Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1453 – 1465
© TÜBİTAK
doi:10.3906/mat-2002-16

An analytical investigatıon on starlikeness and convexity properties for hypergeometric functions

İsmet YILDIZ, Alaattin AKYAR*
Department of Mathematics, Faculty of Arts And Sciences, Düzce University, Düzce, Turkey

Received: 06.02.2020 • **Accepted/Published Online:** 05.06.2020 • **Final Version:** 08.07.2020

Abstract: In this study, we analytically investigate hypergeometric functions having some properties such as convexity and starlike. We fundamentally focus on obtaining desired conditions on the parameters a, b, and c in order that a hypergeometric function to be in various subclasses of starlike and convex functions of order $\alpha = 2^{-r}$ and order $\alpha = 2^{-r}$ type $\beta = 2^{-1}$, with r is a positive integer.

Key words: Analytic function, univalent function, starlike function, convex function, hypergeometric function, gamma function

1. Introduction

First in order to achieve ultimate goal in this study, we consider basic concepts and preliminary results such as certain subclass of univalent analytic functions we need. We know that complex analytic functions have very important features such as having infinitely differentiable and Taylor series representation which, in fact, real analytical functions do not. All of those properties are really amazing and very useful. Perhaps these are the most obvious and remarkable differences between complex analytical functions and real analytical functions. In this regard, the most common an analytic complex function f which used in univalent function theory may be expressed as:

$$w = f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

The functions above are normalized in the sense that $f(0) = f'(0) - 1 = 0$ where origin 0 is the center of unit disk $\mathbb{D} = \{| z |< 1 : z \textbf{ is a complex number}\}$. As usual, the set of all analytic and normalized functions is denoted by $\mathbb{A}$. Intuitively the function f is called as univalent (or one-to-one or schlicht) if it takes the same value once. That is to say, if z_1, z_2 are the two distinct points in the domain (**say** $\mathbb{D}$) of f, then it $f(z_1) \neq f(z_2) \Longleftrightarrow z_1 \neq z_2$. Also, let S denote the class of univalent functions as $f \in \mathbb{A}$. It means that

$$S = \{f \in \mathbb{A} : f \textbf{ is univalent in } \mathbb{D}\}.$$

The studies in univalent function theory are mainly done on S class. Furthermore, a function is known as starlike with regard to the origin, if $tw \in f(\mathbb{D})$ when $w \in f(\mathbb{D})$ and $t \in (0, 1]$. It is clear that f is starlike if

*Correspondence: alaattinakyar@duzce.edu.tr
2010 *AMS Mathematics Subject Classification:* Primary 30C45; Secondary 23584.

and only if it satisfies the inequality of $Re\left(\frac{zf'(z)}{f(z)}\right) > 0$ for all $z \in \mathbb{D}$[2]. The class of such functions will be denoted as S^*. An analytic description of S^* is shown as:

$$S^* = \left\{ f \in \mathbb{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathbb{D} \right\}.$$

Furthermore, a function $f \in \mathbb{A}$ maps $\mathbb{D}$ onto a convex domain $f(\mathbb{D})$ then it is known as a convex function (see [16]). In order that f is a convex function if and only if it should be satisfied with the inequality $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0$ for all $z \in \mathbb{D}$ [9]. The class of such functions will be denoted as C. An analytic description of C is as follows:

$$C = \left\{ f \in \mathbb{A} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{D} \right\}.$$

Thus, it is easy to see that $f \in C$ if and only if $zf' \in S^*$.

The function $f \in \mathbb{A}$ is called as a starlike of order $\alpha \in [0,1)$ if and only if it has $Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha$[2]. The class of such functions will be denoted by $S^*(\alpha)$. Similarly, a function $f \in \mathbb{A}$ is named convex of order $\alpha \in [0,1)$ if and only if $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha$[9]. In this case, the class of such functions will be denoted by $C(\alpha)$. Respectively, two such subclasses are analytically characterized by

$$S^*(\alpha) = \left\{ f \in \mathbb{A} : Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, z \in \mathbb{D}, \alpha \in [0,1) \right\},$$

and

$$C(\alpha) = \left\{ f \in \mathbb{A} : Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha, z \in \mathbb{D}, \alpha \in [0,1) \right\}.$$

Usually, class of starlike and convex functions can also be written as $S^*(0) = S^*$ and $C(0) = C$, respectively. Furthermore, let us show with $S_1^*(\alpha)$ the subclass $S^*(\alpha)$ containing of functions f for which $|(zf'/f) - 1| < 1 - \alpha$ for all $z \in \mathbb{D}$. On the other hand, a function f is said to be in $C_1(\alpha)$ if $zf' \in S_1^*(\alpha)$. One could also find coefficient bounds and different properties of $S_1^*(\alpha)$ and $C_1(\alpha)$ in [3, 6, 11, 14, 17].

Let T be the class consisting of functions of the form as:

$$w = f(z) = z - \sum_{n=2}^{\infty} a_n z^n, (a_n \geq 0). \tag{1.2}$$

The elements of this class are analytic and univalent in the unit disk $\mathbb{D}$. A subclass of T is $S^*(\alpha, \beta)$ that contains the functions providing the following condition:

$$\left| \frac{\frac{zf'(z)}{f(z)} - 1}{\frac{zf'(z)}{f(z)} + 1 - 2\alpha} \right| < \beta, \tag{1.3}$$

where $\alpha \in [0,1)$, $\beta \in (0,1)$ and for all $z \in \mathbb{D}$. In addition, a subclass of T is $C^*(\alpha,\beta)$ that contains the functions providing the following condition:

$$\left| \frac{\frac{zf''(z)}{f'(z)}}{\frac{zf''(z)}{f'(z)} + 2(1-\alpha)} \right| < \beta, \tag{1.4}$$

where $\alpha \in [0,1)$, $\beta \in (0,1]$ and for all $z \in \mathbb{D}$. These classes have already been by Gupta and Jain (see [4]). We note that $S^*(\alpha,1) = S^*(\alpha)$ and $C^*(\alpha,1) = C^*(\alpha)$ (see [13]). Furthermore, from (1.3) and (1.4), we have $f \in C^*(\alpha,\beta) \Leftrightarrow zf' \in S^*(\alpha,\beta)$ (see [8]).

After the analysis of analytic (or regular or holomorphic) functions in $\mathbb{D}$, we discuss the hypergeometric type functions. They were first introduced in 1866 by C.F. Gauss. The ordinary or Gaussian hypergeometric function $z \mapsto {}_2F_1(a,b;c;z)$ with complex parameters a,b,c $(c \neq 0,-1,-2,...)$ is defined for $|z| < 1$ by the power series as:

$$F(z) = {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} z^n, \tag{1.5}$$

where $(a)_n$ is the Pochhammer symbol or shifted factorial defined by

$$(a)_n := \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} a(a+1)(a+2)...(a+n-1), & \text{if } n = 1,2,..., \\ 1, & \text{if } n = 0. \end{cases}$$

One can easily verify that (1.5) satisfies the hypergeometric differential equation

$$z(1-z)\frac{d^2F(z)}{dz^2} + [(a+b+1)z - c]\frac{dF(z)}{dz} + abF(z) = 0.$$

On the other hand, since $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = a(a+1)\cdots(a+n-1)$ (where Γ denotes Gamma function) we can write:

$$\begin{aligned} {}_2F_1(a,b;c;z) &= 1 + \frac{ab}{c1!}z + \frac{a(a+1)b(b+1)}{c(c+1)2!}z^2 + \cdots \\ &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)\Gamma(b+n)}{\Gamma(c+n)(1)_n} z^n. \end{aligned}$$

Most of the mathematical functions, including all the univalent functions, could be defined as hypergeometric functions. Let us take some important examples into consideration as:

1. when $a = 1$ and $b = c$ in (1.5)

$$\begin{aligned} z{}_2F_1(1,b;b;z) &= z\sum_{n=0}^{\infty} \frac{(1)_n (b)_n}{(b)_n (1)_n} z^n \\ &= \sum_{n=0}^{\infty} z^{n+1} = \frac{z}{1-z}. \end{aligned}$$

   Please note that the function $w = f(z) = z(1-z)^{-1}$ is one of the leading examples of the classes S^* and C which maps $\mathbb{D}$ onto the half-plane $Re(w) > -1/2$; thus, it is clear that this function is starlike and convex as well.

2. when $a = 1/2,\ b = 1$ and $c = 3/2$ in (1.5)

$$
\begin{aligned}
z\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};z^2\right) &= z\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n(1)_n}{\left(\frac{3}{2}\right)_n(1)_n}\left(z^2\right)^n = z\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n}{\left(\frac{3}{2}\right)_n}z^{2n}\\
&= z\left[\frac{\left(\frac{1}{2}\right)_0}{\left(\frac{3}{2}\right)_0}+\frac{\left(\frac{1}{2}\right)_1}{\left(\frac{3}{2}\right)_1}z^2+\frac{\left(\frac{1}{2}\right)_2}{\left(\frac{3}{2}\right)_2}z^4+\cdots\right]\\
&= z\left[1+\frac{1}{3}z^2+\frac{1}{5}z^4+\cdots\right]\\
&= z+\frac{1}{3}z^3+\frac{1}{5}z^5+\cdots\\
&= \frac{1}{2}\log\left(\frac{1+z}{1-z}\right).
\end{aligned}
$$

The function above is probably one of the most important members of the S^* class and C class which maps $\mathbb{D}$ onto the horizontal strip $-\frac{\pi}{4} < Im(w) < \frac{\pi}{4}$. This indicates that this function is convex and also starlike with respect to the origin.

3. when $a = c = 1$ and $b = 2$ in (1.5)

$$
\begin{aligned}
z\,{}_2F_1(1,2;1;z) = z\sum_{n=0}^{\infty}\frac{(1)_n(2)_n}{(1)_n(1)_n}z^n &= \sum_{n=0}^{\infty}\frac{(2)_n}{(1)_n}z^{n+1}\\
&= z+2z^2+3z^3+\cdots\\
&= z+\sum_{n=2}^{\infty}nz^n.
\end{aligned}
$$

The function obtained above is well known as the Koebe function. The Koebe function $k(z) = z(1-z)^{-2} = z+\sum_{n=2}^{\infty}nz^n$ is univalent and it is also extremal for many problems in the geometric properties of univalent functions which maps $\mathbb{D}$ onto the entire plane minus the negative real axis from $-\frac{1}{4}$ to $-\infty$. Therefore, it is obvious that this function is starlike with respect to the origin but it is not convex.

To determine and verify our main results, we need to use each of the following results in our investigation.

Remark 1.1 *(see [12]) For real or complex numbers* $a, b, c(c \neq 0, -1, -2, \cdots)$ *and* $Re(c) > Re(b) > 0$ *and* $|\,arg\,(1-z)\,| < \pi$*, we have*

$$
{}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt.
$$

Now, we can list below some elementary properties of ${}_2F_1(a,b;c;z)$ that can be easily verified and can be found in [10]:

$$
{}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z)
$$

$$
\frac{d}{dz}{}_2F_1(a,b;c;z) = \frac{ab}{c}{}_2F_1(a+1,b+1;c+1;z)
$$

$$_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z)$$

$$_2F_1(a,b;b;z) = (1-z)^{-a}.$$

Theorem 1.2 *Let the hypergeometric function $_2F_1(a,b;c;z)$. Then the following hold:*

1. *Converges absolutely if $\mid z \mid < 1$;*
2. *Converges absolutely if $Re(c-a-b) > 0$ if $\mid z \mid = 1$;*
3. *Converges conditionaly if $0 \geq Re(c-a-b) > -1$ if $\mid z \mid = 1$ and $z \neq 1$;*
4. *Diverges if $Re(c-a-b) \leq -1$.*

Theorem 1.3 *(Gauss 1812). Suppose $Re(c-a-b) > 0$. Then*

$$_2F_1(a,b;c;1) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.6}$$

holds.

Many scientists used continuous fractions to determine the sufficient conditions for $z_2F_1(a,b;c;z)$ to be in $S^*(\alpha)$ and $S_1^*(\alpha)$ with $\alpha \in [0,1)$ for various choices of parameters a, b, and c (see [5, 7, 10, 15]). The subject of the present paper is to give some characterizations for a hypergeometric functions to be in various subclasses of starlike and convex functions of order $\alpha = 2^{-r}$ and order $\alpha = 2^{-r}$ type $\beta = 2^{-1}$. To obtain the main results, we will give the following theorems, which may be found in [13].

Theorem 1.4 *A sufficient condition for a function f in form (1.1) to be in $S_1^*(\alpha)(C_1(\alpha))$ is that*

$$\sum_{n=2}^{\infty}(n-\alpha) \mid a_n \mid \leq (1-\alpha)\left(\sum_{n=2}^{\infty} n(n-\alpha) \mid a_n \mid \leq 1-\alpha\right).$$

Theorem 1.5 *Late us take a function f in form (1.2). Then a necessary and sufficient condition for f to be in $S_1^*(\alpha)(C_1(\alpha))$ is that*

$$\sum_{n=2}^{\infty}(n-\alpha)a_n \leq (1-\alpha)\left(\sum_{n=2}^{\infty} n(n-\alpha)a_n \leq 1-\alpha\right).$$

In addition, $f \in S_1^*(\alpha) \Longleftrightarrow f \in S^*(\alpha), f \in C_1(\alpha) \Longleftrightarrow f \in C(\alpha)$, and $f \in S^* \Longleftrightarrow f \in S$.

2. Main results

Theorem 2.1 *If $a, b > 0$ and $c > a+b+1$, then a sufficient condition for $z_2F_1(a,b;c;z)$ to be in $S_1^*(2^{-r})$ with r is a positive integer:*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + \frac{ab}{(1-2^{-r})(c-a-b-1)}\right] \leq 2. \tag{2.1}$$

Also, condition (2.1) is necessary and sufficient for

$$ {}_2F_1\left(a,b;c;z\right) = z\left(2 - {}_2F_1\left(a,b;c;z\right)\right) $$

to be in $S^*(2^{-r})(S_1^*(2^{-r}))$.

Proof Since

$$ z_2F_1(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n, $$

according to Theorem (1.4) we need only to show that

$$ \sum_{n=2}^{\infty} (n - 2^{-r}) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} \leq (1 - 2^{-r}). $$

Now

$$ \sum_{n=2}^{\infty} (n - 2^{-r}) \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} = \sum_{n=1}^{\infty} (n + 1 - 2^{-r}) \frac{(a)_n(b)_n}{(c)_n(1)_n} $$

$$ = \sum_{n=1}^{\infty} (\frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + (1 - 2^{-r}) \sum_{n=1}^{\infty} (\frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{2.2} $$

Noting that $(a)_n = a(a+1)_{n-1}$ and then applying (1.6), we may express (2.2)

$$ = \sum_{n=1}^{\infty} \frac{a(a+1)_{n-1}b(b+1)_{n-1}}{c(c+1)_{n-1}(1)_{n-1}} + (1 - 2^{-r}) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n} $$

$$ = \frac{ab}{c} \sum_{n=1}^{\infty} \frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + (1 - 2^{-r}) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}) $$

$$ = \frac{ab}{c} \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1 - 2^{-r}) \left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1 \right] $$

$$ = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \left[\frac{ab}{c-a-b-1} + (1 - 2^{-r}) \right] - (1 - 2^{-r}). $$

However, this last expression is bounded above by $(1 - 2^{-r})$ if and only if (2.1) holds. Since

$$ {}_2F_1(a,b;c;z) = z - \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n, $$

the necessity of (2.1) for ${}_2F_1\left(a,b;c;z\right)$ to be in $S_1^*(2^{-r})$ and $S^*(2^{-r})$ follows from Theorem (1.5). □

Remark 2.2 *Condition (2.1) as $r \to \infty$ is both necessary and sufficient for ${}_2F_1\left(a,b;c;z\right)$ to be in S.*

In the next theorem, we investigate some constraints on parameters a, b, and c that lead to necessary and sufficient conditions for $z_2F_1(a,b;c;z)$ to be in $S^*(2^{-r})$.

Theorem 2.3 *If* $a, b > -1$, $c > 0$, *and* $ab < 0$, *then a necessary and sufficient condition for* $z_2F_1(a,b;c;z)$ *to be in* $S^*(2^{-r})(S_1^*(2^{-r}))$ *is that* $c \geq a+b+1-ab/(1-2^{-r})$. *The condition* $c \geq a+b+1-ab$ *is necessary and sufficient for* $z_2F_1(a,b;c;z)$ *to be in* S.

Proof Since

$$z_2F_1(a,b;c;z) = z + \sum_{n=2}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} z^n$$

$$= z + \sum_{n=2}^{\infty} \frac{a(a+1)_{n-2}b(b+1)_{n-2}}{c(c+1)_{n-2}(1)_{n-1}} z^n$$

$$= z + \frac{ab}{c} \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n$$

$$= z - \mid \frac{ab}{c} \mid \sum_{n=2}^{\infty} \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n, \tag{2.3}$$

according to Theorem (1.5) we must show that

$$\sum_{n=2}^{\infty} (n-2^{-r}) \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c)_{n-2}(1)_{n-1}} \leq \mid \frac{c}{ab} \mid (1-2^{-r}). \tag{2.4}$$

Note that the left side of (2.4) diverges if $c \leq a+b+1$. Now

$$\sum_{n=2}^{\infty} (n-2^{-r}) \frac{(a+1)_{n-2}(b)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} = \sum_{n=0}^{\infty} (n+2-2^{-r}) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= \sum_{n=0}^{\infty} (n+1) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}} + \sum_{n=0}^{\infty} (1-2^{-r}) \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$= \sum_{n=0}^{\infty} \frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \frac{c}{ab}(1-2^{-r}) \sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + (1-2^{-r})\frac{c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right].$$

Hence, (2.4) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[1 + (1-2^{-r})\frac{c-a-b-1}{ab}\right] \leq$$

$$(1-2^{-r})\left[\frac{c}{\mid ab \mid} + \frac{c}{ab}\right] = 0. \tag{2.5}$$

Thus, (2.5) is valid if and only if $1+(1-2^{-r})(c-a-b-1)/ab \leq 0$ or, equivalently, $c \geq a+b+1-ab/(1-2^{-r})$. Another application of Theorem (1.5) when $r \to \infty$ completes the proof.

Furthermore, the following two theorems are parallel to those (2.1) and (2.3) for the convex case. □

Theorem 2.4 *If $a, b > 0$ and $c > a+b+2$, then a sufficient condition for $z\,{}_2F_1(a,b;c;z)$ to be in $C_1(2^{-r})$ with r is a positive integer:*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\,1+\left(\frac{3-2^{-r}}{1-2^{-r}}\right)\left(\frac{ab}{c-a-b-1}\right)\right.$$

$$\left.+\frac{(a)_2(b)_2}{(1-2^{-r})(c-a-b-2)_2}\right]\leq 2. \tag{2.6}$$

Condition (2.6) is necessary and sufficient for ${}_2F_1(a,b;c;z)=z\,(2-{}_2F_1(a,b;c;z))$ to be in $C(2^{-r})(C_1(2^{-r}))$.

Proof In view of Theorem (1.4) , we only need to show that

$$\sum_{n=2}^{\infty} n(n-2^{-r})\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}})\leq 1-2^{-r}.$$

Now

$$\sum_{n=2}^{\infty} n(n-2^{-r})\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}=\sum_{n=0}^{\infty}(n+2)(n+2-2^{-r})\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$=(n+2)^2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}-(2^{-r})\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \tag{2.7}$$

Letting $n+2=(n+1)+1$, we have

$$\sum_{n=0}^{\infty}(n+2)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}} \tag{2.8}$$

and

$$\sum_{n=0}^{\infty}(n+2)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}=\sum_{n=0}^{\infty}(n+1)^2\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}+2\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+2\sum_{n=0}^{\infty}(n+1)\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}$$

$$+\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n+1}}$$

$$=\sum_{n=1}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n-1}}+3\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_{n}}+\sum_{n=1}^{\infty}\frac{(a)_{n}(b)_{n}}{(c)_{n}(1)_{n}}.$$

Substituting this last equality and (2.8) into right side of (2.7) yields

$$\sum_{n=0}^{\infty}\frac{(a)_{n+2}(b)_{n+2}}{(c)_{n+2}(1)_n}+(3-2^{-r})\sum_{n=0}^{\infty}\frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(1)_n}+(1-2^{-r})\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{2.9}$$

Since $(a)_{n+k}=(a)_k(a+k)_n$, we may rewrite (2.9) as

$$\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+(3-2^{-r})\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+$$

$$(1-2^{-r})\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right].$$

Upon simplification, we can see that this last expression is bounded above by $1-2^{-r}$ if and only if (2.6) holds. That (2.6) is also necessary for ${}_2F_1(a,b;c;z)$ to be in $C(2^{-r})(C_1(2^{-r}))$ follows from Theorem (1.5) . □

Theorem 2.5 *If* $a,b>-1$, $ab<0$, *and* $c>a+b+2$, *then a necessary and sufficient condition for* $z_2F_1(a,b;c;z)$ *to be in* $C(2^{-r})(C_1(2^{-r}))$ *is that*

$$(a)_2(b)_2+(3-2^{-r})ab(c-a-b-2)+(1-2^{-r})(c-a-b-1)_2\geq 0. \tag{2.10}$$

Proof Since $z_2F_1(a,b;c;z)$ has the form (2.3), we can see from Theorem (1.5) that our conclusion is equivalent to

$$\sum_{n=2}^{\infty}n(n-2^{-r})\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}\leq\left|\frac{c}{ab}\right|(1-2^{-r}). \tag{2.11}$$

Note that $c>a+b+2$ if the left side of (2.11) converges. Writing $(n+2)(n+2-2^{-r})=(n+1)^2+(2-2^{-r})(n+1)+(1-2^{-r})$, we see that

$$\sum_{n=2}^{\infty}n(n-2^{-r})\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c)_{n-2}(1)_{n-1}}=\sum_{n=0}^{\infty}(n+2)(n+2-2^{-r})\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+(2-2^{-r})\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+$$

$$(1-2^{-r})\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\frac{(a+1)(b+1)}{(c+1)}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+(3-2^{-r})\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}$$

$$+(1-2^{-r})\frac{c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\Bigg[(a+1)(b+1)$$

.

$$+(3-2^{-r})(c-a-b-2)+\frac{(1-2^{-r})}{ab}(c-a-b-1)_2\Bigg]-\frac{c(1-2^{-r})}{ab}.$$

This last expression is bounded above by $\mid c/ab \mid (1-2^{-r})$ if and only if $(a+1)(b+1)+(3-2^{-r})(c-a-b-2)+((1-2^{-r})/ab)(c-a-b-1)_2 \leq 0$, which is equivalent to (2.10).

Now we need to recall the following necessary and sufficient conditions for functions f to be in the class $S^*(\alpha,\beta)$ and due to Gupta and Jain (see [4]). □

Lemma 2.6

(i) A function f of the form (1.2) is in the class $S^*(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}\left[n\left(1+\beta\right)-1+\beta\left(1-2\alpha\right)\right]a_n \leq 2\beta\left(1-\alpha\right).$$

(ii) A function f of the form (1.2) is in the class $C^*(\alpha,\beta)$ if and only if

$$\sum_{n=2}^{\infty}n\left[n\left(1+\beta\right)-1+\beta\left(1-2\alpha\right)\right]a_n \leq 2\beta\left(1-\alpha\right).$$

Theorem 2.7

(i) If $a,b>-1$, $c>0$ and $ab<0$, then $z_2F_1(a,b;c;z)$ is in $S^*(2^{-r},2^{-1})$ if and only if

$$c > a+b+1-\frac{3ab}{2(1-2^{-r})} \tag{2.12}$$

(ii) If $a,b>0$, $c>a+b+1$,then ${}_2F_1(a,b;c;z)=z\left[2-{}_2F_1\left(a,b;c;z\right)\right]$ is in $S^*(2^{-r},2^{-1})$ if and only if

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{3ab}{2(1-2^{-r})(c-a-b-1)}\right] \leq 2. \tag{2.13}$$

Proof (i) Since

$$z_2F_1(a,b;c;z) = z+\frac{ab}{c}\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n$$

$$= z-\mid\frac{ab}{c}\mid\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}z^n,$$

according to (i) of Lemma (2.6), we must show that

$$\sum_{n=2}^{\infty}\left[n(3/2)-1+(1/2)(1-2^{1-r})\right]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq \left(1-2^{-r}\right)\mid\frac{c}{ab}\mid. \tag{2.14}$$

Note that the left side of (2.14) diverges if $c < a+b+1$ (see[15]). Now

$$\sum_{n=2}^{\infty}\left[n(3/2)-1+(1/2)\left(1-2^{1-r}\right)\right]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}}$$

$$=\frac{3}{2}\sum_{n=0}^{\infty}(n+1)\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}+\left(1-2^{-r}\right)\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_{n+1}}$$

$$=\frac{3}{2}\sum_{n=0}^{\infty}\frac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n}+\frac{(1-2^{-r})c}{ab}\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$=\frac{3}{2}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+\frac{(1-2^{-r})c}{ab}\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right].$$

Hence, (2.14) is equivalent to

$$\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}\left[\frac{3}{2}+\frac{(1-2^{-r})(c-a-b-1}{ab}\right]\leq$$

$$(1-2^{-r})\left[\frac{c}{\mid ab\mid}+\frac{c}{ab}\right]=0. \tag{2.15}$$

Thus, (2.15) is valid if only if

$$\frac{3}{2}+\frac{(1-2^{-r})(c-a-b-1}{ab}\leq 0,$$

or equivalently,

$$c\geq a+b+1-\frac{3ab}{2(1-2^{-r})}.$$

(ii) Since

$${}_2F_1\left(a,b;c;z\right)=z-\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

by(i) of Lemma (2.6), we only need to show that

$$\sum_{n=2}^{\infty}\left[n(3/2)-1+(1/2)\left(1-2^{1-r}\right)\right]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}\leq\left(1-2^{-r}\right).$$

Now,

$$\sum_{n=2}^{\infty}\left[n(3/2)-1+(1/2)\left(1-2^{1-r}\right)\right]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}$$

$$= \frac{3}{2}\sum_{n=1}^{\infty} n\frac{(a)_n(b)_n}{(c)_n(1)_n} + \left(1-2^{-r}\right)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{3}{2}\sum_{n=1}^{\infty} \frac{(a)_n(b)_n}{(c)_n(1)_{n-1}} + \left(1-2^{-r}\right)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}. \tag{2.16}$$

Noting that $(a)_n = a(a+1)_{n-1}$ then, (2.16) may be expressed as

$$\frac{3ab}{2c}\sum_{n=1}^{\infty}\frac{(a+1)_{n-1}(b+1)_{n-1}}{(c+1)_{n-1}(1)_{n-1}} + \left(1-2^{-r}\right)\sum_{n=1}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}$$

$$= \frac{3ab}{2c}\sum_{n=0}^{\infty}\frac{(a+1)_{n}(b+1)_{n}}{(c+1)_{n}(1)_{n}} + \left(1-2^{-r}\right)\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right]$$

$$= \frac{3ab}{2c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)} + \left(1-2^{-r}\right)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} - 1\right]$$

$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[\left(1-2^{-r}\right) + \frac{3ab}{2\left(c-a-b-1\right)}\right] - \left(1-2^{-r}\right).$$

However, this last expression is bounded by $(1-2^{-r})$ if (2.13) holds. □

3. Conclusion

In this manuscript, we have offered a study on "An analytical investigation on starlikeness and convexity properties for hypergeometric function $z_2F_1(a,b;c;z)$". Hypergeometric functions with some properties such as convexity and starlike were analytically investigated. The main subject of the research was to obtain the desired conditions for a, b, and c parameters of interest. Most of the results of this article have been many special results of [1, 5, 8, 10, 15] in some particular cases.

References

[1] Carlson BC, Shaffer DB. Starlike and presatarlike hypergeometric functions. SIAM Journal on Mathematical Analysis 1984; 15: 737-745.

[2] Duren PL. Univalent functions. Berlin, Germany: Springer-Verlag, 1983.

[3] Eenigenburg PJ. A class of starlike mappings of the unit disk. Compositio Mathematica 1972; 24: 235-238.

[4] Jain PK, Gupta VP. Certain classes of univalent functions with negative coefficients. Bulletin of the Australian Mathematical Society 1976; 14: 409-416.

[5] Merkes E, Scott BT. Starlike hypergeometric functions. Proceedings of the American Mathematical Society 1961; 12: 885-888.

[6] Miller SS, Mocanu PT. Differential Subordinations Theory and Applications. New York, NY, USA: Marcel Dekker Inc., 2000.

[7] Miller SS, Mocanu PT. Univalence of Gaussiaan and Confluent Hypergeometric Functions. Proceedings of The American Mathematical Society 1990; 110: 2.

[8] Mostafa AO. Starlikenees and convexity results for hypergeometric functions. Computers and Mathematics with Applications 2010; 59: 2821-2826.

[9] Robertson MS. On the theory of univalent functions. Annals of Mathematics 1936; 37: 374-408.

[10] Ruscheweyh ST, Singh V. On the order of starlikeness of hypergeometric functions. Journal of Mathematical Analysis and Applications 1986; 113: 1-11.

[11] Sakar FM,Aydoğan M. Subclass of m-quasiconformal harmonic functions in association with Janowski starlike functions. Applied Mathematics and Computation 2018; 319: 461-468.

[12] Sansone G, Gerretsen J. Lectures on the theory of functions of a complex variable. Groningen, the Netherlands: Noordhoff Publishing, 1969.

[13] Silverman H. Univalent functions with negative coefficients. Proceedings of the American Mathematical Society 1975; 15: 109-116.

[14] Silverman H. Subclasses of Starlike functions. Revue Roumaine de Mathématique Pures et Appliquées 1978; 23: 1093-1099.

[15] Silverman H. Starlike and convexity properties for hypergeometric functions. Journal of Mathematical Analysis and Applications 1993; 172: 574-581.

[16] Teubner BC. E. Study, Konforme Abbildung Einfachzusammenhangender Bereiche. Leipzig und Berlin, 1913 (in German).

[17] Wright DJ. On a class of starlike functions. Compositio Mathematica 1969; 21: 122-124.