let X , A , and B be the same as in Example (3.1). Consider the soft sets (F, A) and (G, B) given as follows:

$$(F, A) = \{(k_1, \text{thesetofevennumbers}), (k_2, \text{thesetofodddnumbers})\} \text{ and}$$

$(G, B) = \{(n, \{n\}) : n \in \mathcal{N}\}$. Then for $k_2 \in A$, we cannot find a finite subset B' of B such that $F(k_2) \subseteq \bigcup_{n \in B'} G(n)$. Thus, $(F, A) \not\sqsubseteq_{gf} (G, B)$. Hence, $(F, A) \not\cong_{gf} (G, B)$. On the other hand, it can be seen that $(F, A) \sqsubseteq_T (G, B)$ and $(G, B) \sqsubseteq_T (F, A)$. Hence, $(F, A) \cong_T (G, B)$.

Figure 1 illustrates the relationships among the various types of soft equality relations introduced in [1, 2, 15, 18, 20, 26] and this work.

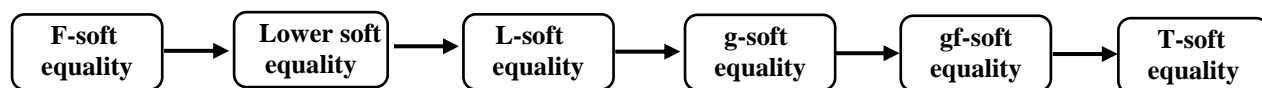


Figure 1. The relationships among some soft equality relations.

Hereinafter, we point out under what conditions a T -soft equality relation implies g -soft equality and gf -soft equality relations.

Proposition 3.6 Let (F, A) and (G, B) be two soft sets over X . Then $(F, A) \cong_T (G, B)$ implies $(F, A) \cong_{gf} (G, B)$ provided that one of the following conditions holds:

- (i) A and B are finite.
- (ii) A is finite and there exists a finite subset $B' \subseteq B$ such that $\bigcup_{b \in B'} G(b) = X$.
- (iii) B is finite and there exists a finite subset $A' \subseteq A$ such that $\bigcup_{a \in A'} F(a) = X$.
- (iv) There exist finite subsets $A' \subseteq A$ and $B' \subseteq B$ such that $\bigcup_{a \in A'} F(a) = \bigcup_{b \in B'} G(b) = X$.

Proof Assume that $(F, A) \cong_T (G, B)$. Then:

- (i) If A and B are finite, then it follows from Definition (3.2) that all subsets of A and B are finite. Thus, we obtain the desired result.
- (ii) Let A be a finite set. Then, by hypothesis, there exists a finite subset $A' \subseteq A$ such that $G(b) \sqsubseteq_{gf} \bigcup_{a \in A'} F(a)$ for each $b \in B$. Thus, $(F, A) \sqsubseteq_{gf} (G, B)$. On the other hand, by the given condition, a set $\bigcup_{b \in B'} G(b) = X$ contains $F(a)$ for each $a \in A$. So $(G, B) \sqsubseteq_{gf} (F, A)$. Hence, $(F, A) \cong_{gf} (G, B)$.
- (iii) Following arguments similar to those above, the result is satisfied.
- (iv) The proof of this result immediately comes from the fact that a set $\bigcup_{a \in A'} F(a) = X$ contains $G(b)$ for each $b \in B$ and a set $\bigcup_{b \in B'} G(b) = X$ contains $F(a)$ for each $a \in A$, where A' and B' are finite.

□

Corollary 3.7 Let (F, A) and (G, B) be two soft sets over X . Then $(F, A) \cong_T (G, B)$ implies $(F, A) \cong_g (G, B)$ provided that one of the following conditions holds:

- (i) A and B are singletons.
- (ii) A is a singleton and there exists $b \in B$ such that $G(b) = X$.

(iii) B is a singleton and there exists $a \in A$ such that $F(a) = X$.

(iv) There exist $a \in A$ and $b \in B$ such that $F(a) = G(b) = X$.

Proposition 3.8 (i) If (G, B) is a full soft set over X , then $(F, A) \sqsubseteq_T (G, B)$ for every soft set (F, A) over X .

(ii) Let Λ be the collection of full soft sets over X . Then $(F, A) \cong_T (G, B)$ for every $(F, A), (G, B) \in \Lambda$.

Proof

(i) Since (G, B) is a full soft set, then $\bigcup_{b \in B} G(b) = X$. Thus, for any soft set (F, A) over X , we find that

$$B \subseteq B \text{ such that } F(a) \subseteq \bigcup_{b \in B} G(b). \text{ Hence, the result is satisfied.}$$

(ii) Following arguments similar to those above, the result is satisfied.

□

Proposition 3.9 (i) A T -soft subset relation \sqsubseteq_T is a partial order relation with reference to \cong_T .

(ii) A T -soft equality relation \cong_T is an equivalence relation.

Proof

(i) Consider (F, A) , (G, B) and (H, C) are arbitrary elements of $S(X_K)$. We obtain, by Definition (3.2), that $(F, A) \sqsubseteq_T (F, A)$. Thus, \sqsubseteq_T is reflexive. Also, if $(F, A) \sqsubseteq_T (G, B)$ and $(G, B) \sqsubseteq_T (F, A)$, then $(G, B) \cong_T (F, A)$. So \sqsubseteq_T is antisymmetric. Assume that $(F, A) \sqsubseteq_T (G, B)$ and $(G, B) \sqsubseteq_T (H, C)$. Then for each $a \in A$, there is a $B' \subseteq B$ such that $F(a) \subseteq \bigcup_{b' \in B'} G(b')$. Now, for each $b' \in B'$, there is a $C_i \subseteq C$ such that $G(b') \subseteq \bigcup_{c' \in C_i} H(c')$. This implies that there exists a subset $L = \bigcup_i C_i$ of C such that $\bigcup_{b' \in B'} G(b') \subseteq \bigcup_{l \in L} H(l)$. So $F(a) \subseteq \bigcup_{l \in L} H(l)$. Thus, $(F, A) \sqsubseteq_T (H, C)$. Hence, \sqsubseteq_T is transitive, as required.

(ii) It can be proved by following arguments similar to those above.

□

Remark 3.10 A T -soft subset relation \sqsubseteq_T is dominance with reference to $=_F, =_g$ and $=_{gf}$. It can be easily seen from Example (3.5) that \sqsubseteq_T with reference to $=_F, =_g$ and $=_{gf}$ is not necessarily antisymmetric.

In what follows, we present a concept of a T -soft union of arbitrary family of soft sets.

Definition 3.11 For $(F_i, A_i) \in S(X_K)$, we define a T -soft union of (F_i, A_i) , denoted by $\sqcup_T(F_i, A_i)$, as the set consisting of all soft sets (H, C) , where $C \subseteq K$, satisfying the following two conditions:

(i) $(F_i, A_i) \sqsubseteq_T (H, C)$ for each $i \in I$.

(ii) If there exists $(J, D) \in S(X_K)$ such that $(F_i, A_i) \sqsubseteq_T (J, D)$ for each $i \in I$, then $(H, C) \sqsubseteq_T (J, D)$.

That is, (H, C) is a minimal T -soft superset of (F_i, A_i) in the sense that if there exists another soft set (J, D) satisfying (i), then (H, C) is a T -soft subset of (J, D) .

Proposition 3.12 For any soft sets (F_i, A_i) over X , we have $\tilde{\cup}(F_i, A_i) \in \sqcup_T(F_i, A_i)$.

Proof Obviously, $(F_i, A_i) \sqsubseteq_T \tilde{\cup}(F_i, A_i)$ for each $i \in I$. Then $\tilde{\cup}(F_i, A_i)$ satisfies the first condition of the above definition. Let $(L, D) \in S(X_K)$ such that $(F_i, A_i) \sqsubseteq_T (L, D)$ for each $i \in I$. Taking $\tilde{\cup}_{i \in C} (F_i, A_i) = (H, C)$, where $C = \bigcup_{i \in I} A_i$. Thus, for each $k \in C$, the next two cases arise:

1. If $k \in \bigcup_{s \in S} A_s \setminus \bigcup_{j \in J} A_j$ for some subsets $S \subseteq I$ and $J \subseteq I$, then $H(k) = \bigcup_{s \in S} F_s(k)$. It follows from the fact $(F_i, A_i) \sqsubseteq_T (L, D)$ that $F_i(k) \subseteq \bigcup_{m \in M_i} L(m)$ for some subsets $M_i \subseteq D$. So $H(k) \subseteq \bigcup_{m \in \bigcup_{i \in S} M_i} L(m)$.
2. If $k \in \bigcap_{i \in I} A_i$, then there exist some subsets $S_i \subseteq D$ satisfying that $F_i(k) \subseteq \bigcup_{k \in S_i} L(k)$. Thus, $H(k) = \bigcup_{i \in C} F_i(k) \subseteq \bigcup_{k \in \bigcup_{i \in C} S_i} L(k)$.

From 1 and 2 above, we conclude that $\tilde{\cup}(F_i, A_i) \sqsubseteq_T L_D$, which implies that the second condition of the above definition holds. Hence, $\tilde{\cup}(F_i, A_i) \in \sqcup_T(F_i, A_i)$. □

Proposition 3.13 For any soft sets (F_i, A_i) over X , we have $\sqcup_T(F_i, A_i) = \{(H, C) \in S(X_K) : (H, C) \cong_T \tilde{\cup}(F_i, A_i)\}$.

Proof Consider that $(H, C) \in \sqcup_T(F_i, A_i)$, then (H, C) satisfies the second condition of Definition (3.11). The above proposition gives that $(F_i, A_i) \sqsubseteq_T \tilde{\cup}(F_i, A_i)$ for each $i \in I$. Thus, we derive that $(H, C) \sqsubseteq_T \tilde{\cup}(F_i, A_i)$. Also, the above proposition gives that $\tilde{\cup}(F_i, A_i)$ satisfies the second condition of Definition (3.11). Therefore, we derive that $\tilde{\cup}(F_i, A_i) \sqsubseteq_T (H, C)$. Hence, we deduce that $(H, C) \cong_T \tilde{\cup}(F_i, A_i)$.

On the other hand, let $(H, C) \cong_T \tilde{\cup}(F_i, A_i)$. Then $(F_i, A_i) \sqsubseteq_T (H, C)$ for each $i \in I$. This implies that (H, C) satisfies the first condition of Definition (3.11). Suppose that $(J, D) \in S(X_K)$ such that $(F_i, A_i) \sqsubseteq_T (J, D)$ for each $i \in I$. Therefore, $\tilde{\cup}(F_i, A_i) \sqsubseteq_T (J, D)$. Thus, $(H, C) \sqsubseteq_T (J, D)$. Hence, (H, C) satisfies the second condition of Definition (3.11) which ultimately implies that $(H, C) \in \sqcup_T(F_i, A_i)$. □

In what follows, we present a concept of a T -soft intersection of arbitrary family of soft sets.

Definition 3.14 For $(F_i, A_i) \in S(X_K)$, we define a T -soft intersection of (F_i, A_i) , denoted by $\sqcap_T(F_i, A_i)$, as the set consisting of all soft sets (H, C) , where $C \subseteq K$, satisfying the following two conditions:

- (i) $(H, C) \sqsubseteq_T (F_i, A_i)$ for each $i \in I$.
- (ii) If there exists $(J, D) \in S(X_K)$ such that $(J, D) \sqsubseteq_T (F_i, A_i)$ for each $i \in I$, then $(J, D) \sqsubseteq_T (H, C)$.

That is, (H, C) is a maximal T -soft subset of (F_i, A_i) in the sense that if there exists another soft set (J, D) satisfying (i), then (J, D) is a T -soft subset of (H, C) .

Before proceeding forward, consider the following:

Example 3.15 Let $A = \{k_1, k_2, k_3\}$ be a set of parameters. For any even number n and odd number m , consider the soft sets (F_n, A) and (F_m, A) over the set of natural numbers \mathcal{N} defined as follows:

$$(F_n, A) = \{(k_1, \{1, 3, n\}), (k_2, \{2\}), (k_3, \text{the set of even numbers})\}.$$

$$(F_m, A) = \{(k_1, \{1, m\}), (k_2, \{2, 3\}), (k_3, \text{the set of even numbers})\}.$$

Then for each $i \in \mathcal{N}$, $\sqcap_{\varepsilon_{i \in I}}(F_i, A) = \{(k_1, \{1\}), (k_2, \{2\}), (k_3, \text{the set of even numbers})\}$. Let a soft set (J, A) over \mathcal{N} be defined as follows: $(J, A) = \{(k_1, \{1\}), (k_2, \{2\}), (k_3, \{3\})\}$. Obviously, $(J, A) \sqsubseteq_T (F_i, A)$ for each $i \in \mathcal{N}$, but $(J, A) \not\sqsubseteq_T \sqcap_{\varepsilon_{i \in I}}(F_i, A)$.

Proposition 3.16 Let (F_i, A_i) be soft sets over X such that $\sqcap_{\varepsilon}(F_i, A_i)$ is a full soft set. Then if $(J, D) \sqsubseteq_T (F_i, A_i)$ for each $i \in I$, we have $(J, D) \sqsubseteq_T \sqcap_{\varepsilon}(F_i, A_i)$.

Proof Consider $(H, C) = \sqcap_{\varepsilon}(F_i, A_i)$, where $C = \bigcup_{i \in I} A_i$. Since $\sqcap_{\varepsilon}(F_i, A_i)$ is a full soft set, then $\bigcup_{c \in C} G(c) = X$.

So $J(d) \subseteq \bigcup_{c \in C} G(c)$ for each $d \in D$. Hence, the desired result is proved. \square

Proposition 3.17 Let (F, A) and (G, B) be two soft sets over X . Then:

(i) $[(F, A) \bigcup_{\mathfrak{R}} (G, B)]^c \cong_T (F^c, A) \sqcap (G^c, B)$.

(ii) $[(F, A) \sqcap (G, B)]^c \cong_T (F^c, A) \bigcup_{\mathfrak{R}} (G^c, B)$.

(iii) $[(F, A) \tilde{\bigcup} (G, B)]^c \cong_T (F^c, A) \sqcap_{\varepsilon} (G^c, B)$.

(iv) $[(F, A) \sqcap_{\varepsilon} (G, B)]^c \cong_T (F^c, A) \tilde{\bigcup} (G^c, B)$.

Proof

(i) Let $(F, A) \bigcup_{\mathfrak{R}} (G, B) = (H, A \cap B)$ and $(F^c, A) \sqcap (G^c, B) = (E, A \cap B)$. Then $[(F, A) \bigcup_{\mathfrak{R}} (G, B)]^c = (H^c, A \cap B)$. For each $k \in A \cap B$, it follows that $H^c(k) = X \setminus H(k) = X \setminus (F(k) \bigcup G(k)) = (X \setminus F(k)) \bigcup (X \setminus G(k)) = E(k)$. This means that E and H^c are the same approximations. Hence, $[(F, A) \bigcup_{\mathfrak{R}} (G, B)]^c \cong_T (F^c, A) \sqcap (G^c, B)$, as required.

(ii) It follows from (i) that $[(F^c, A) \bigcup_{\mathfrak{R}} (G^c, B)]^c \cong_T (F^c, A)^c \sqcap (G^c, B)^c \cong_T (F, A) \sqcap (G, B)$.

Hence, $(F^c, A) \bigcup_{\mathfrak{R}} (G^c, B) \cong_T [(F, A) \sqcap (G, B)]^c$, as required.

(iii) Let $(F, A) \tilde{\bigcup} (G, B) = (H, A \cup B)$ and $(F^c, A) \sqcap_{\varepsilon} (G^c, B) = (E, A \cup B)$. Then $[(F, A) \tilde{\bigcup} (G, B)]^c = (H^c, A \cup B)$. Now, we have the following three cases:

1. If $A \cap B = \emptyset$, then the property holds trivially. Suppose $A \cap B \neq \emptyset$ and let $k \in A \cap B$. Then $H^c(k) = X \setminus H(k) = X \setminus (F(k) \bigcup G(k)) = (X \setminus F(k)) \bigcup (X \setminus G(k)) = E(k)$.
2. If $k \in A \setminus B$, then $H^c(k) = X \setminus H(k) = X \setminus F(k) = E(k)$.
3. If $k \in B \setminus A$, then $H^c(k) = X \setminus H(k) = X \setminus G(k) = E(k)$.

From 1, 2, and 3, we derive that E and H^c are the same approximations. Hence, $[(F, A) \widetilde{\cup} (G, B)]^c \cong_T (F^c, A) \sqcap_\varepsilon (G^c, B)$, as required.

(iv) It follows from (iii) that $[(F^c, A) \widetilde{\cup} (G^c, B)]^c \cong_T (F^c, A)^c \sqcap_\varepsilon (G^c, B)^c \cong_T (F, A) \sqcap_\varepsilon (G, B)$. Hence, $(F^c, A) \widetilde{\cup} (G^c, B) \cong_T [(F, A) \sqcap_\varepsilon (G, B)]^c$, as required.

□

4. Applications of some soft equality relations in soft linear equations

Through this section, let $(F, A) = \{(k_1, x + y), (k_2, x - y)\}$, $(G, B) = \{(k_1, \{5\}), (k_2, \{1\}), (k_3, \{7\})\}$, $(H, C) = \{(k_1, \{5\}), (k_2, \{1\})\}$ and $(L, M) = \{(k_1, \{5\}), (k_2, \{1\}), (k_3, \{5\}), (k_4, \{1\})\}$ be soft real numbers.

Definition 4.1 For each $\lambda \in \{F, L, g\}$, we say that $(F, A) =_\lambda (G, B)$ is a soft linear equation if (F, A) and (G, B) are soft real numbers.

Solve the following two soft linear equations:

$$(F, A) =_F (G, B) \tag{4.1}$$

Solution: Since $A \neq B$, then, by definition of $=_F$, this equation is insolvable.

$$(F, A) =_F (H, C) \tag{4.2}$$

Solution:

$$\begin{aligned} & \begin{cases} x + y = 5 \\ x - y = 1 \end{cases} \\ \iff & \begin{cases} x = 3 \\ y = 2 \end{cases} \end{aligned}$$

Hence, the solution set = $\{(3, 2)\}$.

Proposition 4.2 The solution set of the soft linear equation $(F, A) =_F (H, C)$ is a subset of the solution set of the soft linear equation $(F, A) \cong_g (H, C)$

Proof The proof follows from the fact that $(F, A) =_F (H, C)$ implies $(F, A) \cong_g (H, C)$. □

In the following, we elucidate that the converse of the above proposition fails.

Solve the following soft linear equation:

$$(F, A) \cong_g (H, C) \tag{4.3}$$

Solution: Either

$$\begin{aligned} & \begin{cases} x + y = 5 \\ x - y = 1 \end{cases} \\ \iff & \begin{cases} x = 3 \\ y = 2 \end{cases} \end{aligned}$$

or

$$\begin{cases} x + y = 1 \\ x - y = 5 \end{cases}$$

$$\iff \begin{cases} x = 3 \\ y = -2 \end{cases}$$

Hence, the solution set = $\{(3, 2), (3, -2)\}$.

Proposition 4.3 Let (F, A) and (G, B) be two soft real numbers. Then the soft linear equation $(F, A) \cong_g (G, B)$ is solvable if and only if for each $a \in A$ and $b \in B$, there exist $b' \in B$ and $a' \in A$ such that $F(a) = G(b')$ and $G(b) = F(a')$.

Proof *Necessity:* Let $(F, A) \cong_g (G, B)$ be solvable. Then $(F, A) \sqsubseteq_g (G, B)$, and $(G, B) \sqsubseteq_g (F, A)$. Therefore, for each $a \in A$ and $b \in B$, there exist $b' \in B$ and $a' \in A$ such that $F(a) \subseteq G(b')$ and $G(b) \subseteq F(a')$. Since $F(a)$, $G(b')$, $G(b)$, and $F(a')$ are real numbers, then $F(a) = G(b')$ and $G(b) = F(a')$.

The *sufficient* part is obvious. \square

Solve the following soft linear equation:

$$(F, A) \cong_g (L, M) \quad (4.4)$$

Solution: The above proposition implies that the solution set of this equation and equation (4.3) are equal.

In the following result, we give the sufficient condition to solve the soft linear equation.

Proposition 4.4 If the soft linear equation $(F, A) \cong_g (G, B)$ is solvable such that the distinct components of (F, A) and (G, B) are n and m , respectively. Then $n = m$.

Proof Suppose, to the contrary, that $n \neq m$. Without loss of generality, there exists $a \in A$ such that there does not exist $b \in B$ satisfies $F(a) = G(b)$. Therefore, by Proposition (4.3), we find that $(F, A) \cong_g (G, B)$ is insolvable. This is a contradiction. Hence, it must be that $n = m$. \square

Solve the following soft linear equation:

$$\{(k_1, x + y), (k_2, \{5\})\} \cong_g \{(k_1, \{x\}), (k_2, \{5\})\}. \quad (4.5)$$

Solution: Either $x = 5$ and $x + y = 5 \Rightarrow y = 0$. Then $(5, 0)$ is one of the solutions. Or $x + y = x \Rightarrow y = 0$. Then $(x, 0)$ is another solution, where $x \in \mathcal{R}$. Hence, the solutions set = $\{(5, 0), (x, 0) : x \in \mathcal{R}\}$.

Solve the following soft linear equation:

$$\{(k_1, x + y), (k_2, \{5\})\} \cong_g \{(k_1, \{x\}), (k_2, \{5\}), (k_3, \{2x - 1\})\}. \quad (4.6)$$

Solution: Consider $x + y = 5$. Then it must be as follows: $x = 5$ and $2x - 1 = 5$. However, this linear system is insolvable. Thus, the soft real numbers of the left side contains two distinct components. It follows from Proposition (4.3) that the soft real numbers of the right side must contain exactly two distinct components. This means that either $x = 5$ or $2x - 1 = 5$ or $2x - 1 = x$. So $x \in \{5, 3, 1\}$. Now, if $x = 5$, then $x + y = 2x - 1 \Rightarrow 5 + y = 9 \Rightarrow y = 4$. If $x = 3$, then $x + y = 3 \Rightarrow 3 + y = 3 \Rightarrow y = 0$. Finally, if $x = 1$, then $x + y = 1 \Rightarrow 1 + y = 1 \Rightarrow y = 0$. Hence, the solution set = $\{(5, 4), (3, 0), (1, 0)\}$.

Conclusion

This paper is devoted to introduce new types of soft subset and soft equality relations, namely T -soft subset and T -soft equality. These two relations lead to keep some set-theoretic properties via soft set theory. The concepts of T -soft union and T -soft intersection for arbitrary family of soft sets are formulated based on these relations. Also, we open a way to study and discuss soft linear equations with respect to some soft equality relations. In an upcoming paper, we plan to extend these soft linear equations and to introduce soft linear inequalities.

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