Units and 2-**class field towers of some multiquadratic number fields**

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Abstract: In this paper, we investigate the unit groups, the 2-class groups, the 2-class field towers and the structures of the second 2-class groups of some multiquadratic number fields of degree 8 and 16.

Key words: Multiquadratic number fields, unit groups, 2-class groups, Hilbert 2-class field towers

1. Introduction

Let k be an algebraic number field and $\mathrm{Cl}_2(k)$ its 2-class group, that is the 2-Sylow subgroup of the ideal class group $\mathrm{Cl}(k)$ of k. Let $k^{(1)}$ be the Hilbert 2-class field of k, that is the maximal unramified (including the infinite primes) abelian field extension of k whose degree over k is a 2-power. Put $k^{(0)} = k$ and let $k^{(i)}$ denote the Hilbert 2-class field of $k^{(i-1)}$ for any integer $i \geq 1$. Then the sequence of fields

$$k = k^{(0)} \subset k^{(1)} \subset k^{(2)} \subset \cdots \subset k^{(i)} \cdots$$

is called the 2-class field tower of k. If for all $i \geq 1$, $k^{(i)} \neq k^{(i-1)}$, the tower is said to be infinite; otherwise, the tower is said to be finite, and the minimal integer i satisfying the condition $k^{(i)} = k^{(i-1)}$ is called the length of the tower. Unfortunately, deciding whether or not the 2-class field tower of a number field k is finite is still an open problem and there is no known method to study this finiteness. However, it is known that if the rank of $\mathrm{Cl}_2(k^{(1)}) \leq 2$, then by group theoretical meaning the tower is finite and its length is ≤ 3 (cf.[8]). Furthermore, this problem is closely related to the structure of the Galois group of the tower. In particular, for $\mathrm{Cl}_2(k)$ being cyclic, the Hilbert 2-class field tower of k terminates at the first step $k^{(1)}$, whereas for $\mathrm{Cl}_2(k)$ being of type $(2,2)$, the Hilbert 2-class field tower of k terminates in at most two steps and the structure of the Galois group $G_k = \mathrm{Gal}(k^{(2)}/k)$ is closely related to the capitulation problem in the unramified quadratic extensions of k (cf. [16]), so to the unit groups of these extensions. In fact the number of classes which capitulate in these quadratic extensions of k is given in terms of their unit groups (cf. [12]). In the literature, most studies done in this vein concern quadratic or biquadratic fields (e.g., [2, 7, 16]). In this paper, we are interested in investigating the 2-class field towers of some multiquadratic number fields related to the imaginary triquadratic number field $\mathbb{Q}(\zeta_8, \sqrt{d})$ whenever its 2-class group is of type $(2,2)$, where ζ_8 is a primitive 8th root of unity and d is an odd positive square free integer.

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The layout of this paper is the following. In Section 2, we quote some properties of 2-groups G satisfying $G/G' \simeq (2, 2)$. Next, in Section 3, we characterize the 2-class groups of some imaginary multiquadratic number fields and we compute their 2-class numbers. In Section 4, involving some technical computations, we give unit groups of some multiquadratic number fields of degree 8 and 16. Thereafter, and as applications, in Section 5, we shall investigate the Hilbert 2-class field tower of some families of imaginary triquadratic number fields; and then we deduce the capitulation behaviors in the unramified quadratic extensions of these fields.

Notations

Let k be a number field. Throughout this paper we shall respect the following notations:

- p, p' and q: Three prime integers,

- $Cl_2(k)$: The 2-class group of k,

- $h_2(k)$: The 2-class number of k,

- $h_2(d)$: The 2-class number of the quadratic field $\mathbb{Q}(\sqrt{d})$,

- ε_d: The fundamental unit of the quadratic field $\mathbb{Q}(\sqrt{d})$,

- E_k: The unit group of k,

- FSU: Abbreviation of "fundamental system of units",

- $k^{(1)}$: The Hilbert 2-class field of k,

- $k^{(2)}$: The Hilbert 2-class field of $k^{(1)}$,

- G_k: The Galois group of $k^{(2)}/k$, i.e. $Gal(k^{(2)}/k)$,

- k^*: The genus field of k,

- k^+: The maximal real subfield of k, whenever k is imaginary,

- $q(k) = [E_k : \prod_i E_{k_i}]$: The unit index of k, if k is multiquadratic and k_i are the quadratic subfields of k,

- $N_{k'/k}$: The norm map of an extension k'/k.

2. Some preliminary results of group theory

Let Q_m, D_m, and S_m denote the quaternion, dihedral, and semidihedral groups respectively, of order 2^m, where $m \geq 3$ and $m \geq 4$ for S_m. In addition, let A denote the Klein four-group. Each of these groups is generated by two elements x and y, and admits a representation by generators and relations as follows:

$$
\begin{aligned}
A &= \{x, y \mid x^2 = y^2 = 1, \ y^{-1}xy = x\}, \\
Q_m &= \{x, y \mid x^{2^{m-2}} = y^2 = a, a^2 = 1, \ y^{-1}xy = x^{-1}\}, \\
D_m &= \{x, y \mid x^{2^{m-1}} = y^2 = 1, \ y^{-1}xy = x^{-1}\}, \\
S_m &= \{x, y \mid x^{2^{m-1}} = y^2 = 1, \ y^{-1}xy = x^{2^{m-2}-1}\}.
\end{aligned}
$$

We shall recall some well known properties of 2-groups G such that G/G' is of type $(2,2)$, where G' denotes the commutator subgroup of G. For more details about these properties, we refer the reader to [16, pp. 272-273], [7, p. 162] and [10, Chap. 5].

Let k be an algebraic number field and $\mathrm{Cl}_2(k)$ the 2-Sylow subgroup of its ideal class group $\mathrm{Cl}(k)$. Let $k^{(1)}$ (resp. $k^{(2)}$) be the first (resp. second) Hilbert 2-class field of k and put $G = \mathrm{Gal}(k^{(2)}/k)$, then if G' denotes the commutator subgroup of G, we have by class field theory $G' \simeq \mathrm{Gal}(k^{(2)}/k^{(1)})$ and $G/G' \simeq \mathrm{Gal}(k^{(1)}/k) \simeq \mathrm{Cl}_2(k)$. Assume in all what follows that $\mathrm{Cl}_2(k)$ is of type $(2,2)$, then it is known that G is isomorphic to A, Q_m, D_m, or S_m.

Let x and y be as above. Note that the commutator subgroup G' of G is always cyclic and $G' = \langle x^2 \rangle$. The group G possesses exactly three subgroups of index 2 which are:

$$H_1 = \langle x \rangle, \ H_2 = \langle x^2, y \rangle, \ H_3 = \langle x^2, xy \rangle.$$

Note that for the two cases Q_3 and A, each H_i is cyclic. For the case D_m, with $m > 3$, H_2 and H_3 are also dihedral. For Q_m, with $m > 3$, H_2 and H_3 are quaternion. Finally for S_m, H_2 is dihedral, whereas H_3 is quaternion. Furthermore, if G is isomorphic to A (resp. Q_3), then the subgroups H_i are cyclic of order 2 (resp. 4). If G is isomorphic to Q_m, with $m > 3$, D_m or S_m, then H_1 is cyclic and H_i/H_i' is of type $(2,2)$ for $i \in \{2,3\}$, where H_i' is the commutator subgroup of H_i.

Let F_i be the subfield of $k^{(2)}$ fixed by H_i, where $i \in \{1,2,3\}$. If $k^{(2)} \neq k^{(1)}$, $\langle x^4 \rangle$ is the unique subgroup of G' of index 2. Let L (L is defined only if $k^{(2)} \neq k^{(1)}$) be the subfield of $k^{(2)}$ fixed by $\langle x^4 \rangle$. Then F_1, F_2, and F_3 are the three quadratic subextensions of $k^{(1)}/k$ and L is the unique subfield of $k^{(2)}$ such that L/k is a nonabelian Galois extension of degree 8.

We continue by recalling the definition of Taussky's conditions A and B ([18]). Let k' be a cyclic unramified extension of a number field k and j denotes the basic homomorphism: $j_{k'/k} : \mathrm{Cl}(k) \longrightarrow \mathrm{Cl}(k')$, induced by extension of ideals from k to k'. Thus, we say

1. k'/k satisfies condition A if and only if $|ker(j_{k'/k}) \cap N_{k'/k}(\mathrm{Cl}(k'))| > 1$.

2. k'/k satisfies condition B if and only if $|ker(j_{k'/k}) \cap N_{k'/k}(\mathrm{Cl}(k'))| = 1$.

Set $j_{F_i/k} = j_i$, $i = 1,2,3$. Then we have:

Theorem 2.1 ([16, Theorem 2])

1. If $k^{(1)} = k^{(2)}$, then F_i satisfy condition A, $|ker(j_i)| = 4$, for $i = 1,2,3$, and G is abelian of type $(2,2)$.

2. If $\mathrm{Gal}(L/k) \simeq Q_3$, then F_i satisfy condition A and $|ker(j_i)| = 2$ for $i = 1,2,3$ and $G \simeq Q_3$.

3. If $\mathrm{Gal}(L/k) \simeq D_3$, then F_2, F_3 satisfy condition B and $|ker j_2| = |ker j_3| = 2$. Furthermore, if F_1 satisfies condition B, then $|ker j_1| = 2$ and $G \simeq S_m$, if F_1 satisfies condition A and $|ker j_1| = 2$ then $G \simeq Q_m$. If F_1 satisfies condition A and $|ker j_1| = 4$ then $G \simeq D_m$.

These results are summarized in the following table (see Table 1).

Table 1. Capitulation types.

$\|ker\ j_1\|\ (A/B)$	$\|ker\ j_2\|\ (A/B)$	$\|ker\ j_3\|\ (A/B)$	G
4	4	4	$(2,2)$
$2A$	$2A$	$2A$	Q_3
4	$2B$	$2B$	$D_m, m \geq 3$
$2A$	$2B$	$2B$	$Q_m, m > 3$
$2B$	$2B$	$2B$	$S_m, m > 3$

By Theorem 2.1 and group theoretic properties quoted in the beginning of this section, one can easily deduce the following remark.

Remark 2.2 *The 2-class groups of the three unramified quadratic extensions of k are cyclic if and only if $k^{(1)} = k^{(2)}$ or $k^{(1)} \neq k^{(2)}$ and $G \simeq Q_3$. In the other cases the 2-class group of only one unramified quadratic extension is cyclic and the others are of type $(2,2)$.*

Proposition 2.3 *Let K be a number field and let L be an abelian unramified 2-extension of K. If $G_L = \mathrm{Gal}(L^{(2)}/L)$ is abelian, then L and $K^{(1)}$ have the same Hilbert 2-class field (i.e. $K^{(2)} = L^{(1)}$). Furthermore, $|G_K| = [L : K] \cdot h_2(L)$. In particular, if the 2-class group of K^* (i.e. the genus field of K) is cyclic, then $|G_K| = [K^* : K] \cdot h_2(K^*)$.*

Proof The fact that L/K is unramified implies that $K \subset L \subset K^{(1)} \subset L^{(1)} \subset K^{(2)} \subset L^{(2)}$. Since G_L is abelian, we deduce that $L^{(1)} = L^{(2)}$. Therefore, $K^{(2)} = L^{(1)}$; hence, the first equality. As the 2-class group of K^* is cyclic, the 2-class field tower of K terminates at the first step, which completes the proof. □

3. 2-class groups of some multiquadratic number fields

Let d be an odd positive square-free integer and ζ_8 a primitive 8th root of unity. Note that the triquadratic field $L_d = \mathbb{Q}(\zeta_8, \sqrt{d})$ is the first step of both the cyclotomic $\mathbb{Z}_2$-extension and a noncyclotomic $\mathbb{Z}_2$-extension (e.g., the noncyclotomic $\mathbb{Z}_2$-extension of $\mathbb{Q}(\sqrt{-1})$ (cf. [13]) shifted by $\sqrt{d}$) of the biquadratic field $\mathbb{Q}(\sqrt{-1}, \sqrt{d})$. This fact makes it of particular importance for the triquadratic number fields. In this subsection and the next section, we will make some preparations on the 2-class groups and unit groups of some multiquadratic number fields that will help to study the 2-class field towers and the second 2-class groups of all fields $L_d = \mathbb{Q}(\zeta_8, \sqrt{d})$, such that 2-class group is of type $(2,2)$, together with some different families of mutiquadratic number fields in the last section of this paper. For this, we recall the following results of our earlier paper [3, Theorem 5.7]. Let p, p', and q be three prime integers. The 2-class group of $L_d = \mathbb{Q}(\sqrt{d}, \zeta_8)$ is of type $(2,2)$ if and only if d takes one of the following forms:

$$d = pq \text{ with } p \equiv -q \equiv 1 \pmod{4}, \left(\frac{2}{p}\right) = -1, \left(\frac{2}{q}\right) = 1 \text{ and } \left(\frac{p}{q}\right) = -1. \tag{3.1}$$

$$d = pq \text{ with } p \equiv q \equiv -1 \pmod{4}, \left(\frac{2}{p}\right) = -1, \left(\frac{2}{q}\right) = 1 \text{ and } \left(\frac{p}{q}\right) = -1, \tag{3.2}$$

$$d = p' \text{ with } p' \equiv 1 \pmod{16} \text{ and } \left(\frac{2}{p'}\right)_4 \neq \left(\frac{p'}{2}\right)_4. \tag{3.3}$$

Consider the following notations

1. $L_{pq} = \mathbb{Q}(\sqrt{2}, \sqrt{qp}, \sqrt{-1})$.

2. $L_{pq}^* = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$ the genus field of L_{pq}.

3. $L_{p'} = \mathbb{Q}(\sqrt{2}, \sqrt{p'}, \sqrt{-1})$.

4. $F_{pq} = \mathbb{Q}(\sqrt{2p}, \sqrt{2q}, \sqrt{-2})$ or $\mathbb{Q}(\sqrt{p}, \sqrt{2q}, \sqrt{-2})$, according to whether p and q verify conditions (3.1) or (3.2) .

5. $K_{pq} = \mathbb{Q}(\sqrt{p}, \sqrt{q}, \sqrt{-2})$ or $\mathbb{Q}(\sqrt{q}, \sqrt{2p}, \sqrt{-2})$, according to whether p and q verify conditions (3.1) or (3.2).

6. $k_{pq} = \mathbb{Q}(\sqrt{-2}, \sqrt{pq})$ or $\mathbb{Q}(\sqrt{-2}, \sqrt{2pq})$ according to whether p and q verify conditions (3.1) or (3.2).

7. $m \geq 2$ the positive integer satisfying $h_2(-2q) = 2^m$ (cf. [9]).

Let us start with some lemmas that we shall use in what follows.

Lemma 3.1 ([1]) *Let K_0 be a real number field, $K = K_0(i)$ a quadratic extension of K_0, $n \geq 2$ be an integer and ξ_n a 2^n-th primitive root of unity, then $\xi_n = \frac{1}{2}(\mu_n + \lambda_n i)$, where $\mu_n = \sqrt{2 + \mu_{n-1}}$, $\lambda_n = \sqrt{2 - \mu_{n-1}}$, $\mu_2 = 0$, $\lambda_2 = 2$ and $\mu_3 = \lambda_3 = \sqrt{2}$. Let n_0 be the greatest integer such that ξ_{n_0} is contained in K, $\{\varepsilon_1, ..., \varepsilon_r\}$ a fundamental system of units of K_0 and ε a unit of K_0 such that $(2 + \mu_{n_0})\varepsilon$ is a square in K_0 (if it exists). Then a fundamental system of units of K is one of the following systems:*

1. $\{\varepsilon_1, ..., \varepsilon_{r-1}, \sqrt{\xi_{n_0}\varepsilon}\}$ *if ε exists, in this case $\varepsilon = \varepsilon_1^{j_1}...\varepsilon_{r-1}^{j_1}\varepsilon_r$, where $j_i \in \{0, 1\}$.*

2. $\{\varepsilon_1, ..., \varepsilon_r\}$ *elsewhere.*

Lemma 3.2 *Let p and q be two primes satisfying conditions (3.1). Then,*

1. *A FSU of $\mathbb{Q}(\sqrt{p}, \sqrt{q})$ is $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_q\varepsilon_{pq}}\}$.*

2. *A FSU of $\mathbb{Q}(\sqrt{p}, \sqrt{2q})$ is $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2pq}}\}$.*

Proof To prove this lemma, we will use the algorithm described in [19]. Let $\varepsilon_{pq} = x + y\sqrt{pq}$ for some integers x and y. Since $N(\varepsilon_{pq}) = 1$, then $x^2 - 1 = y^2pq$. Hence, by the unique factorization in $\mathbb{Z}$ there exist y_1, y_2 in $\mathbb{Z}$ such that

$$(1): \begin{cases} x \pm 1 = y_1^2 \\ x \mp 1 = pqy_2^2, \end{cases} \quad (2): \begin{cases} x \pm 1 = py_1^2 \\ x \mp 1 = qy_2^2, \end{cases} \quad (3): \begin{cases} x \pm 1 = 2py_1^2 \\ x \mp 1 = 2qy_2^2, \end{cases} \text{ or } \quad (4): \begin{cases} x \pm 1 = 2y_1^2 \\ x \mp 1 = 2pqy_2^2, \end{cases}$$

Note that $y = y_1y_2$ or $y = 2y_1y_2$.

- System (1) implies $1 = \left(\frac{y_1^2}{p}\right) = \left(\frac{x \pm 1}{p}\right) = \left(\frac{x \mp 1 \pm 2}{p}\right) = \left(\frac{2}{p}\right) = -1$. Thus, this case is impossible.

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- System (3) implies $\left(\frac{2q}{p}\right) = \left(\frac{x \mp 1}{p}\right) = \left(\frac{\pm 2}{p}\right) = \left(\frac{2}{p}\right) = -1$, which contradicts the fact that $\left(\frac{p}{q}\right) = -1$.

- If system (4) holds, then $2\varepsilon_{pq} = 2(x + y\sqrt{pq}) = 2(y_1 + y_2\sqrt{pq})^2$. Thus, ε_{pq} is a square in $\mathbb{Q}(\sqrt{pq})$ which is absurd.

- Suppose that $\begin{cases} x + 1 = py_1^2 \\ x - 1 = qy_2^2, \end{cases}$ then, $-1 = \left(\frac{p}{q}\right) = \left(\frac{x+1}{q}\right) = \left(\frac{x-1+2}{q}\right) = \left(\frac{2}{q}\right) = 1$. This is impossible too.

From the above discussion, we infer that $\begin{cases} x - 1 = py_1^2 \\ x + 1 = qy_2^2. \end{cases}$ Therefore, $2\varepsilon_{pq} = (y_1\sqrt{p} + y_2\sqrt{q})^2$. Therefore, $2\varepsilon_{pq}$ is a square in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$.

Using similar techniques, one can show that $2\varepsilon_q$ is a square in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$. Hence, ε_{pq} (resp. ε_q) is not a square in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$, since, otherwise, we will get $\sqrt{2} \in \mathbb{Q}(\sqrt{p}, \sqrt{q})$, which is not true. As ε_p has norm -1, it follows that $\sqrt{\varepsilon_q \varepsilon_{pq}}$ is the only element of $\{\varepsilon_p^i \varepsilon_q^j \varepsilon_{pq}^k : i, j \text{ and } k \in \{0, 1\}\}$ which is a square in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$. Therefore, the first item (cf. [19]). We similarly prove the second item. □

Remark 3.3 *By the previous proof, we have ε_{pq} is a square in $\mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. One can similarly show that if $q \equiv 3 \pmod 4$, then $2\varepsilon_q$ (resp. $2\varepsilon_{2q}$) is a square in $\mathbb{Q}(\sqrt{q})$ (resp. $\mathbb{Q}(\sqrt{2q})$).*

Lemma 3.4 *Let p and q be two primes satisfying conditions (3.1) or (3.2). Then the class number of $F = \mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$ is odd.*

Proof For values of class numbers of quadratic number fields used below one can see [9, 15].

- If p and q satisfy conditions (3.1), then by class number formula (cf. [19]) we have

$$
\begin{aligned}
h_2(F) &= \frac{1}{2^5} q(F) h_2(p) h_2(q) h_2(-p) h_2(-q) h_2(pq) h_2(-pq) h_2(-1), \\
&= \frac{1}{2^5} q(F) \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 2 \cdot 2 \cdot 1 = \frac{1}{2^2} q(F).
\end{aligned}
$$

Since, by Lemma 3.2, a FSU of $F^+ = \mathbb{Q}(\sqrt{q}, \sqrt{p})$ is given by $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_q \varepsilon_{pq}}\}$ and $\sqrt{2\varepsilon_q}$ is a square in F^+, then by Lemma 3.1, $\{\varepsilon_p, \sqrt{\varepsilon_q \varepsilon_{pq}}, \sqrt{i\varepsilon_q}\}$ is a FSU of F. Thus, $h_2(F) = \frac{1}{2^2} \cdot 4 = 1$.

- If p and q satisfy conditions (3.2), then as above we get

$$
\begin{aligned}
h_2(F) &= \frac{1}{2^5} q(F) h_2(p) h_2(q) h_2(-p) h_2(-q) h_2(pq) h_2(-pq) h_2(-1), \\
&= \frac{1}{2^5} q(F) \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 4 \cdot 1 = \frac{1}{2^3} q(F).
\end{aligned}
$$

By Remark 3.3 we have $\sqrt{2\varepsilon_q}$, $\sqrt{2\varepsilon_p} \in F^+$, so it is easy to see that a FSU of F^+ is $\{\varepsilon_q, \sqrt{\varepsilon_p \varepsilon_q}, \sqrt{\varepsilon_p \varepsilon_{pq}}$ or $\sqrt{\varepsilon_{pq}}\}$ or $\{\varepsilon_q, \varepsilon_{pq}, \sqrt{\varepsilon_p \varepsilon_q}\}$. Thus, by Lemma 3.1 and the fact that $\frac{1}{2^3} q(F) \in \mathbb{N}$, we have $q(F) = 2^3$ (and thus $\{\varepsilon_q, \varepsilon_{pq}, \sqrt{\varepsilon_p \varepsilon_q}\}$ is not a FSU of F^+). Therefore, $h(F)$ is odd.

□

Theorem 3.5 *Let p and q be two primes satisfying conditions (3.1) or (3.2). Then we have*

1. *The 2-class group of L_{pq} is of type $(2,2)$.*

2. *The 2-class group of F_{pq} is of type $(2,2)$.*

3. *The 2-class group of K_{pq} is cyclic of type $\mathbb{Z}/2^{m+1}\mathbb{Z}$, with $h_2(-2q) = 2^m$.*

Proof Assume firstly that p and q verify conditions (3.1). Thus, by [3], the 2-class group of L_{pq} is of type $(2,2)$. Under this assumption, $h_2(p) = h_2(q) = h_2(-2) = 1$, $h_2(-2p) = h_2(pq) = 2$ and $h_2(-2pq) = 4$ (cf.[9, 15]). We claim that the class number of $k' = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ is odd. In fact, we have $h_2(p) = h_2(q) = 1$ and $h_2(pq) = 2$ (cf. [9, 15]). Therefore, Lemma 3.2 and Kuroda's class number formula (cf. [17]) imply that $h_2(k') = \frac{1}{4}q(k')h_2(p)h_2(q)h_2(pq) = 1$. Since there are only two primes which ramify in K_{pq}/k', then by ambiguous class number formula (cf. [11]) the rank of the 2-class group of K_{pq} equals $2 - 1 - e = 1 - e$, where e is an integer defined by $[E_{k'} : E_{k'} \cap N_{K_{pq}/k'}(K_{pq})] = 2^e$. We infer that the rank of the 2-class group of K_{pq} cannot be equal to 2. On the other hand, note that L_{pq}, F_{pq}, and K_{pq} are the three unramified quadratic extensions of $k_{pq} = \mathbb{Q}(\sqrt{-2}, \sqrt{pq})$ that have a 2-class group of type $(2,2)$ (cf. [2, Theorem 1]). Then, by Remark 2.2, the 2-class group of F_{pq} is of type $(2,2)$ and that of K_{pq} is cyclic. From [2, Proposition 6], we deduce that $q(K_{pq}) = 4$. Hence, class number formula (cf. [19]) implies that

$$h_2(K_{pq}) = \frac{1}{2^5}q(K_{pq})h_2(p)h_2(q)h_2(pq)h_2(-2p)h_2(-2q)h_2(-2pq)h_2(-2) = 2h_2(-2q).$$

Thus, we have the result for this case.

Suppose now that p and q satisfy conditions (3.2). Using [2, Proposition 5] and [9, 15, 17], we similarly show that $h_2(K_{pq}) = 2 \cdot h_2(-2q)$. Thus, $h_2(K_{pq})$ is divisible by 8. Thus, as above its 2-class group cannot be of type $(2,2)$, which completes the proof. □

Corollary 3.6 *Keep assumptions of Theorem 3.5, then the group $G_{k_{pq}}$ is neither abelian nor quaternion of order 8.*

Proof Since L_{pq}, F_{pq}, and K_{pq} are the three unramified quadratic extensions of k_{pq}. Therefore, we have the result by Theorem 3.5 and Remark 2.2. □

Corollary 3.7 *Keep assumptions of Theorem 3.5, then the group $G_{K_{pq}}$ is cyclic of order 2^{m+1}.*

Theorem 3.8 *Let p and q be two primes satisfying conditions (3.1) or (3.2). Then the 2-class group of $L_{pq}^* = \mathbb{Q}(\sqrt{2}, \sqrt{q}, \sqrt{p}, i)$ is $\mathbb{Z}/2^m\mathbb{Z}$, where $h_2(-2q) = 2^m$.*

Proof Let $F = \mathbb{Q}(\sqrt{q}, \sqrt{p}, i)$, so we know by Lemma 3.4 that the class number of F is odd. If p and q verify conditions (3.1) (resp. (3.2)) we have 2 unramified in $k' = \mathbb{Q}(\sqrt{p}, \sqrt{-q})$ (resp. $k'' = \mathbb{Q}(\sqrt{-p}, \sqrt{-q})$), then the decomposition group of 2 is a nontrivial cyclic subgroup of $Gal(k'/\mathbb{Q})$ (resp. $Gal(k''/\mathbb{Q})$) (in fact 2 is inert in $\mathbb{Q}(\sqrt{-pq})$ (resp. $\mathbb{Q}(\sqrt{-p})$)). Since a nontrivial cyclic subgroup of $Gal(k'/\mathbb{Q})$ (resp. $Gal(k''/\mathbb{Q})$)

has two elements, there are exactly 2 primes of k' (resp. k'') above 2. As 2 ramify in $\mathbb{Q}(\sqrt{-1})$, it follows that there are exactly 2 primes of F above 2. Therefore, there are exactly two primes that ramify in L_{pq}^*/F. By ambiguous class number formula (cf. [11]) $rank(\mathrm{Cl}_2(L_{pq}^*)) = 2 - 1 - e = 1 - e$, where e is defined by $[E_F : E_F \cap N_{L_{pq}^*/F}(L_{pq}^*)] = 2^e$. Since L_{pq}^* is the genus field of $L_{pq} = \mathbb{Q}(\sqrt{2}, \sqrt{qp}, i)$, so $[L_{pq}^* : L_{pq}] = 2$; moreover, $\mathrm{Cl}_2(L_{pq}) \simeq (2, 2)$, then the 2-class group $\mathrm{Cl}_2(L_{pq}^*)$ is cyclic or of type $(2, 2)$ (cf. Remark 2.2). It follows by the previous equality that $\mathrm{Cl}_2(L_{pq}^*)$ is cyclic. Note that L_{pq}^* is an unramified quadratic extension of K_{pq}. Therefore, by Proposition 2.3 we have $L_{pq}^{*}{}^{(1)} = K_{pq}^{(2)}$ and $G_{K_{pq}} = 2 \cdot h_2(L_{pq}^*)$. Since by Theorem 3.5 the 2-class group of K_{pq} is cyclic of order 2^{m+1}, it follows that $L_{pq}^{*}{}^{(1)} = K_{pq}^{(2)} = K_{pq}^{(1)}$ and $2 \cdot h_2(L_{pq}^*) = h_2(K_{pq}) = 2^{m+1}$. Hence, we have the theorem. $\qquad\square$

Corollary 3.9 *The group $G_{L_{pq}^*}$ is abelian.*

4. Units of some multiquadratic number fields

Let p and q be two prime integers satisfying conditions (3.1), namely

$$p \equiv -q \equiv 1 \pmod 4, \left(\frac{2}{p}\right) = -1, \left(\frac{2}{q}\right) = 1 \text{ and } \left(\frac{p}{q}\right) = -1.$$

Consider the field $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$, and let $\mathbb{K}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$ be its maximal real subfield. The main task of this section is to determine fundamental system of units of $\mathbb{K}^+$ and $\mathbb{K}$, which will be used to prove Theorems 5.2 and 5.5. To prove this result, we have to do some preparations. In the same manner as in the proof of Lemma 3.2, one shows the following lemma.

Lemma 4.1 *Let p and q be two primes satisfying conditions (3.1).*

1. *Let $\varepsilon_{pq} = a + b\sqrt{pq}$, $a, b \in \mathbb{Z}$, then $p(a-1)$ is a square in $\mathbb{N}$, and $\sqrt{2\varepsilon_{pq}} = b_1\sqrt{p} + b_2\sqrt{q}$ and $2 = -pb_1^2 + qb_2^2$, for some integers b_1 and b_2.*

2. *Let $\varepsilon_{2pq} = x + y\sqrt{2pq}$, $x, y \in \mathbb{Z}$, then $2p(x - 1)$ is a square in $\mathbb{N}$, and $\sqrt{2\varepsilon_{2pq}} = y_1\sqrt{2p} + y_2\sqrt{q}$ and $2 = -2py_1^2 + qy_2^2$, for some integers y_1 and y_2.*

3. *Let $\varepsilon_{2q} = c + d\sqrt{2q}$, $c, d \in \mathbb{Z}$, then $c + 1$ is a square in $\mathbb{N}$, and $\sqrt{2\varepsilon_{2q}} = d_1 + d_2\sqrt{2q}$ and $2 = d_1^2 - 2qd_2^2$, for some integers d_1 and d_2.*

4. *Let $\varepsilon_q = c' + d'\sqrt{q}$, $c', d' \in \mathbb{Z}$, then $c' + 1$ is a square in $\mathbb{N}$, and $\sqrt{2\varepsilon_q} = d_1' + d_2'\sqrt{q}$ and $2 = d_1'^2 - qd_2'^2$, for some integers d_1' and d_2'.*

Lemma 4.2 *Let p and q be two primes satisfying conditions (3.1). Let $\mathbb{K}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$, so the unit group of $\mathbb{K}^+$ is one of the following:*

1. *$E_{\mathbb{K}^+} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}} \rangle$,*

2. *$E_{\mathbb{K}^+} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}} \rangle$.*

Proof   To prove this lemma, we use the algorithm described by [19]. Consider the following diagram (see Figure 1):
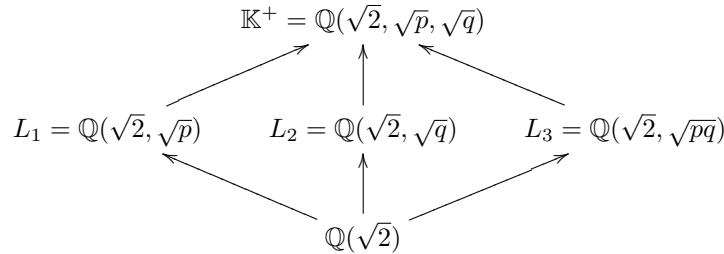

Figure 1. Subfields of $\mathbb{K}^+/\mathbb{Q}(\sqrt{2})$.

By [4], Lemma 3.2, and [3], we have a FSU of L_1 is given by $\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}\}$, a FSU of L_2 is given by $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$ and a FSU of L_3 is given by $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq} \varepsilon_{2pq}}\}$. It follows that

$$E_{L_1} E_{L_2} E_{L_3} = \langle -1, \varepsilon_2, \varepsilon_p, \varepsilon_{pq}, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq} \varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}} \rangle.$$

Note that an FSU of $\mathbb{K}$ consists of seven units chosen from those of L_1, L_2, and L_3, and from the square roots of the units of $E_{L_1} E_{L_2} E_{L_3}$ which are squares in $\mathbb{K}$ (cf. [19]). Thus, we shall determine elements of $E_{L_1} E_{L_2} E_{L_3}$ which are squares in $\mathbb{K}^+$. Suppose that X is an element of $\mathbb{K}^+$ which is a square of an element of $E_{L_1} E_{L_2} E_{L_3}$, so

$$X^2 = \varepsilon_2^a \varepsilon_p^b \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{pq} \varepsilon_{2pq}}^f \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}^g,$$

where a, b, c, d, e, f, and g are in $\{0, 1\}$.

We shall use norm maps from $\mathbb{K}^+$ to its subextensions to eliminate the cases of X^2 which do not occur. Set $G = \mathrm{Gal}(\mathbb{K}^+/\mathbb{Q}) = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$, where

$$\sigma_1(\sqrt{2}) = -\sqrt{2}, \ \sigma_2(\sqrt{p}) = -\sqrt{p} \text{ and } \sigma_3(\sqrt{q}) = -\sqrt{q},$$
$$\text{and } \sigma_i(\sqrt{2}) = \sqrt{2} \text{ for } i \in \{2, 3\},$$
$$\sigma_i(\sqrt{p}) = \sqrt{p} \text{ for } i \in \{1, 3\} \text{ and}$$
$$\sigma_i(\sqrt{q}) = \sqrt{q} \text{ for } i \in \{1, 2\}.$$

Hence, L_1, L_2, and L_3 are the fixed fields of the subgroups of G generated respectively by σ_3, σ_2, and $\sigma_2 \sigma_3$.

Let us firstly do some computations that will help in the computations of these norm maps which we shall use now and for the proof of the next lemmas as well. Let $L_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ and $L_5 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. By Lemma 3.2, a FSU of L_4 is $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_q \varepsilon_{pq}}\}$, and a FSU of L_5 is $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2pq}}\}$.

Let u, v, t, k, and $r \in \{0, 1\}$. Table 2 will be used to compute norm maps. Its 8th line, for example, is constructed as follows (the other lines are constructed in the same manner). By Lemma 4.1, there are two

integers y_1 and y_2 such that $\sqrt{\varepsilon_{2pq}} = \frac{\sqrt{2}}{2}(y_1\sqrt{2p} + y_2\sqrt{q})$, which implies that

$$
\begin{aligned}
\sqrt{\varepsilon_{2pq}}^{\sigma_1} &= \frac{-\sqrt{2}}{2}(-y_1\sqrt{2p} + y_2\sqrt{q}) \\
&= \frac{\frac{-\sqrt{2}}{2}(-y_1\sqrt{2p}+y_2\sqrt{q})(y_1\sqrt{2p}+y_2\sqrt{q})}{(y_1\sqrt{2p}+y_2\sqrt{q})} \\
&= \frac{\frac{-\sqrt{2}}{2}(-y_1^2 2p + y_2^2 q)}{\sqrt{2}\sqrt{\varepsilon_{2pq}}} \\
&= \frac{\frac{-\sqrt{2}}{2}\cdot 2}{\sqrt{2}\sqrt{\varepsilon_{2pq}}} = \frac{-1}{\sqrt{\varepsilon_{2pq}}}.
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\varepsilon_{2pq}}^{\sigma_2} &= \frac{\sqrt{2}}{2}(-y_1\sqrt{2p} + y_2\sqrt{q}) && \sqrt{\varepsilon_{2pq}}^{\sigma_3} = \frac{\sqrt{2}}{2}(y_1\sqrt{2p} - y_2\sqrt{q}) \\
&= -\frac{-\sqrt{2}}{2}(-y_1\sqrt{2p} + y_2\sqrt{q}) && \phantom{\sqrt{\varepsilon_{2pq}}^{\sigma_3}} = \frac{-\sqrt{2}}{2}(-y_1\sqrt{2p} + y_2\sqrt{q}) \\
&= -\frac{-1}{\sqrt{\varepsilon_{2pq}}} = \frac{1}{\sqrt{\varepsilon_{2pq}}}. && \phantom{\sqrt{\varepsilon_{2pq}}^{\sigma_3}} = \frac{-1}{\sqrt{\varepsilon_{2pq}}}.
\end{aligned}
$$

$$
\begin{aligned}
\sqrt{\varepsilon_{2pq}}^{1+\sigma_1} &= \sqrt{\varepsilon_{2pq}} \cdot \frac{-1}{\sqrt{\varepsilon_{2pq}}} = -1. \\
\sqrt{\varepsilon_{2pq}}^{1+\sigma_2} &= \sqrt{\varepsilon_{2pq}} \cdot \frac{1}{\sqrt{\varepsilon_{2pq}}} = 1. \\
\sqrt{\varepsilon_{2pq}}^{1+\sigma_3} &= \sqrt{\varepsilon_{2pq}} \cdot \frac{-1}{\sqrt{\varepsilon_{2pq}}} = -1. \\
\sqrt{\varepsilon_{2pq}}^{1+\sigma_1\sigma_2} &= \sqrt{\varepsilon_{2pq}} \cdot \left(\frac{1}{\sqrt{\varepsilon_{2pq}}}\right)^{\sigma_1} = -\varepsilon_{2pq}. \\
\sqrt{\varepsilon_{2pq}}^{1+\sigma_1\sigma_3} &= \sqrt{\varepsilon_{2pq}} \cdot \left(\frac{-1}{\sqrt{\varepsilon_{2pq}}}\right)^{\sigma_1} = \varepsilon_{2pq}. \\
\sqrt{\varepsilon_{2pq}}^{1+\sigma_2\sigma_3} &= \sqrt{\varepsilon_{2pq}} \cdot \left(\frac{-1}{\sqrt{\varepsilon_{2pq}}}\right)^{\sigma_2} = -\varepsilon_{2pq}.
\end{aligned}
$$

Now we return to our square $X^2 = \varepsilon_2^a \varepsilon_p^b \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f \sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}^g$, by applying the norm $N_{\mathbb{K}^+/L_2} = 1 + \sigma_2$ (see Table 2 page 1478) we get:

$$
\begin{aligned}
N_{\mathbb{K}^+/L_2}(X^2) &= \varepsilon_2^{2a} \cdot (-1)^b \cdot 1 \cdot \varepsilon_q^d \cdot \varepsilon_{2q}^e \cdot 1 \cdot (-1)^{vg} \cdot \varepsilon_2^g \\
&= \varepsilon_2^{2a} \varepsilon_q^d \varepsilon_{2q}^e (-1)^{b+vg} \varepsilon_2^g.
\end{aligned}
$$

As ε_q, ε_{2q} are squares in L_2 and $N_{\mathbb{K}^+/L_2}(X^2) > 0$, so $b + vg \equiv 0 \pmod{2}$ and ε_2^g is a square in L_2. However, ε_2 is not a square in L_2, then $g = 0$ and thus $b = 0$. Therefore, X^2 become

$$
X^2 = \varepsilon_2^a \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^f.
$$

Similarly, by applying $N_{\mathbb{K}^+/L_3} = 1 + \sigma_2\sigma_3$ (see Table 2 page 1478) one gets:

$$
N_{\mathbb{K}^+/L_3}(X^2) = \varepsilon_2^{2a} \cdot \varepsilon_{pq}^{2c} \cdot (\varepsilon_{pq}\varepsilon_{2pq})^f,
$$

unfortunately, here we conclude nothing. Therefore, we will use the norm map over $L_4 = \mathbb{Q}(\sqrt{p}, \sqrt{q})$ which is $N_{\mathbb{K}^+/L_4} = 1 + \sigma_1$. Note that $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_q\varepsilon_{pq}}\}$ is a FSU of L_4. Thus,

$$
\begin{aligned}
N_{\mathbb{K}^+/L_4}(X^2) &= (-1)^a \cdot \varepsilon_{pq}^{2c} \cdot (-\varepsilon_q)^d \cdot (-1)^e \cdot (\varepsilon_{pq})^f \\
&= \varepsilon_{pq}^{2c} \cdot (-1)^{a+d+e} \cdot \varepsilon_q^d \cdot \varepsilon_{pq}^f > 0.
\end{aligned}
$$

Since $N_{\mathbb{K}^+/L_4}(X^2) > 0$, then $a + d + e \equiv 0 \pmod{2}$. By Remark 3.3, $2\varepsilon_q$ is a square in $\mathbb{Q}(\sqrt{q})$ and $2\varepsilon_{pq}$ is a square in $\mathbb{Q}(\sqrt{p}, \sqrt{q})$. Therefore, $d = f$, since otherwise we will get ε_{pq} or ε_q is a square in L_4. Therefore,

$$
X^2 = \varepsilon_2^a \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{2q}}^e \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d.
$$

Now we apply $N_{\mathbb{K}^+/L_5} = 1 + \sigma_1 \sigma_3$, where $L_5 = \mathbb{Q}(\sqrt{p}, \sqrt{2q})$. Note that a FSU of L_5 is given by $\{\varepsilon_p, \varepsilon_{2q}, \sqrt{\varepsilon_{2pq}}\}$. We have

$$
\begin{aligned}
N_{\mathbb{K}^+/L_5}(X^2) &= (-1)^a \cdot 1 \cdot (-1)^d \cdot (-\varepsilon_{2q})^e \cdot \varepsilon_{2pq}^d \\
&= \varepsilon_{2pq}^d \cdot (-1)^{a+d+e} \cdot \varepsilon_{2q}^e > 0.
\end{aligned}
$$

Therefore, $a + d + e \equiv 0 \pmod 2$. As ε_{2q} is not a square in L_5, then $e = 0$ and $a + d \equiv 0 \pmod 2$. Therefore, $a = d$ and

$$
X^2 = \varepsilon_2^d \varepsilon_{pq}^c \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d.
$$

Remark that ε_{pq} is a square in $\mathbb{K}^+$, so we may put

$$
X^2 = \varepsilon_2^d \sqrt{\varepsilon_q}^d \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}^d.
$$

Applying the norm maps from $\mathbb{K}^+$ to all the rest of its subextensions, no contradiction is obtained and we conclude nothing about d. Therefore, $d = 0$ or 1; thus, we have the result. □

Lemma 4.3 *Suppose that the unit group of $\mathbb{K}^+$ takes the form in the second item of Lemma 4.2. Then the unit group of $\mathbb{K}$ is one of the following:*

1. $E_{\mathbb{K}} = \langle \zeta_8, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}} \rangle$, *or*

2. $E_{\mathbb{K}} = \langle \zeta_8, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}}, \sqrt[4]{\zeta_8^2 \varepsilon_q \varepsilon_{2q}} \rangle$.

Proof   We shall make use of Table 2 page 1478, and respect the same notations of the proof of the previous Lemma 4.2. According to Lemma 3.1, set

$$
Y^2 = (2 + \sqrt{2})\varepsilon_2^a \varepsilon_p^b \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}^f \sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}}^g.
$$

We have $N_{\mathbb{K}^+/L_2} = 1 + \sigma_2$. Thus,

$$
\begin{aligned}
N_{\mathbb{K}^+/L_2}(Y^2) &= (2 + \sqrt{2})^2 \cdot \varepsilon_2^{2a} \cdot (-1)^b \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot 1 \cdot (-1)^{fv} \cdot \varepsilon_2^f \cdot (-1)^{gt} \cdot \varepsilon_2^g \cdot \sqrt{\varepsilon_q}^g, \\
&= (2 + \sqrt{2})^2 \cdot \varepsilon_2^{2a} \cdot \varepsilon_q^c \cdot \varepsilon_{2q}^d \cdot (-1)^{fv+b+gt} \cdot \varepsilon_2^{g+f} \cdot \sqrt{\varepsilon_q}^g > 0,
\end{aligned}
$$

We have $fv + b + gt = 0 \pmod 2$. Recall that a FSU of L_2 is $\{\varepsilon_2, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}\}$, so

- the case $g = 0$ and $f = 1$ is impossible. In fact $\sqrt{\varepsilon_2} \notin L_2$,

- the case $g = 1$ and $f = 0$ is impossible too. In fact $\sqrt{\varepsilon_2 \sqrt{\varepsilon_q}} \notin L_2$,

- the case $g = 1$ and $f = 1$ is impossible too. In fact $\sqrt[4]{\varepsilon_q} \notin L_2$.

It follows that $f = g = 0$ and $b = 0$. Thus,

$$
Y^2 = (2 + \sqrt{2})\varepsilon_2^a \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e.
$$

We have $N_{\mathbb{K}^+/L_4} = 1 + \sigma_1$. Hence,

$$
\begin{aligned}
N_{\mathbb{K}^+/L_4}(Y^2) &= (4-2) \cdot (-1)^a \cdot (-1)^c \cdot \varepsilon_q^c \cdot (-1)^d \cdot (-1)^e \cdot \varepsilon_{pq}^e, \\
&= (-1)^{a+c+d+e} \cdot 2 \cdot \varepsilon_q^c \varepsilon_{pq}^e > 0.
\end{aligned}
$$

We have $a + c + d + e = 0 \pmod 2$. Note that $\{\varepsilon_p, \varepsilon_q, \sqrt{\varepsilon_q \varepsilon_{pq}}\}$ is a FSU of L_4. Since 2 is not a square in L_4, then $e \neq c$. It follows that $a + d + 1 = 0 \pmod 2$. Therefore, $a \neq d$ and $e \neq c$. To summarize we have

$$
Y^2 = (2+\sqrt{2})\varepsilon_2^a \sqrt{\varepsilon_q}^c \sqrt{\varepsilon_{2q}}^d \sqrt{\varepsilon_{pq}}^e,
$$

with $a \neq d$ and $e \neq c$. Let us now apply $N_{\mathbb{K}^+/L_3} = 1 + \sigma_2\sigma_3$. Thus,

$$
N_{\mathbb{K}^+/L_3}(Y^2) = (2+\sqrt{2})^2 \cdot \varepsilon_2^{2a} \cdot 1 \cdot 1 \cdot (-1)^e \cdot \sqrt{\varepsilon_{pq}}^e > 0.
$$

Thus, $e = 0$ and so $c = 1$. Then we have

$$
Y^2 = (2+\sqrt{2})\varepsilon_2^a \sqrt{\varepsilon_q} \sqrt{\varepsilon_{2q}}^d.
$$

By applying $N_{\mathbb{K}^+/L_5} = 1 + \sigma_1\sigma_3$ with $\mathbb{Q}(\sqrt{p}, \sqrt{2q})$, we get

$$
\begin{aligned}
N_{\mathbb{K}^+/L_5}(Y^2) &= (4-2) \cdot (-1)^a \cdot (-1) \cdot (-1)^d \cdot \varepsilon_{2q}^d \\
&= (-1)^{a+d+1} \cdot 2 \cdot \varepsilon_{2q}^d > 0.
\end{aligned}
$$

Since 2 is not a square in L_5, then $d = 1$ and so $a = 0$. Thus, $Y^2 = (2+\sqrt{2})\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}$. Therefore, the results are obtained by applying Lemma 3.1. □

Now we are able to state and prove the main theorem of this section.

Theorem 4.4 *Let p and q be two primes satisfying conditions (3.1). Let $\mathbb{K} = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q}, \sqrt{-1})$ and $\mathbb{K}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p}, \sqrt{q})$. Then we have:*

1. $E_{\mathbb{K}^+} = \langle -1, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}} \rangle.$

2. $E_{\mathbb{K}} = \langle \zeta_8, \varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}, \sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}}, \sqrt[4]{\zeta_8^2 \varepsilon_q \varepsilon_{2q}} \rangle.$

Table 2. Image of units by σ_i

ε	ε^{σ_1}	ε^{σ_2}	ε^{σ_3}	$\varepsilon^{1+\sigma_1}$	$\varepsilon^{1+\sigma_2}$	$\varepsilon^{1+\sigma_3}$	$\varepsilon^{1+\sigma_1\sigma_2}$	$\varepsilon^{1+\sigma_1\sigma_3}$	$\varepsilon^{1+\sigma_2\sigma_3}$
ε_2	$\frac{-1}{\varepsilon_2}$	ε_2	ε_2	-1	ε_2^2	ε_2^2	-1	-1	ε_2^2
ε_p	ε_p	$\frac{-1}{\varepsilon_p}$	ε_p	ε_p^2	-1	ε_p^2	-1	ε_p^2	-1
ε_{pq}	ε_{pq}	$\frac{1}{\varepsilon_{pq}}$	$\frac{1}{\varepsilon_{pq}}$	ε_{pq}^2	1	1	1	1	ε_{pq}^2
$\sqrt{\varepsilon_q}$	$-\sqrt{\varepsilon_q}$	$\sqrt{\varepsilon_q}$	$\frac{1}{\sqrt{\varepsilon_q}}$	$-\varepsilon_q$	ε_q	1	$-\varepsilon_q$	-1	1
$\sqrt{\varepsilon_{2q}}$	$\frac{-1}{\sqrt{\varepsilon_{2q}}}$	$\sqrt{\varepsilon_{2q}}$	$\frac{1}{\sqrt{\varepsilon_{2q}}}$	-1	ε_{2q}	1	-1	$-\varepsilon_{2q}$	1
$\sqrt{\varepsilon_{pq}}$	$-\sqrt{\varepsilon_{pq}}$	$\frac{1}{\sqrt{\varepsilon_{pq}}}$	$\frac{-1}{\sqrt{\varepsilon_{pq}}}$	$-\varepsilon_{pq}$	1	-1	-1	1	$-\varepsilon_{pq}$
$\sqrt{\varepsilon_{2pq}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{1}{\sqrt{\varepsilon_{2pq}}}$	$\frac{-1}{\sqrt{\varepsilon_{2pq}}}$	-1	1	-1	$-\varepsilon_{2pq}$	ε_{2pq}	$-\varepsilon_{2pq}$
$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}$	$(-1)^u\sqrt{\frac{\varepsilon_p}{\varepsilon_2\varepsilon_{2p}}}$	$(-1)^v\sqrt{\frac{\varepsilon_2}{\varepsilon_p\varepsilon_{2p}}}$	$\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}$	$(-1)^u\varepsilon_p$	$(-1)^v\varepsilon_2$	$\varepsilon_2\varepsilon_p\varepsilon_{2p}$			
$\sqrt[4]{\varepsilon_2^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}$	$(-1)^k\sqrt[4]{\frac{\varepsilon_q\varepsilon_{pq}}{\varepsilon_2^2\varepsilon_{2pq}}}$	$(-1)^t\sqrt[4]{\frac{\varepsilon_2^2\varepsilon_q}{\varepsilon_{pq}\varepsilon_{2pq}}}$	$(-1)^r\sqrt[4]{\frac{\varepsilon_2^2}{\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}}$	$(-1)^k\sqrt{\varepsilon_q\varepsilon_{pq}}$	$(-1)^t\varepsilon_2\sqrt{\varepsilon_q}$	$(-1)^r\varepsilon_2$			

Proof Under conditions (3.1), we have $h_2(p) = h_2(q) = h_2(-q) = h_2(2q) = h_2(-1) = h_2(2) = h_2(-2) = 1$, $h_2(-p) = h_2(2p) = h_2(-2p) = h_2(pq) = h_2(-pq) = h_2(2pq) = 2$ and $h_2(-2pq) = 4$ (cf. [9, 15]). Therefore, by Wada's class number formula (cf. [19]) one gets

$$
\begin{aligned}
h_2(\mathbb{K}) &= \tfrac{1}{2^{16}} q(\mathbb{K}) h_2(-1) h_2(2) h_2(-2) h_2(p) h_2(-p) h_2(q) \\
&\qquad h_2(-q) h_2(2p) h_2(-2p) h_2(2q) h_2(-2q) h_2(pq) h_2(-pq) h_2(2pq) h_2(-2pq) \\
&= \tfrac{1}{2^{16}} \cdot q(\mathbb{K}) \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 1 \cdot 1 \cdot 2 \cdot 2 \cdot 1 \cdot h_2(-2q) \cdot 2 \cdot 2 \cdot 2 \cdot 4, \\
&= \tfrac{1}{2^8} . q(\mathbb{K}) . h_2(-2q).
\end{aligned}
$$

On the other hand, by Theorem 3.8 we have $h_2(\mathbb{K}) = h_2(-2q)$. Therefore, obviously we must have $q(\mathbb{K}) = 2^8$.

- Suppose that the unit group of $\mathbb{K}^+$ takes the form in the first item of Lemma 4.2, then a FSU of $\mathbb{K}^+$ is

$$
\{\varepsilon_2, \varepsilon_p, \sqrt{\varepsilon_q}, \sqrt{\varepsilon_{2q}}, \sqrt{\varepsilon_{pq}}, \sqrt{\varepsilon_{2pq}}, \sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}\} = \{\alpha_1, \alpha_2, ..., \alpha_7\}.
$$

Thus, by Lemma 3.1 a FSU of $\mathbb{K}$ is $\{\alpha_1, \alpha_2, ..., \alpha_7\}$ or $\{\alpha_{i_1}, ..., \alpha_{i_5}, \alpha_{i_0}, \sqrt{\zeta_8 \alpha}\}$ with $i_k \in \{1, ..., 7\}$ and $\alpha = \alpha_1^{r_1} \alpha_2^{r_2} \cdots \alpha_7^{r_7}$, where $r_k \in \{0, 1\}$, and $\alpha_{i_0} \in \{\varepsilon_2, \varepsilon_p\}$, for some i_0. Thus, $q(\mathbb{K}) \leq 2^7$, which is absurd.

- Assume now that the unit group of $\mathbb{K}$ takes the form in the first item of Lemma 4.3, then $q(\mathbb{K}) \leq 2^7$, which is also absurd.

Thus, the only possible case is the one which is given by the second item of Lemma 4.3. This completes the proof. □

5. 2-class field towers of some multiquadratic number fields

Keep the notations of the previous sections. Now we can investigate the structure of the second 2-class groups of L_{pq} and F_{pq} (i.e. $G_{L_{pq}}$ and $G_{F_{pq}}$) defined in Section 3.

Lemma 5.1 *Let p and q be two primes satisfying conditions (3.1). Then*

- $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_q}) = 1$ *and* $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_{pq}}) = -\varepsilon_{pq}$.

- $N_{L_{pq}^*/L_{pq}}(\varepsilon_2) = \varepsilon_2^2$, $N_{L_{pq}^*/L_{pq}}(\zeta_8) = i$ *and* $N_{L_{pq}^*/L_{pq}}(\varepsilon_p) = -1$.

- $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}) = \pm\varepsilon_2$, $N_{L_{pq}^*/L_{pq}}(\sqrt[4]{\zeta_8^2 \varepsilon_q \varepsilon_{2q}}) = \pm\zeta_8$ *and* $N_{L_{pq}^*/L_{pq}}(\sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}}) = \pm\varepsilon_2 \sqrt{\varepsilon_{pq} \varepsilon_{2pq}}$.

Proof We shall use Lemma 4.1 and keep its notations. Note that $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq} \varepsilon_{2pq}}\}$ is a FSU of L_{pq}.

- $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_q}) = \frac{1}{\sqrt{2}}(d_1' + d_2'\sqrt{q}) \cdot \frac{1}{\sqrt{2}}(d_1' - d_2'\sqrt{q}) = 1$.
  $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_{pq}}) = \frac{1}{\sqrt{2}}(b_1\sqrt{2p} + b_2\sqrt{q}) \cdot \frac{1}{\sqrt{2}}(-b_1\sqrt{2p} - b_2\sqrt{q}) = -\sqrt{\varepsilon_{pq}} \cdot \sqrt{\varepsilon_{pq}} = -\varepsilon_{pq}$.

- The norms in the second point are direct.

- We have $N_{L_{pq}^*/L_{pq}}(\varepsilon_2 \varepsilon_p \varepsilon_{2p}) = \varepsilon_2^2 \cdot (-1) \cdot (-1)$. Thus, $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_2 \varepsilon_p \varepsilon_{2p}}) = \pm\varepsilon_2$.
  Since $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_{2q}}) = \frac{1}{\sqrt{2}}(d_1 + d_2\sqrt{2q}) \cdot \frac{1}{\sqrt{2}}(d_1 - d_2\sqrt{2q}) = 1$, then $N_{L_{pq}^*/L_{pq}}(\zeta_8\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}) = \zeta_8^2 \cdot 1 \cdot 1$. Thus, $N_{L_{pq}^*/L_{pq}}(\sqrt[4]{\zeta_8^2 \varepsilon_q \varepsilon_{2q}}) = \pm\zeta_8$.
  $N_{L_{pq}^*/L_{pq}}(\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}\varepsilon_{2pq}}) = \varepsilon_2^2 \cdot 1 \cdot \varepsilon_{pq} \cdot \varepsilon_{2pq}$. Thus, $N_{L_{pq}^*/L_{pq}}(\sqrt[4]{\varepsilon_2^2 \varepsilon_q \varepsilon_{pq} \varepsilon_{2pq}}) = \pm\varepsilon_2 \sqrt{\varepsilon_{pq} \varepsilon_{2pq}}$.

□

Theorem 5.2 *Let p and q be two primes satisfying conditions* (3.1) *or* (3.2) *and let p' be a prime satisfying conditions* (3.3)*. Let $m \geq 2$ be an integer such that $h_2(-2q) = 2^m$. Then the group $G_{L_{pq}} \simeq Q_{m+1}$ and the group $G_{L_{p'}}$ is of type $(2,2)$.*

Proof

1. By Proposition 2.3 and Theorem 3.8 we have $|G_{L_{pq}}| = 2 \cdot h_2(L_{pq}^*) = 2^{m+1}$. Assume that p and q verify conditions (3.1). According to [2, Corollaire 17], the group $G_{k_{pq}}$ is quaternion or semidihedral. By Galois theory we have

$$Gal(L_{pq}^{(2)}/k_{pq}) \simeq Gal(L_{pq}/k_{pq}) \times G_{L_{pq}}.$$

   Thus, $G_{L_{pq}}$ is a subgroup of $G_{k_{pq}}$ of index 2. Therefore, $G_{L_{pq}}$ is dihedral or quaternion. Since a FSU of L_{pq} is $\{\varepsilon_2, \varepsilon_{pq}, \sqrt{\varepsilon_{pq}\varepsilon_{2pq}}\}$ (cf.[3]), then by [12], Theorem 4.4 and Lemma 5.1, the number of classes of $\text{Cl}_2(L_{pq})$ which capitulate in L_{pq}^* is $[L_{pq}^* : L_{pq}][E_{L_{pq}} : N_{L_{pq}^*/L_{pq}}(E_{L_{pq}^*})] = 2 \cdot 1 = 2$. Therefore, from Table 1, we deduce that $G_{L_{pq}}$ cannot be dihedral. Hence, $G_{L_{pq}}$ is quaternion.

   Suppose now that p and q verify the condition (3.2). As previously we show that $G_{L_{pq}}$ is a subgroup of $G_{k_{pq}}$ of index 2 and by [2] $G_{k_{pq}}$ is quaternion. Therefore, $G_{L_{pq}}$ is quaternion.

2. Let $k' = \mathbb{Q}(\sqrt{-1}, \sqrt{2p'})$. As $L_{p'} = \mathbb{Q}(\sqrt{-1}, \sqrt{p'}, \sqrt{2})$ is the genus field of k', so by [6, Théorème 5.2], the 2-class group of k' is of type $(2,4)$. Hence, [5, Corollaire 1] implies that the Hilbet 2-class field tower of k terminates at the first step. Therefore, the Hilbert 2-class field tower of its genus field $L_{p'}$ terminates at the first step. Thus, we have the result.

□

Corollary 5.3 *Keep the assumptions of the previous Theorem* 5.2*. Then*

1. *There are exactly 2 classes of $\text{Cl}_2(L_{pq})$ which capitulate in each of the three unramified quadratic extensions of L_{pq}.*

2. *There are 4 classes of $\text{Cl}_2(L_{p'})$ which capitulate in each of the three unramified quadratic extensions of $L_{p'}$.*

Lemma 5.4 *Let p and q be two primes satisfying conditions* (3.1)*. Then*

- $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_q}) = -1$ *and* $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_{pq}}) = \varepsilon_{pq}$.

- $N_{L_{pq}^*/F_{pq}}(\varepsilon_2) = -1$, $N_{L_{pq}^*/F_{pq}}(\varepsilon_p) = -1$ *and* $N_{L_{pq}^*/F_{pq}}(\zeta_8) = -1$.

- $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}) = \pm\varepsilon_{2p}$, $N_{L_{pq}^*/F_{pq}}(\sqrt[4]{\zeta_8^2\varepsilon_q\varepsilon_{2q}}) = \pm\sqrt{-\varepsilon_{2q}}$ *and* $N_{L_{pq}^*/F_{pq}}(\sqrt[4]{\varepsilon_2^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}) = \pm\sqrt{\varepsilon_{pq}}$.

Proof

- Let us use and keep the notations notations of Lemma 4.1. We have:
  $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_q}) = \frac{1}{\sqrt{2}}(d_1' + d_2'\sqrt{q}) \cdot \frac{1}{-\sqrt{2}}(d_1' - d_2'\sqrt{q}) = \frac{-1}{2}(d_1'^2 - d_2'^2 q) = -1$.
  $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_{pq}}) = \frac{1}{\sqrt{2}}(b_1\sqrt{2p} + b_2\sqrt{q}) \cdot \frac{1}{-\sqrt{2}}(-b_1\sqrt{2p} - b_2\sqrt{q}) = \sqrt{\varepsilon_{pq}} \cdot \sqrt{\varepsilon_{pq}} = \varepsilon_{pq}$.

- The first norm and the second norm in the second point are direct. We have $N_{L_{pq}^*/F_{pq}}(\zeta_8) = N_{L_{pq}^*/F_{pq}}(\frac{1}{\sqrt{2}}(1+i)) = \frac{-1}{2}(1-i^2) = -1$.

- Note that $N_{L_{pq}^*/L_{pq}}(\varepsilon_{2p}) = \varepsilon_{2p}^2$. Then $N_{L_{pq}^*/L_{pq}}(\varepsilon_2\varepsilon_p\varepsilon_{2p}) = (-1)\cdot(-1)\cdot\varepsilon_{2p}^2$. Thus, $N_{L_{pq}^*/L_{pq}}(\sqrt{\varepsilon_2\varepsilon_p\varepsilon_{2p}}) = \pm\varepsilon_{2p}$.

  We have $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_{2q}}) = \frac{1}{\sqrt{2}}(d_1 + d_2\sqrt{2q}) \cdot \frac{1}{-\sqrt{2}}(d_1 + d_2\sqrt{2q}) = -\varepsilon_{2q}$, then $N_{L_{pq}^*/F_{pq}}(\zeta_8\sqrt{\varepsilon_q}\sqrt{\varepsilon_{2q}}) = (-1)\cdot(-1)\cdot(-\varepsilon_{2q}) = -\varepsilon_{2q}$. Therefore, $N_{L_{pq}^*/F_{pq}}(\sqrt[4]{\zeta_8^2\varepsilon_q\varepsilon_{2q}}) = \pm\sqrt{-\varepsilon_{2q}}$.

  We have $N_{L_{pq}^*/F_{pq}}(\sqrt{\varepsilon_{2pq}}) = \frac{1}{\sqrt{2}}(y_1\sqrt{2p} + y_2\sqrt{q}) \cdot \frac{1}{-\sqrt{2}}(y_1\sqrt{2p} - y_2\sqrt{q}) = \frac{-1}{2}(y_1^2 2p - y_2^2 q) = 1$. So

  $$N_{L_{pq}^*/F_{pq}}(\varepsilon_2\sqrt{\varepsilon_q}\sqrt{\varepsilon_{pq}}\sqrt{\varepsilon_{2pq}}) = (-1)\cdot(-1)\cdot\varepsilon_{pq}\cdot 1 = \varepsilon_{pq}.$$ Therefore, $N_{L_{pq}^*/F_{pq}}(\sqrt[4]{\varepsilon_2^2\varepsilon_q\varepsilon_{pq}\varepsilon_{2pq}}) = \pm\sqrt{\varepsilon_{pq}}$.

  $\square$

Theorem 5.5 *Let $m \geq 2$ such that $h_2(-2q) = 2^m$.*

1. *Let p and q be two primes satisfying conditions (3.1). Then $G_{F_{pq}} \simeq D_{m+1}$.*

2. *Let p and q be two primes satisfying conditions (3.2). Then $G_{F_{pq}} \simeq Q_{m+1}$.*

Proof

1. Since, by the third point of Lemma 5.4, $\sqrt{\varepsilon_{pq}} \in F_{pq}$, then according to [2, Proposition 5], a FSU of F_{pq} is given by $\{\sqrt{\varepsilon_{pq}}, \varepsilon_{2p}, \sqrt{-\varepsilon_{2q}}\}$. As in the proof of Theorem 5.2 and using the same references we deduce that $G_{F_{pq}} \simeq Q_{m+1}$ or D_{m+1}. By Lemma 5.4, [12] and Theorem 4.4 , the number of classes of $\mathrm{Cl}_2(F_{pq})$ which capitulate in L_{pq}^* is $[L_{pq}^* : F_{pq}][E_{F_{pq}} : N_{L_{pq}^*/F_{pq}}(E_{L_{pq}^*})] = 2\cdot 2 = 4$. Thus, we prove the first item.

2. The proof of the second item is similar to that of Theorem 5.2.

$\square$

Corollary 5.6

1. *Let p and q be two primes satisfying conditions (3.1). Then there are 4 classes of $\mathrm{Cl}_2(F_{pq})$ which capitulate in L_{pq}^* and there are exactly 2 classes of $\mathrm{Cl}_2(F_{pq})$ which capitulate in each of the other quadratic unramified extensions of F_{pq}.*

2. *Let p and q be two primes satisfying conditions (3.2). Then, there are exactly 2 classes of $\mathrm{Cl}_2(F_{pq})$ which capitulate in each of the three unramified quadratic extensions of F_{pq}.*

Remark 5.7 • *Assume that p and q verify conditions (3.1) or (3.2). The authors of [2] did not determine the order of $G_{k_{pq}}$, but now by the above results it is easy to see that $|G_{k_{pq}}| = 2^{m+2}$, with $m \geq 2$ such that $h_2(-2q) = 2^m$, and so we have the following diagram (see Figure 2):*
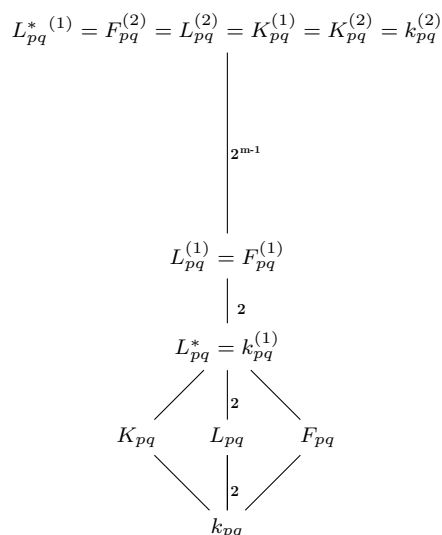
$$L_{pq}^{*}{}^{(1)} = F_{pq}^{(2)} = L_{pq}^{(2)} = K_{pq}^{(1)} = K_{pq}^{(2)} = k_{pq}^{(2)}$$

$2^{m\text{-}1}$

$$L_{pq}^{(1)} = F_{pq}^{(1)}$$

2

$$L_{pq}^{*} = k_{pq}^{(1)}$$

2

$$K_{pq} \qquad L_{pq} \qquad F_{pq}$$

2

$$k_{pq}$$

Figure 2. The Hilbert 2-class field towers.

- *Note also that the authors of [2] did not determine the exact structure of $G_{k_{pq}}$ whenever p and q satisfy conditions (3.1). Now by our above results it is easy to see that, under conditions (3.1), $G_{k_{pq}}$ is semidihedral of order 2^{m+2}.*

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