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On the variational curves due to the ED-frame field in Euclidean 4-space

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Abstract: In this study, we define a variational field for constructing a family of Frenet curves of the length l lying on a connected oriented hypersurface and calculate the length of the variational curves due to the ED-frame field in Euclidean 4-space. And then, we derive the intrinsic equations for the variational curves and also obtain boundary conditions for this type of curves due to the ED-frame field in Euclidean 4-space.

Key words: Euclidean 4-space, ED-frame field, variational curve

1. Introduction

An elastic line of the length l is defined as a curve with associated energy

$$K = \int_0^l k_1^2 ds,$$

where s is the arc-length parameter and k_1 is the first curvature of the curve [8]. The integral K is called the total square curvature. If the curve is an extremal for the variation problem that minimizes the value of K, then this curve is called a relaxed elastic line [8]. Besides, Nickerson and Manning [10] derived the boundary conditions for a relaxed elastic line on an oriented surface in Euclidean 3-space. In addition, similar applications of the elasticity theory in different spaces can be found in [2, 5, 7]. So far, studies have been about elastic energy problems that occur in the absence of any external force on the elastic line mentioned. Manning [8] solved the problem of minimizing the sum of the energies of elasticity and the energy that is created due to the stationary force when the force acts on an elastic line. The physical motivation for the problem of elastic lines on surfaces may be found in the investigation of the nucleosome core particle of a DNA molecule [9, 12].

Frame fields constitute a very useful tool for studying curves and surfaces. It is well known that a regular space curve in R^3 is 3-times continuously differentiable for the construction of its Frenet frame. In general, the Frenet–Serret frame field was well defined by Guggenheimer [6]. Another one of the most important frame fields of the differential geometry is the Darboux frame field which is a natural moving frame constructed on a surface. It is the analogous of the Frenet–Serret frame field as applied to surface geometry [3, 11]. There are many studies about the Frenet–Serret frame field into higher dimensional spaces but the Darboux frame field even into 4-space was not available in the literature. Recently, Düldül et al. studied the generalization of

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Darboux frame field to higher dimensional spaces and they called this new frame field ”the extended Darboux frame field (ED-frame field)” [4].

In this study, we define a variational field for constructing a family of Frenet curves of the length l lying on a connected oriented hypersurface and calculate the length of the variational curves due to the ED-frame field in Euclidean 4-space. And then, we derive the intrinsic equations for the variational curves and also obtain boundary conditions for this type of curves due to the ED-frame field in Euclidean 4-space.

2. Preliminaries

Definition 2.1 *Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis of R^4. The vector product of the vectors $\mathbf{x} = \sum_{i=1}^{4} x_i \mathbf{e}_i$, $\mathbf{y} = \sum_{i=1}^{4} y_i \mathbf{e}_i$, $\mathbf{z} = \sum_{i=1}^{4} z_i \mathbf{e}_i$ is defined by [14]*

$$\mathbf{x} \times \mathbf{y} \times \mathbf{z} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}.$$

Definition 2.2 *A unit-speed curve $\alpha : I \to E^n$ of class C^n is called a Frenet curve if the vectors $\alpha'(s), \alpha''(s), ..., \alpha^{(n-1)}(s)$ are linearly independent at each point along the curve [1].*

Let M be a hypersurface oriented by the unit normal vector field N in E^4 and α be a Frenet curve of class C^n $(n \geq 4)$ with the arc-length parameter s lying on M. We denote the unit tangent vector field of the curve by $\mathbf{T}$ and denote unit normal vector field of the hypersurface restricted to the curve by $\mathbf{N}(s)=N(\alpha(s))$.

Case 1. If the set $\{\mathbf{N}, \mathbf{T}, \alpha''\}$ is linearly independent, then the Gram–Schmidt orthonormalization method gives the orthonormal set $\{\mathbf{N}, \mathbf{T}, \mathbf{E}\}$, where $\mathbf{E} = \dfrac{\alpha'' - \langle \alpha'', \mathbf{N} \rangle \mathbf{N}}{\| \alpha'' - \langle \alpha'', \mathbf{N} \rangle \mathbf{N} \|}$.

Case 2. If the set $\{\mathbf{N}, \mathbf{T}, \alpha''\}$ is linearly dependent, i.e. if α'' is in the direction of the normal vector $\mathbf{N}$, applying the Gram–Schmidt orthonormalization method to $\{\mathbf{N}, \mathbf{T}, \alpha'''\}$ yields the orthonormal set $\{\mathbf{N}, \mathbf{T}, \mathbf{E}\}$, where $\mathbf{E} = \dfrac{\alpha''' - \langle \alpha''', \mathbf{N} \rangle \mathbf{N} - \langle \alpha''', \mathbf{T} \rangle \mathbf{T}}{\| \alpha''' - \langle \alpha''', \mathbf{N} \rangle \mathbf{N} - \langle \alpha''', \mathbf{T} \rangle \mathbf{T} \|}$.

In each case, if we define $\mathbf{D} = \mathbf{N} \times \mathbf{T} \times \mathbf{E}$, the orthonormal frame field $\{\mathbf{T}, \mathbf{E}, \mathbf{D}, \mathbf{N}\}$ is obtained along the curve α instead of its Frenet frame field. It is obvious that $\mathbf{E}(s)$ and $\mathbf{D}(s)$ are also tangent to the hypersurface M for all s. Thus, the set $\{\mathbf{T}(s), \mathbf{E}(s), \mathbf{D}(s)\}$ spans the tangent hyperplane of the hypersurface at the point $\alpha(s)$. The derivative formulas of the ED-frame field (the extended Darboux frame field) have the matrix form for Case 1:

$$\frac{d}{ds} \begin{bmatrix} \mathbf{T} \\ \mathbf{E} \\ \mathbf{D} \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g^1 & 0 & \kappa_n \\ -\kappa_g^1 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & \tau_g^2 \\ -k_n & -\tau_g^1 & -\tau_g^2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{E} \\ \mathbf{D} \\ \mathbf{N} \end{bmatrix}, \tag{2.1}$$

and for Case 2:

$$\frac{d}{ds} \begin{bmatrix} \mathbf{T} \\ \mathbf{E} \\ \mathbf{D} \\ \mathbf{N} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & \kappa_n \\ 0 & 0 & \kappa_g^2 & \tau_g^1 \\ 0 & -\kappa_g^2 & 0 & 0 \\ -\kappa_n & -\tau_g^1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{E} \\ \mathbf{D} \\ \mathbf{N} \end{bmatrix}, \tag{2.2}$$

where κ_n, κ_g^i, and τ_g^i are the normal curvature, geodesic curvature, and geodesic torsion of order i, $(i=1,2)$, respectively, such that $\langle \mathbf{T}', \mathbf{N}\rangle = \kappa_n$, $\langle \mathbf{E}', \mathbf{N}\rangle = \tau_g^1$, $\langle \mathbf{D}', \mathbf{N}\rangle = \tau_g^2$, $\langle \mathbf{T}', \mathbf{E}\rangle = \kappa_g^1$ and $\langle \mathbf{E}', \mathbf{D}\rangle = \kappa_g^2$ [4].

Theorem 2.3 *Let α be a unit-speed curve parameterized by the arc-length s on an oriented hypersurface M in Euclidean 4-space. If α is a line of curvature on M if and only if*

$$\tau_g^1(s) = \tau_g^2(s) = 0 \tag{2.3}$$

for Case 1 [4].

Theorem 2.4 *Let α be a unit-speed curve parameterized by the arc-length s on an oriented hypersurface M in Euclidean 4-space. If α is a geodesic curve on M, then*

$$\kappa_g^2(s) = k_3(s),\ \tau_g^1(s) = -k_2(s),\ \kappa_n(s) = k_1(s), \tag{2.4}$$

where k_i $(i=1,2,3)$ denotes the $i-th$ curvature functions of α [4].

Theorem 2.5 *Let α be a unit-speed curve parameterized by the arc-length s on an oriented hypersurface M in Euclidean 4-space. If α is an asymptotic curve on M, then*

$$\kappa_g^1(s) = k_1(s),\ \kappa_g^2(s) = k_2(s)\cos\varphi,\ \tau_g^1(s) = -k_2(s)\sin\varphi,\ \tau_g^2(s) = k_3(s) + \frac{d\varphi}{ds},\ \kappa_n(s) = 0, \tag{2.5}$$

where k_i $(i=1,2,3)$ denotes the $i-th$ curvature functions of α [4].

Definition 2.6 *A variation of a curve segment $\alpha : [a,b] \to M$ is a two-parameter mapping*

$$\mathbf{x} : [a,b] \times (-\delta, \delta) \to M$$

such that $\alpha(u) = \mathbf{x}(u,0)$ for all $a \leq u \leq b$. [11].

3. On the variational curves due to the ED-frame field in Euclidean 4-space

Let x be a coordinate patch of $M(u_1,u_2,u_3)$ and the partial velocities of x are given by

$$x_{u_1} = \frac{\partial x}{\partial u_1},\ x_{u_2} = \frac{\partial x}{\partial u_2},\ x_{u_3} = \frac{\partial x}{\partial u_3} \tag{3.1}$$

in Euclidean 4-space.

A curve $\alpha \in M$ can be written $\alpha(s) = x(u_1(s), u_2(s), u_3(s))$, $0 \leq s \leq l$. Then, the unit tangent vector of α is expressed with

$$\mathbf{T}(s) = \alpha'(s) = x_{u_1}\frac{du_1}{ds} + x_{u_2}\frac{du_2}{ds} + x_{u_3}\frac{du_3}{ds}, \tag{3.2}$$

and for any suitable scalar functions $p(s)$, $q(s)$, $r(s)$, we can write

$$\mathbf{E}(s) = p(s)x_{u_1} + q(s)x_{u_2} + r(s)x_{u_3}, \tag{3.3}$$

where s is the arc-length parameter.

To obtain variational curves of the length l, it is generally necessary to extend α to an arc $\alpha^*(s)$ defined for $0 \le s \le l^*$ with $l^* > l$, but sufficiently close to l so that α^* lies in the coordinate patch.

Let $\mu(s),\ 0 \le s \le l^*$ be a sufficiently smooth scalar, non-vanishing function. Then it can be defined as

$$\eta(s) = \mu(s)p^*(s), \quad \xi(s) = \mu(s)q^*(s), \quad \sigma(s) = \mu(s)r^*(s), \tag{3.4}$$

where $p^*(s),\ q^*(s),\ r^*(s)$ are the coefficients of $\mathbf{E}(s)$ for a newly obtained curve. In this case, we denote the variational vector field

$$\eta(s)x_{u_1} + \xi(s)x_{u_2} + \ \sigma(s)x_{u_3} = \mu(s)\mathbf{E}(s) \tag{3.5}$$

along α.
Suppose that

$$\mu(0) = 0 \text{ and } \mu'(0) = 0. \tag{3.6}$$

For $0 \le \delta \le l^*$, a variational of α is defined by

$$\beta(\delta; t) = x\left(u_1(\delta) + t\eta(\delta), u_2(\delta) + t\xi(\delta), u_3(\delta) + t\ \sigma(\delta)\right). \tag{3.7}$$

Because of (3.6), the variational (3.7) has $\beta(0;t) = \alpha(0)$, $\left.\dfrac{\partial \beta(\delta;t)}{\partial \delta}\right|_{\delta=0} = \left.\dfrac{\partial \alpha(\delta)}{\partial \delta}\right|_{\delta=0} = \alpha'(0)$. It means that the variational curves have the same initial point and initial direction.
Let $L^*(t)$ denote the length of the variational $\beta(\delta;t)$, for fixed t, such that $|t| < \epsilon,\ 0 \le \delta \le l^*$. Then, we have

$$L^*(t) = \int_0^{l^*} \left| \left\langle \frac{\partial \beta}{\partial \delta}(\delta;t), \frac{\partial \beta}{\partial \delta}(\delta;t) \right\rangle \right|^{\frac{1}{2}} d\delta \tag{3.8}$$

with

$$L^*(0) = l^* > l. \tag{3.9}$$

It is clear from the expressions (3.7) and (3.8) that $L^*(t)$ is continuous in t. It follows from the expression (3.9), we have

$$L^*(t) > \frac{l + l^*}{2} > l, \quad |t| < \epsilon^*, \tag{3.10}$$

for a suitable ϵ^* satisfying $0 < \epsilon^* \le \epsilon$.
If the parameter δ is restricted to an interval $0 \le \delta \le \gamma(t) \le l^*$ with the condition

$$\int_0^{\gamma(t)} \left| \left\langle \frac{\partial \beta}{\partial \delta}, \frac{\partial \beta}{\partial \delta} \right\rangle \right|^{\frac{1}{2}} d\delta = l, \tag{3.11}$$

then $\beta(\delta;t)$ is bounded to an arc of length l.
Note that $\gamma(0) = l$, where the function $\gamma(t)$ need not be determined explicitly, but we shall need its derivative (given in Lemma 3.1).

3.1. The variational calculations due to the ED-frame field for Case 1 in Euclidean 4-space

Some partial derivatives of the variational $\beta(\delta;t)$ with respect to δ are calculated as

$$\left.\frac{\partial\beta}{\partial\delta}\right|_{t=0} = \mathbf{T}, \tag{3.12}$$

and

$$\left.\frac{\partial^2\beta}{\partial\delta^2}\right|_{t=0} = \mathbf{T}' = \kappa_g^1\mathbf{E} + \kappa_n\mathbf{N}. \tag{3.13}$$

The first derivative of the variational $\beta(\delta;t)$ with respect to t is obtained as

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0} = \mu\mathbf{E}. \tag{3.14}$$

Using the expressions (3.14) and (2.1), then we have the following mixed derivatives of the variational $\beta(\delta;t)$

$$\left.\frac{\partial^2\beta}{\partial t\partial\delta}\right|_{t=0} = \left.\frac{\partial^2\beta}{\partial\delta\partial t}\right|_{t=0} = -\mu\kappa_g^1\mathbf{T} + \mu'\mathbf{E} + \mu\kappa_g^2\mathbf{D} + \mu\tau_g^1\mathbf{N}, \tag{3.15}$$

and

$$\begin{aligned}\left.\frac{\partial^3\beta}{\partial t\partial\delta^2}\right|_{t=0} = \; & (-2\mu'\kappa_g^1 - \mu(\kappa_g^1)' - \mu\tau_g^1\kappa_n)\mathbf{T} \\ & +(\mu'' - \mu(\kappa_g^1)^2 - \mu(\kappa_g^2)^2 - \mu(\tau_g^1)^2)\mathbf{E} \\ & +(2\mu'\kappa_g^2 + \mu(\kappa_g^2)' - \mu\tau_g^1\tau_g^2)\mathbf{D} \\ & +(2\mu'\tau_g^1\kappa_g^2 + \mu(\tau_g^1)' + \mu\kappa_g^2\tau_g^2 - \mu\kappa_g^1\kappa_n)\mathbf{N}.\end{aligned} \tag{3.16}$$

Lemma 3.1 *Due to the ED-frame field for Case 1, the following relation is obtained:*

$$\left.\frac{d\gamma}{dt}\right|_{t=0} = \int_0^l \mu\kappa_g^1 ds. \tag{3.17}$$

Proof Differentiating the expression (3.11) with respect to t, then we find

$$\left.\frac{d\gamma}{dt}\right|_{t=0} \sqrt{\left|\left\langle\frac{\partial\beta}{\partial\delta},\frac{\partial\beta}{\partial\delta}\right\rangle\right|}\Bigg|_{\delta=\gamma(t)} + \int_0^l \frac{\left\langle\frac{\partial\beta}{\partial\delta},\frac{\partial^2\beta}{\partial\delta\partial t}\right\rangle}{\sqrt{\left|\left\langle\frac{\partial\beta}{\partial\delta},\frac{\partial\beta}{\partial\delta}\right\rangle\right|}} ds = 0. \tag{3.18}$$

Using the expressions (3.12) and (3.15) with $\gamma(0) = l$ at $t = 0$ into the expression (3.18), the proof is completed. □

We can obtain equations which satisfy condition $K'(0)=0$ for arbitrary μ satisfying (3.6). The omitted terms are those with the factor $\left\langle \frac{\partial\beta}{\partial\delta}, \frac{\partial^2\beta}{\partial\delta^2} \right\rangle$ which vanishes at $t=0$. Hence, we have

$$\begin{aligned} K'(t) = \ & \frac{d\gamma}{dt}\left\{ \left\langle \frac{\partial\beta}{\partial\delta}, \frac{\partial\beta}{\partial\delta} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\delta^2}, \frac{\partial^2\beta}{\partial\delta^2} \right\rangle \right\} \\ & -3\int_0^l \left\langle \frac{\partial^2\beta}{\partial\delta\partial t}, \frac{\partial\beta}{\partial\delta} \right\rangle \left\langle \frac{\partial^2\beta}{\partial\delta^2}, \frac{\partial^2\beta}{\partial\delta^2} \right\rangle \left\langle \frac{\partial\beta}{\partial\delta}, \frac{\partial\beta}{\partial\delta} \right\rangle^{-\frac{5}{2}} d\delta \\ & +2\int_0^l \left\langle \frac{\partial^3\beta}{\partial\delta^2\partial t}, \frac{\partial^2\beta}{\partial\delta^2} \right\rangle \left\langle \frac{\partial\beta}{\partial\delta}, \frac{\partial\beta}{\partial\delta} \right\rangle^{-\frac{3}{2}} d\delta. \end{aligned} \tag{3.19}$$

Using the expressions (3.12), (3.13), (3.15), and (3.16) into the expression (3.19), then we find

$$\begin{aligned} K'(t) = \ & \left(\int_0^l \mu\kappa_g^1 ds \right) \left((\kappa_g^1(l))^2 + (\kappa_n(l))^2 \right) \\ & + \int_0^l \mu \left((\kappa_g^1)^3 + \kappa_g^1\kappa_n^2 - 2\kappa_g^1(\kappa_g^2)^2 - 2\kappa_g^1(\tau_g^1)^2 + 2\kappa_g^2\tau_g^2\kappa_n + 2(\tau_g^1)'\kappa_n \right) ds \\ & +4\int_0^l \mu'(\tau_g^1\kappa_n) ds + 2\int_0^l \mu''\kappa_g^1 ds. \end{aligned} \tag{3.20}$$

Integrating by parts together with (3.6), we get

$$\int_0^l \mu''\kappa_g^1 ds. = \kappa_g^1(l)\mu'(l) - (\kappa_g^1(l))'\mu(l) + \int_0^l \mu(\kappa_g^1)'' ds, \tag{3.21}$$

and

$$\int_0^l \mu'(\tau_g^1\kappa_n) ds = \tau_g^1(l)\kappa_n(l)\mu(l) - \int_0^l \mu \left((\tau_g^1)'\kappa_n + \tau_g^1\kappa_n' \right) ds. \tag{3.22}$$

If we use partial integration to remove derivative of μ, we obtain the following equation

$$\begin{aligned} K'(0) \ = \ & \int_0^l \mu[(\kappa_g^1)^3 + \kappa_g^1\kappa_n^2 - 2\kappa_g^1(\kappa_g^2)^2 - 2\kappa_g^1(\tau_g^1)^2 + 2\kappa_g^2\tau_g^2\kappa_n \\ & -2(\tau_g^1)'\kappa_n - 4\tau_g^1\kappa_n' + 2(\kappa_g^1)'' + \kappa_g^1(\kappa_g^1(l))^2 + (\kappa_n(l))^2] ds \\ & +\mu(l)[4\tau_g^1(l)\kappa_n(l) - 2(\kappa_g^1(l))'] + \mu'(l)[2\kappa_g^1(l)]. \end{aligned} \tag{3.23}$$

The given curve must satisfy the following conditions and the differential equation for all of μ, since α is to minimize K:

$$\begin{aligned} &\text{(I)} \quad \kappa_g^1(l) = 0 \\ &\text{(II)} \quad (\kappa_g^1)'(l) = 2\tau_g^1(l)\kappa_n(l), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}(\mathrm{DE}) \qquad & \kappa_g^1[(\kappa_g^1)^2+\kappa_n^2-2(\kappa_g^2)^2-2(\tau_g^1)^2+(\kappa_g^1(l))^2+(\kappa_n(l))^2] \\ & +2\kappa_g^2\tau_g^2\kappa_n-2(\tau_g^1)'\kappa_n-4\tau_g^1\kappa_n'+2(\kappa_g^1)''=0.\end{aligned} \tag{3.25}$$

Consequently, we can say that the given curve is a relaxed elastic line:

Theorem 3.2 *The intrinsic equations for a relaxed elastic line due to the ED-frame field for Case 1 on a connected oriented hypersurface M in Euclidean 4-space are given by the differential equation (3.25) and the boundary conditions (3.24) at the free end, where κ_g^i, τ_g^i, and κ_n are the functions giving the geodesic curvature of order i, geodesic torsion of order i and normal curvature.*

3.2. The variational calculations due to the ED-frame field for Case 2 in Euclidean 4-space

Some partial derivatives of the variational $\beta(\delta;t)$ with respect to δ are calculated as

$$\left.\frac{\partial\beta}{\partial\delta}\right|_{t=0}=\mathbf{T}, \tag{3.26}$$

and

$$\left.\frac{\partial^2\beta}{\partial\delta^2}\right|_{t=0}=\mathbf{T}'=\kappa_n\mathbf{N}. \tag{3.27}$$

The first derivative of the variational $\beta(\delta;t)$ with respect to t is obtained as

$$\left.\frac{\partial\beta}{\partial t}\right|_{t=0}=\mu\mathbf{E}. \tag{3.28}$$

From the expressions (3.28) and (2.2), we have the following derivatives of the variational $\beta(\delta;t)$

$$\left.\frac{\partial^2\beta}{\partial t\partial\delta}\right|_{t=0}=\left.\frac{\partial^2\beta}{\partial\delta\partial t}\right|_{t=0}=\mu'\mathbf{E}+\mu\kappa_g^2\mathbf{D}+\mu\tau_g^1\mathbf{N}, \tag{3.29}$$

and

$$\begin{aligned}\left.\frac{\partial^3\beta}{\partial t\partial\delta^2}\right|_{t=0} = & \ (-\mu\tau_g^1\kappa_n)\mathbf{T}+(\mu''-\mu(\kappa_g^2)^2-\mu(\tau_g^1)^2)\mathbf{E} \\ & +(2\mu'\kappa_g^2+\mu(\kappa_g^2)')\mathbf{D}+(2\mu'\tau_g^1+\mu(\tau_g^1)')\mathbf{N}.\end{aligned} \tag{3.30}$$

Lemma 3.3 *Due to the ED-frame field for Case 2, the following relation is obtained:*

$$\left.\frac{d\gamma}{dt}\right|_{t=0}=0. \tag{3.31}$$

Proof Using the expressions (3.26) and (3.29) into the expression (3.18), then the proof is completed. □

When we write obtained equations for $t = 0$ in the expression (3.19), we have the following:

$$K'(t) = \int_0^l (4\mu'\tau_g^1\kappa_n + 2\mu(\tau_g^1)'\kappa_n)ds. \tag{3.32}$$

However, using integration by parts and the relation (3.6), we can write

$$\int_0^l \mu'(\tau_g^1\kappa_n)ds = \tau_g^1(l)\kappa_n(l)\mu(l) - \int_0^l \mu\left((\tau_g^1)'\kappa_n + \tau_g^1\kappa_n'\right)ds. \tag{3.33}$$

Using the partial integration (3.33) to remove derivative of μ, we arrive at the following equation:

$$K'(0) = -\int_0^l \mu(2(\tau_g^1)'\kappa_n + 4\tau_g^1\kappa_n')ds + 4\mu(l)\tau_g^1(l)\kappa_n(l). \tag{3.34}$$

According to the expression (3.34) and the relation (3.6), we obtain

$$\text{(I)} \quad \tau_g^1(l)\kappa_n(l) = 0. \tag{3.35}$$

and

$$\text{(DE)} \quad 2(\tau_g^1)'\kappa_n + 4\tau_g^1\kappa_n' = 0. \tag{3.36}$$

Then, we can give the following theorem:

Theorem 3.4 *Let α be a curve on a connected oriented hypersurface M in Euclidean 4-space. If the curve α lying on M is a line of curvature or an asymptotic curve, then the given curve is a relaxed elastic line due to the ED-frame field for Case 2 in Euclidean 4-space.*

4. Geometrical interpretations

Corollary 4.1 *Let α be a curve on a connected oriented hypersurface M in Euclidean 4-space. If the curve α lying on M is a line of curvature, then we obtain the following intrinsic equations for a relaxed elastic line due to the ED-frame field for Case 1:*

$$\begin{aligned} &(I) \quad \kappa_g^1(l) = 0 \\ &(II) \quad (\kappa_g^1)'(l) = 0, \end{aligned} \tag{4.1}$$

and

$$(DE) \quad \kappa_g^1[(\kappa_g^1)^2 + \kappa_n^2 - 2(\kappa_g^2)^2 + (\kappa_n(l))^2] + 2(\kappa_g^1)'' = 0. \tag{4.2}$$

Proof We know that $\tau_g^1 = \tau_g^2 = 0$. Hence, the intrinsic equations for a relaxed elastic line due to the ED-frame field for Case 1 on a connected oriented hypersurface in Euclidean 4-space provide the conditions (4.1) and (4.2). □

Corollary 4.2 *Let α be a curve on a connected oriented hypersurface M in Euclidean 4-space. If the curve α lying on M is a geodesic curve, then we have the following boundary conditions and differential equation for a relaxed elastic line due to the ED-frame field for Case 1:*

$$\begin{aligned} &(I) \quad \kappa_g^1(l) = 0 \\ &(II) \quad (\kappa_g^1)'(l) = -2k_2(l)k_1(l), \end{aligned} \tag{4.3}$$

and

$$(DE) \quad 2k_3k_1\tau_g^2 + 2k_1k_2' + 4k_2k_1' = 0. \tag{4.4}$$

Proof According to Theorem 2.4, the proof is completed. □

Corollary 4.3 *Let α be a curve on a connected oriented hypersurface M in Euclidean 4-space. If the curve α lying on M is an asymptotic curve, then we have the following boundary conditions and differential equation for a relaxed elastic line due to the ED-frame field for Case 1:*

$$\begin{aligned} &(I) \quad k_1(l) = 0 \\ &(II) \quad k_1'(l) = 0, \end{aligned} \tag{4.5}$$

and

$$(DE) \quad k_1^3 - 2k_1k_2^2 + k_1^2 - 2k_3^2 + 2k_1'' = 0. \tag{4.6}$$

Proof Using the expression (2.5) into the expressions (3.24) and (3.25), we obtain the expressions (4.5) and (4.6). □

5. Applications

In this section, we shall give the following applications for the hypersphere and hypercylinder.

Example 5.1 *Let us consider the unit-speed curve*

$$\gamma(s) = \left(\cos\sqrt{\frac{2}{3}}s, \sin\sqrt{\frac{2}{3}}s, \cos\sqrt{\frac{1}{3}}s, \sin\sqrt{\frac{1}{3}}s\right)$$

lying on the hypersphere M given by its implicit equation $x^2 + y^2 + z^2 + w^2 = 2$.

The unit normal vector of M along γ is $\mathbf{N}(s) = \sqrt{\frac{1}{2}}(\gamma(s))$ *and the unit tangent vector field of the curve γ is as follows:*

$$\mathbf{T}(s) = \left(-\sqrt{\frac{2}{3}}\sin\sqrt{\frac{2}{3}}s, \sqrt{\frac{2}{3}}\cos\sqrt{\frac{2}{3}}s, -\sqrt{\frac{1}{3}}\sin\sqrt{\frac{1}{3}}s, \sqrt{\frac{1}{3}}\cos\sqrt{\frac{1}{3}}s\right).$$

Differentiating the tangent vector field $\mathbf{T}$ with respect to s, then we find

$$\mathbf{T}'(s) = \left(-\frac{2}{3}\cos\sqrt{\frac{2}{3}}s, -\frac{2}{3}\sin\sqrt{\frac{2}{3}}s, -\frac{1}{3}\cos\sqrt{\frac{1}{3}}s, -\frac{1}{3}\sin\sqrt{\frac{1}{3}}s\right).$$

Since $\mathbf{T}'$ *is linear independent with* $\mathbf{N}$, *then Case 1 is valid.*
Thus, we obtain

$$\mathbf{E}(s) = \sqrt{\frac{1}{2}}\left(-\cos\sqrt{\frac{2}{3}}s, -\sin\sqrt{\frac{2}{3}}s, \cos\sqrt{\frac{1}{3}}s, \sin\sqrt{\frac{1}{3}}s\right).$$

From $\mathbf{T}' = \kappa_g^1\mathbf{E}+\kappa_n\mathbf{N}$, *we get* $\kappa_g^1 = \sqrt{\frac{1}{18}}$ *and* $\kappa_n = -\sqrt{\frac{1}{2}}$.

According to Theorems 2.4, 2.5, and 3.2, the given curve is not a geodesic curve, an asymptotic curve, a relaxed elastic line.

Example 5.2 *Assume that the unit-speed curve*

$$\gamma(s) = \left(\sqrt{2}\cos\frac{s}{3}, \frac{2s}{3}, \sqrt{3}\cos\frac{s}{3}, \sqrt{5}\sin\frac{s}{3}\right)$$

lying on the hypercylinder $f(x,y,z,w) = x^2+z^2+w^2 = 5$ *[13].*

The unit normal vector of the hypercylinder is $\mathbf{N}(s) = \frac{1}{\sqrt{5}}(x,0,z,w)$, *namely,*

$$\mathbf{N}(s) = \left(\sqrt{\frac{2}{5}}\cos\frac{s}{3}, 0, \sqrt{\frac{3}{5}}\cos\frac{s}{3}, \sin\frac{s}{3}\right).$$

The unit tangent vector field of γ *is as follows:*

$$\mathbf{T}(s) = \frac{\gamma'(s)}{\|\gamma'(s)\|} = \left(-\frac{\sqrt{2}}{3}\sin\frac{s}{3}, \frac{2}{3}, -\frac{\sqrt{3}}{3}\sin\frac{s}{3}, \frac{\sqrt{5}}{3}\cos\frac{s}{3}\right).$$

Since the curvature vector field $\mathbf{T}'(s) = \gamma''(s) = \left(-\frac{\sqrt{2}}{9}\cos\frac{s}{3}, 0, -\frac{\sqrt{3}}{9}\cos\frac{s}{3}, -\frac{\sqrt{5}}{9}\sin\frac{s}{3}\right)$ *is linear dependent with* $\mathbf{N}(s)$; *Case 2 is valid along the curve* γ.
Thus, we obtain

$$\mathbf{E}(s) = \left(\frac{2\sqrt{2}}{3\sqrt{5}}\sin\frac{s}{3}, \frac{5}{3\sqrt{5}}, \frac{2\sqrt{3}}{3\sqrt{5}}\sin\frac{s}{3}, -\frac{2}{3}\cos\frac{s}{3}\right).$$

Since $\mathbf{T}' = \kappa_n\mathbf{N}$, *then we get* $\kappa_n = -\frac{\sqrt{5}}{9}$.

According to Theorem 2.5, the curve γ *is not an asymptotic curve.*

Now, calculate $\tau_g^1 = \langle\mathbf{E}',\mathbf{N}\rangle$. *Differentiating* $\mathbf{E}$ *and using* $\mathbf{N}$, *then we find* $\tau_g^1 = \frac{2}{9}$.

From Theorem 2.3, the curve γ *is not a line of curvature. According to Theorem 3.4, the curve is not a relaxed elastic line.*

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