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A new solution to the discontinuity problem on metric spaces

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Abstract: We study on the Rhoades' question concerning the discontinuity problem at fixed point for a self-mapping T of a metric space. We obtain a new solution to this question. Our result generalizes some recent theorems existing in the literature and implies the uniqueness of the fixed point. However, there are also cases where the fixed point set of a self-mapping contains more than 1 element. Therefore, by a geometric point of view, we extend the Rhoades' question to the case where the fixed point set is a circle. We also give a solution to this extended version.

Key words: Metric space, fixed point, fixed circle, discontinuity

1. Introduction

In [29], Rhoades asked the question whether there exists a contractive condition which is strong enough to generate a fixed point, but which does not force the map to be continuous at the fixed point. Some solutions of this open problem have been presented. For more details, we refer the interested readers to [1–7, 13, 18, 21, 24–27, 32]. For example, in [24] and [1], the following results were obtained as solutions to this open problem on metric spaces.

Theorem 1.1 *[24] If a self-mapping T of a complete metric space (X, d) satisfies the conditions*

1. $d(Tx, Ty) \leq \psi(\max\{d(x, Tx), d(y, Ty)\})$, where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a self-mapping such that $\psi(t) < t$ for each $t > 0$,
2. For a given $\varepsilon > 0$, there exists a $\delta(\varepsilon) > 0$ such that

$$\varepsilon < \max\{d(x, Tx), d(y, Ty)\} < \varepsilon + \delta$$

implies $d(Tx, Ty) \leq \varepsilon$,

then T has a unique fixed point z. Moreover, T is continuous at z if and only if

$$\lim_{x \to z} \max\{d(x, Tx), d(z, Tz)\} = 0.$$

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Theorem 1.2 *[1] If a self-mapping T of a complete metric space (X,d) satisfies the conditions*

1. $d(Tx,Ty) \leq \psi(N(x,y))$, where $\psi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ is a self-mapping such that $\psi(t) < t$ for each $t > 0$ with
$$N(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), a[d(x,Ty)+d(y,Tx)]/2\}, 0 \leq a < 1,$$

2. There exists a $\delta(\varepsilon) > 0$ such that $\varepsilon < M(x,y) < \varepsilon + \delta$ implies $d(Tx,Ty) \leq \varepsilon$ for a given $\varepsilon > 0$ with
$$M(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\},$$

then T has a unique fixed point z. Moreover, T is continuous at z if and only if
$$\lim_{x \to z} M(x,z) = 0.$$

Let (X,d) be a metric space, $T : X \to X$ be a self-mapping and $Fix(T) = \{x \in X : Tx = x\}$ be the fixed point set of the self-mapping T. In this paper, we investigate new contractive conditions. In Section 2, we give a solution to the Rhoades' question by means of 2 auxiliary numbers. In Section 3, we investigate geometric properties of the fixed point sets of some discontinuous activation functions. By a geometric point of view, we extend the Rhoades' question. We also give a solution to this extended version and provide necessary illustrative examples to support our theoretical results.

2. New discontinuity results

From now on, we assume that $0 \leq \theta < 1$, $\alpha, \beta, \mu \in \mathbb{R}^{+} \cup \{0\}$ and $\gamma = \alpha + \beta + \mu > 0$. We define the following numbers:

$$\begin{aligned} N_d(x,y) &= \alpha \max\{d(x,Tx), d(y,Ty)\} \\ &\quad + \beta \max\{d(x,y), d(x,Tx), d(y,Ty), \theta[d(x,Ty)+d(y,Tx)]/2\} \\ &\quad + \mu \max\left\{d(x,Tx), d(y,Ty), \frac{d(x,y)d(y,Ty)}{1+d(x,Tx)}, \frac{d(x,y)d(y,Ty)}{1+d(Tx,Ty)}\right\} \end{aligned}$$

and

$$\begin{aligned} M_d(x,y) &= \alpha \max\{d(x,Tx), d(y,Ty)\} \\ &\quad + \beta \max\{d(x,y), d(x,Tx), d(y,Ty), [d(x,Ty)+d(y,Tx)]/2\} \\ &\quad + \mu \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(y,Ty)[d(x,Ty)+d(y,Tx)]}{1+d(x,Tx)+d(y,Ty)}\right\}. \end{aligned}$$

By means of these numbers, we give a new solution to the Rhoades' question in the following theorem.

Theorem 2.1 *Let (X,d) be a complete metric space and $\gamma = \alpha + \beta + \mu > 0$ for some $\alpha, \beta, \mu \in \mathbb{R}^{+} \cup \{0\}$. If T is a self-mapping on X satisfying the following conditions,*

1. There exists a function $\psi : \mathbb{R}^{+} \to \mathbb{R}^{+}$ such that $\psi(t) < t$ for each $t > 0$ and
$\gamma d(Tx,Ty) \leq \psi(N_d(x,y))$,

2. For a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < \frac{1}{\gamma} M_d(x,y) < \varepsilon + \delta$ implies $d(Tx, Ty) \leq \varepsilon$,

then T has a unique fixed point $z \in X$ and $T^n x \to z$ for each $x \in X$. Additionally, T is continuous at z if and only if $\lim_{x \to z} M_d(x,z) = 0$.

Proof Let $x_0 \in X$ be any point such that $x_0 \notin Fix(T)$ and the sequence $\{x_n\}$ in X be defined by the rule $x_{n+1} = Tx_n$ for $n = 0, 1, 2, 3, \ldots$. Using the condition (1) and definition of the number $N_d(x,y)$ we get

$$\begin{aligned}
\gamma d(x_n, x_{n+1}) &= \gamma d(Tx_{n-1}, Tx_n) \leq \psi(N_d(x_{n-1}, x_n)) < N_d(x_{n-1}, x_n) \\
&= \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\quad + \beta \max\left\{ \begin{array}{c} d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ \theta[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]/2 \end{array} \right\} \\
&\quad + \mu \max\left\{ \begin{array}{c} d(x_{n-1}, x_n), d(x_n, x_{n+1}), \\ \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_{n-1}, x_n)}, \frac{d(x_{n-1}, x_n) d(x_n, x_{n+1})}{1 + d(x_n, x_{n+1})} \end{array} \right\} \\
&= \alpha \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} + \beta \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\quad + \mu \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.
\end{aligned} \tag{2.1}$$

Assume that $d(x_{n-1}, x_n) \leq d(x_n, x_{n+1})$. Using the inequality (2.1) we get

$$d(x_n, x_{n+1}) < d(x_n, x_{n+1}),$$

which is a contradiction. So we find $d(x_n, x_{n+1}) < d(x_{n-1}, x_n)$ and

$$N_d(x_{n-1}, x_n) = (\alpha + \beta + \mu) d(x_{n-1}, x_n) = \gamma d(x_{n-1}, x_n).$$

If we put $d(x_n, x_{n+1}) = s_n$ then by the inequality (2.1) we get

$$s_n < s_{n-1}, \tag{2.2}$$

and so, $\{s_n\}$ is a strictly decreasing sequence of positive real numbers. The sequence $\{s_n\}$ tends to a limit $s \geq 0$. Suppose $s > 0$. Then there exists a positive integer k such that $n \geq k$ implies

$$s < s_n < s + \delta(s). \tag{2.3}$$

Combining the condition (2) and the inequality (2.2), we have

$$d(Tx_{n-1}, Tx_n) = d(x_n, x_{n+1}) = s_n < s, \tag{2.4}$$

for $n \geq k$. But, the inequality (2.4) contradicts to the inequality (2.3) and hence we obtain $s = 0$.

Let us fix an $\varepsilon > 0$ to show that $\{x_n\}$ is a Cauchy sequence. We may assume that $\delta = \delta(\varepsilon) < \varepsilon$ without loss of generality. Since $s_n \to 0$, there exists a positive integer k satisfying the following inequality for $n \geq k$:

$$d(x_n, x_{n+1}) = s_n < \frac{\delta}{2} \ (0 < \delta < 1).$$

Following Jachymski's technique (see [11, 12]), we use the mathematical induction to show that

$$d(x_k, x_{k+n}) < \varepsilon + \frac{\delta}{2}, \tag{2.5}$$

for any $n \in \mathbb{N}$. Since we have

$$d(x_k, x_{k+1}) = s_k < \frac{\delta}{2} < \varepsilon + \frac{\delta}{2},$$

inequality (2.5) holds for $n = 1$. Suppose that the inequality (2.5) is true for some n. Using the triangle inequality, we get

$$d(x_k, x_{k+n+1}) \leq d(x_k, x_{k+1}) + d(x_{k+1}, x_{k+n+1}).$$

If we show

$$d(x_{k+1}, x_{k+n+1}) \leq \varepsilon,$$

then we deduce that the inequality (2.5) holds for $n+1$. Now we show that

$$M_d(x_k, x_{k+n}) \leq \varepsilon + \delta,$$

where

$$\begin{aligned} M_d(x_k, x_{k+n}) &= \alpha \max\{d(x_k, Tx_k), d(x_{k+n}, Tx_{k+n})\} \\ &\quad + \beta \max\left\{ \begin{array}{c} d(x_k, x_{k+n}), d(x_k, Tx_k), d(x_{k+n}, Tx_{k+n}), \\ [d(x_k, Tx_{k+n}) + d(x_{k+n}, Tx_k)]/2 \end{array} \right\} \\ &\quad + \mu \max\left\{ \begin{array}{c} d(x_k, x_{k+n}), d(x_k, Tx_k), d(x_{k+n}, Tx_{k+n}), \\ \frac{d(x_{k+n}, Tx_{k+n})[d(x_k, Tx_{k+n}) + d(x_{k+n}, Tx_k)]}{1 + d(x_k, Tx_k) + d(x_{k+n}, Tx_{k+n})} \end{array} \right\}. \end{aligned} \tag{2.6}$$

Then by the mathematical induction hypothesis, we obtain

$$\begin{aligned} d(x_k, x_{k+n}) &< \varepsilon + \frac{\delta}{2}, \\ d(x_k, x_{k+1}) &< \frac{\delta}{2}, \\ d(x_{k+n}, x_{k+n+1}) &< \frac{\delta}{2}, \\ [d(x_k, x_{k+n+1}) + d(x_{k+n}, x_{k+1})]/2 &< \varepsilon + \delta, \\ \frac{d(x_{k+n}, x_{k+n+1})[d(x_k, x_{k+n+1}) + d(x_{k+n}, x_{k+1})]}{1 + d(x_k, x_{k+1}) + d(x_{k+n}, x_{k+n+1})} &< \varepsilon + \delta. \end{aligned} \tag{2.7}$$

Using (2.6) and (2.7), we get $M_d(x_k, x_{k+n}) < \varepsilon + \delta$ and considering the condition (2), we find

$$d(Tx_k, Tx_{k+n}) = d(x_{k+1}, x_{k+n+1}) \leq \varepsilon.$$

Consequently, the inequality (2.5) indicate that $\{x_n\}$ is a Cauchy sequence. Then there exists a point $z \in X$ such that $x_n \to z$ as $n \to \infty$ by the completeness hypothesis on the metric space X. Furthermore, we have $Tx_n \to z$.

Now assume $Tz \neq z$, that is, z is not a fixed point of T. Using the condition (1) we have

$$\begin{aligned}\gamma d(Tz,Tx_n) &\leq \psi(N_d(z,x_n)) < N_d(z,x_n) \\ &= \alpha\max\{d(z,Tz),d(x_n,Tx_n)\} \\ &\quad +\beta\max\left\{\begin{array}{c} d(z,x_n),d(z,Tz),d(x_n,Tx_n), \\ \theta[d(z,Tx_n)+d(x_n,Tz)]/2 \end{array}\right\} \\ &\quad +\mu\max\left\{\begin{array}{c} d(z,Tz),d(x_n,Tx_n),\frac{d(z,x_n)d(x_n,Tx_n)}{1+d(z,Tz)} \\ \frac{d(z,x_n)d(x_n,Tx_n)}{1+d(Tz,Tx_n)} \end{array}\right\}\end{aligned}$$

and taking limit for $n \to \infty$ we obtain

$$\gamma d(Tz,z) < (\alpha+\beta+\mu)\, d(z,Tz) = \gamma d(Tz,z),$$

a contradiction. Hence, we get $Tz = z$, that is, z is a fixed point of T.

Now, assume that w is another fixed point of T such that $z \neq w$. Using the condition (1), we find

$$\begin{aligned}\gamma d(Tz,Tw) &= \gamma d(z,w) \leq \psi(N_d(z,w)) \\ &< N_d(z,w) = \alpha\max\{d(z,Tz),d(w,Tw)\} \\ &\quad +\beta\max\left\{\begin{array}{c} d(z,w),d(z,Tz),d(w,Tw), \\ \theta[d(z,Tw)+d(w,Tz)]/2 \end{array}\right\} \\ &\quad +\mu\max\left\{\begin{array}{c} d(z,Tz),d(w,Tw), \\ \frac{d(z,w)d(w,Tw)}{1+d(z,Tz)},\frac{d(z,w)d(w,Tw)}{1+d(Tz,Tw)} \end{array}\right\} \\ &< (\alpha+\beta+\mu)\, d(z,w) = \gamma d(z,w),\end{aligned}$$

which is a contradiction. This shows that z is the unique fixed point of the self-mapping T.

For the last part of the proof, assume that T is continuous at the fixed point z. If $x_n \to z$ then $Tx_n \to Tz = z$ and

$$d(x_n,Tx_n) \leq d(x_n,z) + d(Tx_n,z) \to 0.$$

Hence we get $\lim\limits_{x_n\to z} M_d(x_n,z) = 0$. Conversely, if $\lim\limits_{x_n\to z} M_d(x_n,z) = 0$ then $d(x_n,Tx_n) \to 0$ as $x_n \to z$. This implies $Tx_n \to z = Tz$, that is, T is continuous at z. □

Corollary 2.2 *Let $\gamma = \alpha+\beta+\mu > 0$ for some $\alpha,\beta,\mu \in \mathbb{R}^+ \cup \{0\}$ and T be a self-mapping on a complete metric space (X,d). If T satisfies the following conditions,*

1. *$\gamma d(Tx,Ty) < N_d(x,y)$, for any $x,y \in X$ and $N_d(x,y) > 0$,*
2. *For a given $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $\varepsilon < \frac{1}{\gamma}M_d(x,y) < \varepsilon+\delta$ implies $d(Tx,Ty) \leq \varepsilon$,*

then T has a unique fixed point $z \in X$ and $T^n x \to z$ for each $x \in X$. Also, T is continuous at z if and only if $\lim\limits_{x\to z} M_d(x,z) = 0$.

Corollary 2.3 *Let (X,d) be a complete metric space and T a self-mapping on X satisfying the following conditions:*

1. *There exists a function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\psi(d(x,y)) < d(x,y)$ *for each* $d(x,y) > 0$ *and* $d(Tx,Ty) \leq \psi(d(x,y))$,

2. *For a given* $\varepsilon > 0$, *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < t < \varepsilon + \delta$ *implies* $\psi(t) \leq \varepsilon$ *for any* $t > 0$.

Then T *has a unique fixed point* $z \in X$ *and* $T^n x \to z$ *for each* $x \in X$.

Now we give an example for $\alpha = \beta = 0$ and $\mu = 1$.

Example 2.4 *Let* $X = [0, 2\lambda]$ $(\lambda \in \mathbb{R}^+)$, d *be the usual metric on* X *and* T *be the self-mapping defined by*

$$Tx = \begin{cases} \lambda & ; \quad x \leq \lambda \\ 0 & ; \quad x > \lambda \end{cases}.$$

It is easy to verify that T *satisfies the conditions of Theorem 2.1 with the functions*

$$\psi(t) = \begin{cases} \lambda & ; \quad t > \lambda \\ \frac{t}{3} & ; \quad t \leq \lambda \end{cases}$$

and

$$\delta(\varepsilon) = \begin{cases} 2\lambda & ; \quad \varepsilon \geq \lambda \\ 2\lambda - \varepsilon & ; \quad \varepsilon < \lambda \end{cases}.$$

Then T *has the unique fixed point* $x = \lambda$. T *is discontinuous at the fixed point* $x = \lambda$ *since the limit* $\lim\limits_{x \to \lambda} M_d(x, \lambda)$ *does not exist.*

Remark 2.5 1) *In Theorem 2.1, if we take* $\alpha = 1$ *and* $\beta = \mu = 0$, *we obtain Theorem 1.1 and if we take* $\beta = 1$ *and* $\alpha = \mu = 0$, *we obtain Theorem 1.2.*

2) *We note that* T^m *has also a fixed point under the hypothesis of Theorem 2.1. This fixed point can be unique. For example, if we consider the self-mapping* T *defined in Example 2.4, we find* $T^m x = \lambda$ $(m \geq 2)$ *for all* $x \in X$ *and hence* T^m *has a unique fixed point* $x = \lambda$.

Theorem 2.6 *Let* $\gamma = \alpha + \beta + \mu > 0$ *for some* $\alpha, \beta, \mu \in \mathbb{R}^+ \cup \{0\}$ *and* T *be a self-mapping on a complete metric space* (X, d). *If* T *satisfies the following conditions*

1. *There exits a function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that* $\psi(t) < t$ *for each* $t > 0$, $\gamma = \alpha + \beta + \mu$ *and* $\gamma d(T^m x, T^m y) \leq \psi(N_d^*(x,y))$, *where*

$$\begin{aligned} N_d^*(x,y) &= \alpha \max\{d(x, T^m x), d(y, T^m y)\} \\ &\quad + \beta \max\{d(x,y), d(x, T^m x), d(y, T^m y), \theta[d(x, T^m y) + d(y, T^m x)]/2\} \\ &\quad + \mu \max\left\{d(x, T^m x), d(y, T^m y), \frac{d(x,y)\, d(y, T^m y)}{1 + d(x, T^m x)}, \frac{d(x,y)\, d(y, T^m y)}{1 + d(T^m x, T^m y)}\right\}, \end{aligned}$$

2. *For a given* $\varepsilon > 0$, *there exists a* $\delta = \delta(\varepsilon) > 0$ *such that* $\varepsilon < \frac{1}{\gamma} M_d^*(x, y) < \varepsilon + \delta$ *implies* $d(T^m x, T^m y) \leq \varepsilon$, *where*

$$\begin{aligned} M_d^*(x,y) &= \alpha \max\{d(x,T^m x), d(y,T^m y)\} \\ &+ \beta \max\{d(x,y), d(x,T^m x), d(y,T^m y), [d(x,T^m y) + d(y,T^m x)]/2\} \\ &+ \mu \max\left\{d(x,y), d(x,T^m x), d(y,T^m y), \frac{d(y,T^m y)[d(x,T^m y) + d(y,T^m x)]}{1 + d(x,T^m x) + d(y,T^m y)}\right\}, \end{aligned}$$

then T *has a unique fixed point* $z \in X$. *Also,* T *is continuous at* z *if and only if* $\lim_{x \to z} M_d^*(x, z) = 0$.

Proof From Theorem 2.1, it is clear that the function T^m has a unique fixed point z, that is, $T^m z = z$. Consequently, we have

$$Tz = TT^m z = T^m Tz,$$

and so Tz is a fixed point of T^m. From the uniqueness of the fixed point, then we get $Tz = z$. Hence, T has a unique fixed point. □

3. Fixed points of discontinuous activation functions and an extended version of the Rhoades' open problem

In the previous section, our obtained results imply the uniqueness of the fixed point and hence the set $Fix(T)$ is a singleton. In this section, we deal with the geometric properties of the set $Fix(T)$ in case that it is not a singleton. We note that the number $M_d(x, y)$ can be also used to determine discontinuity (or continuity) of a self-mapping T on its fixed points without any hypothesis on the metric space and the self-mapping. We give the following proposition.

Proposition 3.1 *Let* (X, d) *be a metric space and* T *a self-mapping on* X. *Then* T *is continuous at* $z \in Fix(T)$ *if and only if* $\lim_{x\to z} M_d(x, z) = 0$.

Corollary 3.2 *Let* (X, d) *be a metric space and* T *a self-mapping on* X. *Then* T *is discontinuous at* $z \in Fix(T)$ *if and only if* $\lim_{x\to z} M_d(x, z) \neq 0$ *when the limit* $\lim_{x\to z} M_d(x, z)$ *exists. If the limit* $\lim_{x\to z} M_d(x, z)$ *does not exist then* T *is discontinuous at* z.

It is known that the structures of activation functions are important in the dynamical analysis of recurrent neural networks [10]. In recent years, discontinuous functions and fixed points of self-mappings are gained importance in the study of several types of neural networks (see for example [8–10, 16, 17, 33] and the references therein).

In [16], a general class of discontinuous activation functions were considered to discuss the stability problem of multiple equilibria for delayed neural networks with discontinuous activation functions. Any member of this class has the form

$$f(x) = \begin{cases} u & ; \quad -\infty < x < p \\ l_1 x + c_1 & ; \quad p \leq x \leq r \\ l_2 x + c_2 & ; \quad r < x \leq q \\ v & ; \quad q < x < +\infty \end{cases},$$

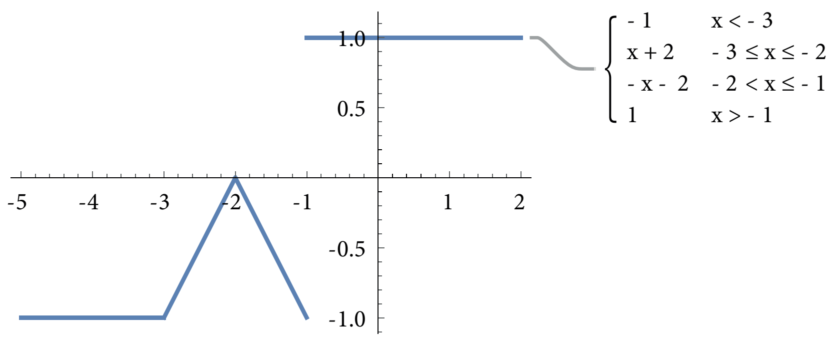

Figure 1. The discontinuous activation function g.

where p, r, q, u, v, l_1, l_2, c_1, c_2 are constants with $-\infty < p < r < q < +\infty$, $l_1 > 0$, $l_2 < 0$, $u = f(p) = f(q)$, $f(r) = l_2 r + c_2$ and $v > f(r)$.

It was shown that the storage capacity of the neural networks can be considerably expanded by use of discontinuous activation functions (see [16] for more details and the usage of the Brouwer's fixed point theorem). By choosing $u = -1$, $l_1 = 1$, $l_2 = -1$, $c_1 = 2$, $c_2 = -2$, $p = -3$, $r = -2$, $q = -1$, $v = 1$, we obtain the following discontinuous activation function $g(x)$ belonging to this class:

$$g(x) = \begin{cases} -1 & ; \quad -\infty < x < -3 \\ x+2 & ; \quad -3 \leq x \leq -2 \\ -x-2 & ; \quad -2 < x \leq -1 \\ 1 & ; \quad -1 < x < +\infty \end{cases}.$$

Notice that the fixed point set of g is not a singleton, especially we have $Fix(g) = \{-1, 1\}$. We determine the continuity of g at its fixed points by use of the number $M_d(x, y)$. Since $\lim_{x \to -1} M_d(x, -1)$ does not exit, g is discontinuous at $x = -1$. We have $\lim_{x \to 1} M_d(x, 1) = 0$ and hence g is continuous at $x = 1$. Let us consider the circle $C_{0,1}$. Clearly, we have $C_{0,1} = Fix(g)$ and the function g fixes the circle $C_{0,1}$ by a different point of view (see figure which is drawn by Mathematica [34]). This is the basis of a recent problem called *fixed-circle problem*. A circle $C_{x_0,r}$ is called the fixed circle of a self-mapping T if $Tx = x$ for all $x \in C_{x_0,r}$. For more details about this problem on metric (resp. generalized metric) spaces, one can see [14, 15, 18–23, 26, 27, 30, 31].

We have observed that the fixed point set of the discontinuous activation function g is a circle and g is not continuous on its fixed circle. At this point, we extend the Rhoades' question to the case where the set $Fix(T)$ contains a circle for any self-mapping T on a metric space X as follows:

Is there a contractive condition which is strong enough to generate a fixed circle but which does not force the map to be continuous on its fixed circle?

Now, we give a solution to this extended version of the Rhoades' problem using the number $M_d(x, y)$. To do this, we fix the second variable y as $y = x_0$ in the definition of the number $M_d(x, y)$. Then we have

$$\begin{aligned} M_d(x, x_0) \quad = \quad & \alpha \max\{d(x, Tx), d(x_0, Tx_0)\} \\ & +\beta \max\{d(x, x_0), d(x, Tx), d(x_0, Tx_0), [d(x, Tx_0) + d(x_0, Tx)]/2\} \\ & +\mu \max\left\{d(x, x_0), d(x, Tx), d(x_0, Tx_0), \frac{d(x_0, Tx_0)[d(x, Tx_0) + d(x_0, Tx)]}{1 + d(x, Tx) + d(x_0, Tx_0)}\right\}. \end{aligned}$$

Definition 3.3 *Let T be a self-mapping on a metric space (X,d). If there exists a point $x_0 \in X$ and a function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\psi(t) < t$ for each $t > 0$ satisfying*

$$d(x,Tx) > 0 \Rightarrow \gamma d(x,Tx) \leq \psi(M_d\,(x,x_0)),$$

for all $x \in X$, then T is called an M_{x_0}-type contraction.

Using the M_{x_0}-type contractive property of the self-mapping and a geometric condition, we prove the following fixed-circle theorem without any assumption on X.

Theorem 3.4 *Let (X,d) be a metric space, T a self-mapping on X and the number ρ defined as follows:*

$$\rho = \inf\left\{d\left(x,Tx\right) : x \in X, x \notin Fix(T)\right\}.$$

If T is an M_{x_0}-type contraction with $x_0 \in X$ and $d\left(x_0,Tx\right) = \rho$ for all $x \in C_{x_0,\rho}$ then the circle $C_{x_0,\rho}$ is a fixed circle of T, that is, the set $Fix(T)$ contains the circle $C_{x_0,\rho}$. T is continuous at any $z \in C_{x_0,\rho}$ if and only if $\lim_{x\to z} M_d\left(x,z\right) = 0$.

Proof At first, suppose $x_0 \notin Fix(T)$. Then considering the definition of an M_{x_0}-type contraction, we get

$$\gamma d(x_0,Tx_0) \leq \psi(M_d(x_0,x_0)) < M_d(x_0,x_0) = \left(\alpha+\beta+\mu\right)d(x_0,Tx_0),$$

a contradiction since we have $\gamma = \alpha+\beta+\mu$.

Let $x \in C_{x_0,\rho}$ be any point such that $x \neq x_0$. Assume $x \notin Fix(T)$ and hence $d(Tx,x) > 0$. By the definitions of an M_{x_0}-type contraction and the number ρ, we get

$$\begin{aligned}
\gamma d(x,Tx) &\leq \psi(M_d(x,x_0)) < M_d(x,x_0) \\
&= \alpha \max\left\{d\left(x,Tx\right),0\right\} + \beta \max\left\{d\left(x,x_0\right), d\left(x,Tx\right), 0, \left[d\left(x,x_0\right)+d\left(x_0,Tx\right)\right]/2\right\} \\
&\quad +\mu \max\left\{d\left(x,x_0\right), d\left(x,Tx\right), 0, 0\right\} \\
&= \alpha \max\left\{d\left(x,Tx\right),0\right\} + \beta \max\left\{\rho, d\left(x,Tx\right), 0, \rho\right\} + \mu \max\left\{\rho, d\left(x,Tx\right), 0, 0\right\} \\
&< \left(\alpha+\beta+\mu\right)d(x,Tx) = \gamma d(x,Tx),
\end{aligned}$$

which is a contradiction. Consequently, we have $Tx = x$ for all $x \in C_{x_0,\rho}$ and hence $C_{x_0,\rho}$ is a fixed circle of T.

The last part of the proof is clear by Proposition 3.1. □

Corollary 3.5 *Let (X,d) be a metric space, T a self-mapping on X and the number ρ defined as follows*:

$$\rho = \inf\left\{d\left(x,Tx\right) : x \in X, x \notin Fix(T)\right\}.$$

If T is an M_{x_0}-type contraction with $x_0 \in X$ and $d\left(x_0,Tx\right) \leq \rho$ for all $x \in D_{x_0,\rho}$ then the disc $D_{x_0,\rho}$ is a fixed disc of T, that is, the set $Fix(T)$ contains the disc $D_{x_0,\rho}$.

Corollary 3.6 *Let (X,d) be a metric space, T a self-mapping on X and the number ρ defined as follows* :

$$\rho = \inf\left\{d\left(x,Tx\right) : x \in X, x \notin Fix(T)\right\}.$$

If there exists $x_0 \in X$ *such that* $d(x_0, Tx) \le \rho$ *for all* $x \in D_{x_0,\rho}$ *and*

$$d(x,Tx) > 0 \text{ implies } \gamma d(x,Tx) < M_d(x, x_0),$$

for all $x \in X$, *then the set* $Fix(T)$ *contains the circle* $C_{x_0,\rho}$ *and the disc* $D_{x_0,\rho}$.

We note that any circle $C_{x_0,r}$ $(r \le \rho)$ is fixed by the self-mapping T in Theorem 3.4 (resp. Corollary 3.5 and Corollary 3.6). Also, the converse statement of Theorem 3.4 (resp. Corollary 3.5 and Corollary 3.6) is not true everywhen (see Example 3.7).

Now, we give some examples.

Example 3.7 *Let* $(\mathbb{C}, d)$ *be the usual metric space with the metric* $d(u,v) = |u - v|$. *Let us consider the self-mapping* $T : \mathbb{C} \longrightarrow \mathbb{C}$ *defined by*

$$Tz = \begin{cases} \frac{z}{2} & ; \quad |z| \ge 2 \\ z & ; \quad |z| < 2 \end{cases},$$

for all $z \in \mathbb{C}$. *We have*

$$\rho = \inf\{d(z,Tz) : z \ne Tz\} = \inf\left\{\frac{|z|}{2} : |z| \ge 2\right\} = 1.$$

Then, it is easy to verify that the self-mapping T *satisfies the conditions of Theorem 3.4 with* $z_0 = 0$, $\psi(t) = \frac{9}{13}t$, $\alpha = \mu = \frac{1}{2}$, $\beta = \frac{1}{3}$ *and* $\gamma = \frac{4}{3}$. *Consequently,* $Fix(T)$ *contains the circle* $C_{0,1}$ *and the disc* $D_{0,1}$.

If we define another self-mapping $T_r : \mathbb{C} \longrightarrow \mathbb{C}$ *by*

$$T_r z = \begin{cases} z_0 & ; \quad |z - z_0| > r \\ z & ; \quad |z - z_0| \le r \end{cases},$$

for all $z \in \mathbb{C}$ *with* $r \in (0, \infty)$, *then self-mapping* T_r *is not an* M_{z_0}*-type contraction with* $z_0 \in \mathbb{C}$. *But* T_r *fixes the circle* $C_{z_0,r}$ *and the disc* $D_{z_0,r}$.

Example 3.8 *Let* $(\mathbb{R}, d)$ *be the usual metric space with* $d(x,y) = |x - y|$. *Let us define the self-mapping* $S : \mathbb{R} \to \mathbb{R}$ *as*

$$Sx = \begin{cases} x + 1 & ; \quad x > 1 \\ x & ; \quad x \le 1 \end{cases},$$

for all $x \in \mathbb{R}$. *Clearly,* $\rho = 1$ *and it is easy to check that the self-mapping* S *satisfies the conditions of Theorem 3.4 with* $x_0 = 0$, $\psi(t) = \frac{6}{7}t$, $\alpha = 0$, $\mu = \beta = \frac{1}{2}$ *and* $\gamma = 1$.

Then the circle $C_{0,1} = \{-1, 1\}$ *is a fixed circle of* S *(particularly,* $Fix(S)$ *contains the disc* $D_{0,1} = [-1,1]$*). Furthermore, we have* $\lim_{x \to -1} M_d(x, -1) = 0$ *and hence* S *is continuous at the fixed point* -1. *Since the limit* $\lim_{x \to 1} M_d(x, 1)$ *does not exist,* S *is discontinuous at the fixed point* 1 *(we have* $\lim_{x \to 1^-} M_d(x,1) = 0$ *while* $\lim_{x \to 1^+} M_d(x,1) \ne 0$*).*

S *is also an* M_{x_0}*-type contraction with* $x_0 = -1$, $\psi(t) = \frac{3}{4}t$, $\alpha = \frac{1}{4}$, $\beta = 0$, $\mu = \frac{1}{2}$ *and* $\gamma = \frac{3}{4}$. *Clearly, the circle* $C_{-1,1} = \{-2, 0\}$ *is a fixed circle of* S *and also we have* $D_{-1,1} \subset Fix(S)$. *It is easy to see that* $\lim_{x \to -2} M_d(x, -2) = 0$ *and* $\lim_{x \to 0} M_d(x, 0) = 0$ *and hence* S *is continuous on the fixed circle* $C_{-1,1}$.

From this last example, we deduce that the radius ρ of the fixed circle (resp. fixed disc) is independent from the center x_0 in Theorem 3.4 (resp. Corollary 3.5 and Corollary 3.6).

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