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On Ulam's type stability criteria for fractional integral equations including Hadamard type singular kernel

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Abstract: In this paper, we deal with the Hyers-Ulam-Rassias (HUR) and Hyers-Ulam (HU) stability of Hadamard type fractional integral equations on compact intervals. The stability conditions are developed using a new generalized metric (GM) definition and the fixed point technique by motivating Wang and Lin Ulam's type stability of Hadamard type fractional integral equations. Filomat 2014; 28(7): 1323-1331. Moreover, our approach is efficient and ease in use than to the previously studied approaches. Finally, we give two examples to explain our main results.

Key words: Hyers-Ulam stability, Hyers-Ulam-Rassias stability, fractional integral equation, fixed point theory, Hadamard type singular kernel

1. Introduction

Although the subject fractional calculus was introduced more than 300 years ago, researches are still working in the development of both theory and application [21, 26, 29, 32]. The applications of fractional calculus has been observed in almost every field of sciences, such as mechanics, electricity, biology, economics, physics, biophysics, control theory, signal processing and image processing (see [13, 25, 30, 31, 33]). In recent years, the HU stability of various fractional differential equations has been widely studied (see [5, 7, 14, 19, 20, 24, 38–40]). The study on HU stability was initiated in 1940 and later on, it was extended to Banach spaces [9]. After that, many researchers put their effort to develop the generalized theory to study the HU stability of various type of differential phenomena (see [2–4, 8, 9, 11, 12, 16–18, 22, 23, 27, 28, 36, 38] and the other stability results [15, 34, 35]).

In 2013, Wang et al.[38] gave Ulam's type stability of fractional differential equations involving Hadamard derivative. They obtained some Ulam-Hyers stability conditions by using the method studied in [28].

In 2014, Wang and Lin [37] investigated the Ulam's type stability for fractional integral equations involving Hadamard type singular kernel on a compact interval by using fixed point method. They extended the developed results of [38] by choosing the closed interval based on the method of [27].

In 2016, Abbas et al.[1], developed the Ulam's stability results for partial integral equations using the Schauder's fixed-point results by taking the Hadamard's fractional integral.

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Kilbas et al.[21] examined the Cauchy problem for the following nonlinear fractional differential equation with the Hadamard fractional derivative of complex order $\alpha_1 \in \mathbb{C}$ $(\Re(\alpha_1) > 0)$:

$$\left(D^{\alpha_1}_{a_1+}y\right)(x) = u\left(x, y\left(x\right)\right), \left(a_1 \le x \le b_1\right) \tag{1.1}$$

$$\left(D^{\alpha_1-k}_{a_1+}y\right)(a_1) = b_k, (k = 1, ..., m) \tag{1.2}$$

where $m = [\Re(\alpha_1)] + 1$ for $\alpha_1 \notin \mathbb{N}$ and $m = \alpha_1$ for $m \in \mathbb{N}$, $[a_1, b_1]$ is a finite interval of $\mathbb{R}$, and

$$\left(D^{\alpha 1}_{a_1+}y\right)(x) = \left(x\frac{d}{dx}\right)^m \frac{1}{\Gamma\left(m-\alpha_1\right)} \int_{a_1}^{x} (\ln\frac{x}{\eta})^{m-\alpha_1+1} y\left(\eta\right)\frac{d\eta}{\eta}.$$

They proved the equivalence of (1.1)-(1.2) and a Volterra integral equation in the following form:

$$y(x) = \sum_{j=1}^{m} \frac{c_j}{\Gamma(\alpha_1 - j + 1)}(\ln\frac{x}{a_1})^{\alpha_1 - j} + \frac{1}{\Gamma\left(\alpha_1\right)}\int_{a_1}^{x}(\ln\frac{x}{\eta})^{\alpha_1-1}u(\eta, y\left(\eta\right))\frac{d\eta}{\eta}, \tag{1.3}$$

and applied this results to establish conditions for a unique solution of the Cauchy problem (1.1)-(1.2). Here $m-1 < \alpha_1 \le m$ $(m = 1, 2, ...)$, a_1 and b_1 are given constants such that $0 < a_1 \le x \le b_1 < \infty$. Also, c_j are fixed real numbers for $j = 1, 2, ..., m$, $\Gamma(.)$ is the Gamma function and $u : [a_1, b_1] \times \mathbb{R} \to \mathbb{R}$. In this paper, we deal with the Ulam's type stability for the fractional integral equations (1.3).

We organized this study as follows: In Section 1, we have given introduction. In Section 2, we introduce some definitions and some theorems which will be useful in proofs of our results. In Section 3, by using the motivation of [37], we investigate HUR and HU stability of the Eq.(1.3) on a compact interval with the help of a new GM definition. In Section 4, we give several examples for our results.

2. Preliminaries

Below, we give some definitions and some theorems which will be useful in the proofs of our main results.

Definition 2.1 *Let $M > 0$ be a given constant and ψ be a nonnegative function. For every function y satisfying*

$$\left| y(x) - \sum_{j=1}^{m} \frac{c_j}{\Gamma(\alpha_1 - j + 1)}\left(\ln\frac{x}{a_1}\right)^{\alpha_1 - j} - \frac{1}{\Gamma(\alpha_1)}\int_{a_1}^{x}\left(\ln\frac{x}{\eta}\right)^{\alpha_1-1}u(\eta, y(\eta))\frac{d\eta}{\eta}\right| \times \mathrm{e}^{-M(x-a_1)} \le \psi(x),$$

there is a solution y_1 of the Eq.(1.3) and $c > 0$ is a constant which is independent of y and y_1 such that

$$\left|y(x) - y_1(x)\right| \mathrm{e}^{-M(x-a_1)} \le c\psi\left(x\right), x \in \left[a_1, b_1\right],$$

then the Eq.(1.3) is called HUR stable.

In Definition 2.1, if $\psi\left(x\right)$ takes an arbitrary constant function, then the Eq.(1.3) is called HU stable.

Now, the function d is called a GM on Y if and only if $d : Y \times Y \to [0, \infty]$ satisfies the following conditions for $Y \neq 0$:

(M_1) $d(u, v) = 0$ if and only if $u = v$,
(M_2) $d(u, v) = d(v, u)$ for all $u, v \in Y$,
(M_3) $d(u, w) \leq d(u, v) + d(v, w)$ for all $u, v, w \in Y$.

Theorem 2.2 *[10] Let (Y, d) be a generalized complete metric (GCM) space. Suppose that $T : Y \to Y$ is a strictly contractive operator with the Lipschitz constant $L < 1$. If there exists a nonnegative integer k such that $d\left(T^{k+1}x, T^k x\right) < \infty$ for some $x \in Y$, then the following are true:*
(H_1) The sequence $\{T^n x\}$ converges to a fixed point x^* of T,
(H_2) x^* is the unique fixed point of T in

$$Y^* = \left\{y \in Y \ : \ d\left(T^k x, y\right) < \infty\right\},$$

(H_3) If $y \in Y^*$, then

$$d\left(y, x^*\right) \leq \frac{1}{1-L} d\left(Ty, y\right).$$

HU stability of various equations is examined using Theorem 2.2 and d (GM) defined by

$$d\left(g, h\right) := \inf\left\{C \in [0, \infty] \ : \ |g(x) - h(x)| \leq C\psi(x), \ x \in [a_1, b_1]\right\}, \tag{2.1}$$

see [17, 18, 36] and therein references.

3. Main results

Below, we are interested in HUR and HU stability of the Eq.(1.3) on $[a_1, b_1]$ such that $[a_1, b_1]$ is a compact interval. Now, we will give the following lemma which will be used in the proofs of our main results. In this lemma, we use a new GM which is different from (2.1).

Let $I := [a_1, b_1]$ be an interval for $a_1 > 0$ and the set Y be such as

$$Y = C\left(I, \mathbb{R}\right). \tag{3.1}$$

Lemma 3.1 *[6] Let $d : Y \times Y \to [0, \infty]$ be a function defined by*

$$d\left(g, h\right) := \inf\left\{C \in [0, \infty] \ : \ |g(x) - h(x)|\, \mathrm{e}^{-M(x-a_1)} \leq C\psi(x), \ x \in I\right\}, \tag{3.2}$$

where $M > 0$ and $\psi \in C\left[I, (0, \infty)\right]$ are given constant and function, respectively. Then (Y, d) is a GCM space.

Proof Firstly, we show that the function d defined in (3.2) is a GM on Y. It is clear that the conditions M_1 and M_2 are satisfied. Now, we show that the condition M_3 also satisfies. For some $g, h, f \in Y$, we can find an $x \in I$ such that

$$\begin{aligned} |g(x) - h(x)|\, \mathrm{e}^{-M(x-a_1)} &= |g(x) - f(x) + f(x) - h(x)|\, \mathrm{e}^{-M(x-a_1)} \\ &\leq |g(x) - f(x)|\, \mathrm{e}^{-M(x-a_1)} + |f(x) - h(x)|\, \mathrm{e}^{-M(x-a_1)}. \end{aligned}$$

Then, M_3 holds.

Now, we will show that (Y,d) is complete metric space. Let $\{g_n\}$ be a Cauchy sequence on (Y,d). Then, we can find an integer $N_1(\varepsilon)>0$ for any $\varepsilon>0$ such that $d(g_m,g_n)\leq\varepsilon$ for all $m,n\geq N_1(\varepsilon)$. In other words, there exists an integer $N_1(\varepsilon)>0$ for any $\varepsilon>0$ such that

$$|g_m(x)-g_n(x)|\,\mathrm{e}^{-M(x-a_1)}\leq\varepsilon\psi(x) \tag{3.3}$$

for all $m,n\geq N_1(\varepsilon)$ and all $x\in I$. This means that $\{g_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ for any fixed x. Because of $\mathbb{R}$ is complete, $\{g_n(x)\}$ converges for all $x\in I$. Thus, we have a function

$$g(x):=\lim_{n\to\infty}g_n(x),$$

where $g:I\to\mathbb{R}$. In (3.3), letting $m\to\infty$, we can find an integer $N_1(\varepsilon)>0$ for any $\varepsilon>0$ such that

$$|g(x)-g_n(x)|\,\mathrm{e}^{-M(x-a_1)}\leq\varepsilon\psi(x) \tag{3.4}$$

for all $x\in I$ and $n\geq N_1(\varepsilon)$. This means that, it can be find an integer $N_1(\varepsilon)>0$ for any $\varepsilon>0$ such that $d(g,g_n)\leq\varepsilon$ for all $n>N_1(\varepsilon)$. Additionally, because of ψ is bounded on I, $\{g_n(x)\}$ converges uniformly to g with the aid of (3.4) and so that $g\in Y$. Thus, the proof is complete. □

Let $0<p_1<1$, $m-1<\alpha_1\leq m$, $p_1\leq m$ and the following conditions hold:

$[A_1]$: The function $u:[a_1,b_1]\times\mathbb{R}\to\mathbb{R}$ is continuous and for any $\eta\in[a_1,b_1]$ and $y,z\in\mathbb{R}$,

$$|u(\eta,y)-u(\eta,z)|\leq L^*\eta^{p_1}|y-z|. \tag{3.5}$$

$[A_2]$: For all $x\in[a_1,b_1]$, the continuous function $y:[a_1,b_1]\to\mathbb{R}$ satisfies the following inequality:

$$\left|y(x)-\sum_{j=1}^{m}\frac{c_j}{\Gamma(\alpha_1-j+1)}\left(\ln\frac{x}{a_1}\right)^{\alpha_1-j}-\frac{1}{\Gamma(\alpha_1)}\int_{a_1}^{x}\left(\ln\frac{x}{\eta}\right)^{\alpha_1-1}u(\eta,y(\eta))\frac{d\eta}{\eta}\right|$$
$$\times\mathrm{e}^{-(L^*+1)(x-a_1)}\leq\psi(x), \tag{3.6}$$

and $\psi:[a_1,b_1]\to(0,\infty)$ satisfies

$$\left(\int_{a_1}^{x}(\psi(\eta))^{2/p_1}\,d\eta\right)^{p_1/2}\leq K^*\psi(x). \tag{3.7}$$

$[A_3]$: Let $0<K^*L^*M^*<1$, where

$$M^*=\frac{1}{\Gamma(\alpha_1)}\left(\frac{1-p_1}{\alpha_1-p_1}\right)^{1-p_1}\left(\frac{p_1}{2(L^*+1)}\right)^{p_1/2}\left(\ln\frac{b_1}{a_1}\right)^{\alpha_1-p_1}.$$

Here K^*, L^*, M^* values were obtained by (3.2) metric. Similar conditions also exist by using (2.1) metric in [39].

Theorem 3.2 *Suppose that* $[A_1]$, $[A_2]$ *and* $[A_3]$ *are satisfied. Then, there exists an unique continuous function* $y_1 : [a_1, b_1] \to \mathbb{R}$ *which satisfies* (1.3) *and*

$$y_1(x) = \sum_{j=1}^{m} \frac{c_j}{\Gamma(\alpha_1 - j + 1)} \left(\ln \frac{x}{a_1}\right)^{\alpha_1 - j} + \frac{1}{\Gamma(\alpha_1)} \int_{a_1}^{x} \left(\ln \frac{x}{\eta}\right)^{\alpha_1 - 1} u(\eta, y_1(\eta)) \frac{d\eta}{\eta} \tag{3.8}$$

and

$$|y(x) - y_1(x)| \, \mathrm{e}^{-(L^*+1)(x-a_1)} \le \frac{\psi(x)}{1 - K^* L^* M^*}, \tag{3.9}$$

for all $x \in [a_1, b_1]$

Proof Let the set Y be defined as (3.1). Also, let $d : Y \times Y \to [0, \infty]$ be a function defined as

$$d(g, h) := \inf \left\{ C \in [0, \infty] \ : \ |g(x) - h(x)| \, \mathrm{e}^{-(L^*+1)(x-a_1)} \le C\psi(x), \ x \in I \right\}. \tag{3.10}$$

From Lemma 3.1, (Y, d) is a GCM space. Now, define the operator $T : Y \to Y$ by

$$(Ty)(x) = \sum_{j=1}^{m} \frac{b_j}{\Gamma(\alpha_1 - j + 1)} \left(\ln \frac{x}{a_1}\right)^{\alpha_1 - j} + \frac{1}{\Gamma(\alpha_1)} \int_{a_1}^{x} \left(\ln \frac{x}{\eta}\right)^{\alpha_1 - 1} u(\eta, y(\eta)) \frac{d\eta}{\eta}, \tag{3.11}$$

for all $y \in Y$ and $x \in [a_1, b_1]$. It is obvious that the operator T is well defined. Now, we will show that T is strictly contractive on Y. From the definition of (Y, d), for any $g, h \in Y$, we find a $C_{g,h} \in [0, \infty)$ such that

$$|g(x) - h(x)| \, \mathrm{e}^{-(L^*+1)(x-a_1)} \le C_{g,h} \psi(x), \tag{3.12}$$

for any $x \in [a_1, b_1]$. Then, from the definition T in (3.11), and (3.5), (3.7), (3.12), Hölder's and Cauchy-Schwarz inequalities, we have

$$|(Tg)(x) - (Th)(x)| = \frac{1}{\Gamma(\alpha_1)} \left| \int_{a_1}^{x} (\ln x - \ln \eta)^{\alpha_1 - 1} \frac{1}{\eta} \left(u(\eta, g(\eta)) - u(\eta, h(\eta)) \right) d\eta \right|$$

$$\le \frac{L^*}{\Gamma(\alpha_1)} \int_{a_1}^{x} (\ln x - \ln \eta)^{\alpha_1 - 1} \eta^{-1} \eta^{p_1} |g(\eta) - h(\eta)| \, d\eta$$

$$\le \frac{L^* C_{g,h}}{\Gamma(\alpha_1)} \int_{a_1}^{x} (\ln x - \ln \eta)^{\alpha_1 - 1} \eta^{p_1 - 1} \psi(\eta) \, \mathrm{e}^{(L^*+1)(\eta - a_1)} d\eta$$

$$= \frac{L^* C_{g,h}}{\Gamma(\alpha_1)} \int_{a_1}^{x} (\ln x - \ln \eta)^{\alpha_1 - 1} \left(\eta^{\frac{p_1 - 1}{\alpha_1 - 1}} \right)^{\alpha_1 - 1} \psi(\eta) \, \mathrm{e}^{(L^*+1)(\eta - a_1)} d\eta$$

$$= \frac{L^* C_{g,h}}{\Gamma(\alpha_1)} \int_{a_1}^{x} \left((\ln x - \ln \eta) \, \eta^{\frac{p_1 - 1}{\alpha_1 - 1}} \right)^{\alpha_1 - 1} \psi(\eta) \, \mathrm{e}^{(L^*+1)(\eta - a_1)} d\eta$$

$$\le \frac{L^* C_{g,h}}{\Gamma(\alpha_1)} \left(\int_{a_1}^{x} \left((\ln x - \ln \eta) \eta^{\frac{p_1 - 1}{\alpha_1 - 1}} \right)^{\frac{\alpha_1 - 1}{1 - p_1}} d\eta \right)^{1 - p_1} \left(\int_{a_1}^{x} \left(\psi(\eta) \mathrm{e}^{(L^*+1)(\eta - a_1)} \right)^{1/p_1} d\eta \right)^{p_1}$$

$$\leq \frac{L^*C_{g,h}}{\Gamma(\alpha_1)}\left(\int_{a_1}^{x}(\ln x-\ln\eta)^{\frac{\alpha_1-1}{1-p_1}}\eta^{-1}d\eta\right)^{1-p_1}\left(\int_{a_1}^{x}(\psi(\eta))^{2/p_1}d\eta\right)^{p_1/2}$$

$$\times\left(\int_{a_1}^{x}e^{\frac{2(L^*+1)(\eta-a_1)}{p_1}}d\eta\right)^{p_1/2}$$

$$\leq \frac{K^*L^*C_{g,h}\psi(x)}{\Gamma(\alpha_1)}\left(\frac{1-p_1}{\alpha_1-p_1}(\ln x-\ln a_1)^{\frac{\alpha_1-p_1}{1-p_1}}\right)^{1-p_1}\left(\frac{p_1}{2(L^*+1)}e^{\frac{2(L^*+1)(x-a_1)}{p_1}}\right)^{p_1/2}$$

$$\leq \frac{K^*L^*C_{g,h}\psi(x)}{\Gamma(\alpha_1)}\left(\frac{1-p_1}{\alpha_1-p_1}\right)^{1-p_1}(\ln b_1-\ln a_1)^{\alpha_1-p_1}\left(\frac{p_1}{2(L^*+1)}\right)^{p_1/2}e^{(L^*+1)(x-a_1)}$$

for all $x\in[a_1,b_1]$. Thus, for all $x\in[a_1,b_1]$, we get

$$|(Tg)(x)-(Th)(x)|\,e^{-(L^*+1)(x-a_1)}\leq K^*L^*M^*C_{g,h}\psi(x).$$

So, we can write

$$d(Tg,Th)\leq K^*L^*M^*d(g,h)$$

for any $g,h\in Y$. By the condition $[A_3]$, T is a strictly contractive on Y. Let $g_1\in Y$. Since g_1 and Tg_1 are continuous, u and g_1 are bounded on $[a_1,b_1]$ and $\psi(x)>0$, then we can find a constant $C^*>0$ such that for

$$|(Tg_1)(x)-g_1(x)|\,e^{-(L^*+1)(x-a_1)}=\left|\sum_{j=1}^{m}\frac{c_j}{\Gamma(\alpha_1-j+1)}\left(ln\frac{x}{a_1}\right)^{\alpha_1-j}\right.$$

$$\left.+\frac{1}{\Gamma(\alpha_1)}\int_{a_1}^{x}\left(ln\frac{x}{\eta}\right)^{\alpha_1-1}u(\eta,g_1(\eta))\frac{d\eta}{\eta}-g_1(x)\right|e^{-(L^*+1)(x-a_1)}$$

$$\leq C^*\psi(x).$$

Therefore, we show that $d(Tg_1,g_1)<\infty$. Now, using Theorem 2.2, there exists $y_1\in C([a_1,b_1],\mathbb{R})$ such that $T^ng_1\to y_1$ in (Y,d) as $n\to\infty$ and $Ty_1=y_1$. That is, for every $x\in[a_1,b_1]$, y_1 satisfies (3.8). Now, we will show $\{g\in Y\ :\ d(g_1,g)<\infty\}=Y$. Since g and g_1 are bounded on $[a_1,b_1]$ and $min_{x\in[a_1,b_1]}\psi(x)>0$, for any $g\in Y$ and any $x\in[a_1,b_1]$ we can find a constant $C_g>0$ such that

$$|g_1(x)-g(x)|\,e^{-(L^*+1)(x-a_1)}\leq C_g\psi(x).$$

So, we obtain $d(g_1,g)<\infty$ for all $g\in Y$. That is, $\{g\in Y\ :\ d(g_1,g)<\infty\}=Y$. Hence, the function y_1 is unique continuous with (3.8). Also, since (3.6) we obtain that

$$d(y,Ty)\leq 1. \tag{3.13}$$

So, we obtain

$$d(y,y_1)\leq\frac{1}{1-K^*L^*M^*}d(Ty,y)\leq\frac{1}{1-K^*L^*M^*},$$

for all $x\in[a_1,b_1]$. That is, the inequality (3.9) holds. □

Corollary 3.3 *If $\psi(x)$ is an arbitrary constant function in Theorem 3.2, then we obtain HU stability of the equation (1.3).*

Now, instead of $[A_1]$ consider the following condition $[A_1^*]$.

$[A_1^*]$: Let $u:[a_1,b_1]\times\mathbb{R}\to\mathbb{R}$ be Carathéodory function. Also, there exists $M_*>0$ such that

$$|u(\eta,y)|\le M_*\eta^{p_1}(\ln b_1-\ln\eta)^{q_1}, p_1-\alpha_1\le q_1\le 0.$$

Also, for any $\eta\in[a_1,b_1]$ and $y,z\in\mathbb{R}$, let

$$|u(\eta,y)-u(\eta,z)|\le L_1^*\eta^{p_1}(\ln b_1-\ln\eta)|y-z|, p_1-\alpha_1\le q_1\le 0.$$

Theorem 3.4 *Let*

$$M_1^*=\frac{1}{\Gamma(\alpha_1)}\left(\frac{1-p_1}{\alpha_1+q_1-p_1}\right)^{1-p_1}\left(\frac{p_1}{2(L_1^*+1)}\right)^{p_1/2}(\ln b_1-\ln a_1)^{\alpha_1+q_1-p_1}.$$

Suppose that $[A_1^*]$, $[A_2]$ and $0<K^*L_1^*M_1^*<1$ are satisfied. Then for all $x\in[a_1,b_1]$, there exists an unique continuous function $y_1:[a_1,b_1]\to\mathbb{R}$ such that y_1 satisfies (1.3) and

$$|y(x)-y_1(x)|\,\mathrm{e}^{-(L^*+1)(x-a_1)}\le\frac{\psi(x)}{1-K^*L_1^*M_1^*}.$$

Proof First, we consider the second integral term in (3.8) and we show that it is bounded. That is, using $[A_1^*]$, we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha_1)}\int_{a_1}^{x}\left(\ln\frac{x}{\eta}\right)^{\alpha_1-1}u(\eta,y(\eta))\frac{d\eta}{\eta}\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1-1}\eta^{-1}\eta^{p_1}(\ln b_1-\ln\eta)^{q_1}M_*d\eta\\
&\le\frac{1}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1-1}\eta^{-1}\eta^{p_1}(\ln x-\ln\eta)^{q_1}M_*d\eta\\
&=\frac{M_*}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1+q_1-1}\left(\eta^{\frac{p_1-1}{\alpha_1+q_1-1}}\right)^{\alpha_1+q_1-1}d\eta\\
&=\frac{M_*}{\Gamma(\alpha_1)}\int_{a_1}^{x}\left[(\ln x-\ln\eta)\,\eta^{\frac{p_1-1}{\alpha_1+q_1-1}}\right]^{\alpha_1+q_1-1}d\eta\\
&\le\frac{M_*}{\Gamma(\alpha_1)}\left(\int_{a_1}^{x}\left[(\ln x-\ln\eta)\,\eta^{\frac{p_1-1}{\alpha_1+q_1-1}}\right]^{\frac{\alpha_1+q_1-1}{1-p_1}}d\eta\right)^{p_1-1}\left(\int_{a_1}^{x}1d\eta\right)^{p_1}\\
&=\frac{M_*(b_1-a_1)^{p_1}}{\Gamma(\alpha_1)}\left[\frac{1-p_1}{\alpha_1+q_1-p_1}(\ln x-\ln a_1)^{\frac{\alpha_1+q_1-p_1}{1-p_1}}\right]^{p_1-1}.
\end{aligned}$$

Proof is made by using the similar arguments to the proof of Theorem 3.2. Then, we have

$$\begin{aligned}
\left|(Tg)(x)-(Th)(x)\right| &= \frac{1}{\Gamma(\alpha_1)}\left|\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1-1}\frac{1}{\eta}\left(u(\eta,g(\eta))-u(\eta,h(\eta))\right)d\eta\right| \\
&\le \frac{L_1^*}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1-1}\eta^{-1}\eta^{p_1}(\ln b_1-\ln\eta)^{q_1}\left|g(\eta)-h(\eta)\right|d\eta \\
&\le \frac{L_1^*}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1-1}\eta^{-1}\eta^{p_1}(\ln x-\ln\eta)^{q_1}\left|g(\eta)-h(\eta)\right| \\
&\qquad\times \mathrm{e}^{-(L_1^*+1)(\eta-a_1)}\mathrm{e}^{(L_1^*+1)(\eta-a_1)}d\eta \\
&\le \frac{L_1^*C_{g,h}}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1+q_1-1}\eta^{p_1-1}\psi(\eta)\,\mathrm{e}^{(L_1^*+1)(\eta-a_1)}d\eta \\
&= \frac{L_1^*C_{g,h}}{\Gamma(\alpha_1)}\int_{a_1}^{x}(\ln x-\ln\eta)^{\alpha_1+q_1-1}\left(\eta^{\frac{p_1-1}{\alpha_1+q_1-1}}\right)^{\alpha_1+q_1-1}\psi(\eta)\,\mathrm{e}^{(L_1^*+1)(\eta-a_1)}d\eta \\
&= \frac{L_1^*C_{g,h}}{\Gamma(\alpha_1)}\int_{a_1}^{x}\left((\ln x-\ln\eta)\,\eta^{\frac{p_1-1}{\alpha_1+q_1-1}}\right)^{\alpha_1+q_1-1}\psi(\eta)\,\mathrm{e}^{(L_1^*+1)(\eta-a_1)}d\eta \\
&\le \frac{L_1^*C_{g,h}}{\Gamma(\alpha_1)}\left(\int_{a_1}^{x}\left((\ln x-\ln\eta)\eta^{\frac{p_1-1}{\alpha_1+q_1-1}}\right)^{\frac{\alpha_1+q_1-1}{1-p_1}}d\eta\right)^{1-p_1} \\
&\qquad\times\left(\int_{a_1}^{x}\left(\psi(\eta)\mathrm{e}^{(L_1^*+1)(\eta-a_1)}\right)^{1/p_1}d\eta\right)^{p_1} \\
&\le \frac{L_1^*C_{g,h}}{\Gamma(\alpha_1)}\left(\int_{a_1}^{x}(\ln x-\ln\eta)^{\frac{p_1-1}{\alpha_1+q_1-1}}\eta^{-1}d\eta\right)^{1-p_1}\left(\int_{a_1}^{x}(\psi(\eta))^{2/p_1}d\eta\right)^{p_1/2} \\
&\qquad\times\left(\int_{a_1}^{x}\mathrm{e}^{\frac{2(L_1^*+1)(\eta-a_1)}{p_1}}d\eta\right)^{p_1/2} \\
&\le \frac{K^*L_1^*C_{g,h}\psi(x)}{\Gamma(\alpha_1)}\left(\frac{1-p_1}{\alpha_1+q_1-p_1}(\ln x-\ln a_1)^{\frac{\alpha_1+q_1-p_1}{1-p_1}}\right)^{1-p_1} \\
&\qquad\times\left(\frac{p_1}{2(L_1^*+1)}\mathrm{e}^{\frac{2(L_1^*+1)(x-a_1)}{p_1}}\right)^{p_1/2} \\
&\le \frac{K^*L_1^*C_{g,h}\psi(x)}{\Gamma(\alpha_1)}\left(\frac{1-p_1}{\alpha_1+q_1-p_1}\right)^{1-p_1}\left(\frac{p_1}{2(L_1^*+1)}\right)^{p_1/2} \\
&\qquad\times\mathrm{e}^{(L_1^*+1)(x-a_1)}(\ln b_1-\ln a_1)^{\alpha_1+q_1-p_1} \\
&= K^*L_1^*M_1^*C_{g,h}\psi(x),
\end{aligned}$$

for all $x\in[a_1,b_1]$. Thus, we get

$$\left|(Tg)(x)-(Th)(x)\right|\mathrm{e}^{-(L^*+1)(x-a_1)}\le K^*L_1^*M_1^*C_{g,h}\psi(x),$$

for all $x\in[a_1,b_1]$. Therefore, we can complete the rest of the proof by applying standard process in proof of Theorem 3.2 which is quite similar. □

4. Examples

In this section, we will give the application of our main results with two examples. We will see that examples are based on the results Section 3, although these examples cannot be demonstrated with results of Wang and Lin [37].

Example 4.1. Consider the equation (1.3) with $u(\eta, y(\eta)) = \eta^2 y(\eta)$. Let $a_1 \in \mathbb{R}^+$, $b_1 = a_1\mathrm{e}$, $p_1 = \frac{1}{2}$, $\alpha_1 = 1$, $n = 1$. Then, $M^* = \frac{1}{2}$ and $L^* = 3$. Also, we assume that for all $x \in [a_1, a_1\mathrm{e}]$, $y : [a_1, a_1\mathrm{e}] \to \mathbb{R}$ is a continuous function such that

$$\left| y(x) - c_1 - \int_{a_1}^{x} \eta^2 y(\eta) \frac{d\eta}{\eta} \right| \mathrm{e}^{-4(x-a_1)} \leq \mathrm{e}^{4x}.$$

Here $\psi(x) = \mathrm{e}^{4x}$. So, for all $x \in [a_1, a_1\mathrm{e}]$ we have

$$\left(\int_{a_1}^{x} \mathrm{e}^{16\eta} d\eta \right)^{1/4} = \left(\frac{1}{16} \left(\mathrm{e}^{16x} - \mathrm{e}^{16a_1} \right) \right)^{1/4} \leq \frac{1}{2} \mathrm{e}^{4x}.$$

That is, we find $K^* = \frac{1}{2}$. Then, we obtain $0 < K^* L^* M^* = \frac{3}{4} < 1$ and $\Gamma(1) = 1$.

Because of Theorem 3.2, we say that there exists an unique continuous function $y_1 : [a_1, a_1\mathrm{e}] \to \mathbb{R}$ such that

$$y_1(x) = c_1 + \int_{a_1}^{x} \eta^2 y_1(\eta) \frac{d\eta}{\eta}$$

and

$$|y(x) - y_1(x)| \, \mathrm{e}^{-4(x-a_1)} \leq 4\mathrm{e}^{x}.$$

for all $x \in [a_1, a_1\mathrm{e}]$. Thus by Theorem 3.2, Eq.(1.3) is HUR stable. However, in this example if we use Theorem 2.1 in [37], we get $K = \frac{1}{2\sqrt{2}}$, $M = 1$, $L = 3$ and $KML = \frac{3}{2\sqrt{2}} > 1$. Therefore, since the KLM is not smaller than one, this example cannot apply to Theorem 2.1 in [37].

Example 4.2. Consider the equation (1.3) with $u(\eta, y) = 3\eta^{\frac{1}{2}} \sin y(\eta) \left(\ln a_1\mathrm{e} - \ln \eta \right)^{-\frac{1}{3}}$. Let $a_1 \in \mathbb{R}^+$, $b_1 = a_1\mathrm{e}$, $p_1 = \frac{1}{2}$, $\alpha_1 = 1$, $q_1 = -\frac{1}{3}$, $n = 1$. Then, we find

$$|u(\eta, y)| \leq 3\eta^{\frac{1}{2}} \left(\ln a_1\mathrm{e} - \ln \eta \right)^{-\frac{1}{3}}.$$

Also, for all $\eta \in [a_1, a_1\mathrm{e}]$ and $y, z \in \mathbb{R}$, we obtain

$$|u(\eta, y) - u(\eta, z)| \leq 3\eta^{\frac{1}{2}} \left(\ln a_1\mathrm{e} - \ln \eta \right) |y - z|.$$

So, $[A_1^*]$ satisfies. Since $L_1^* = 3$, we get $M_1^* = \frac{\sqrt{3}}{2}$. Also, we assume that for all $x \in [a_1, a_1\mathrm{e}]$, $y \in ([a_1, a_1\mathrm{e}], \mathbb{R})$ satisfies

$$\left| y(x) - c_1 - \int_{a_1}^{x} 3\eta^{\frac{1}{2}} \sin y(\eta) \left(\ln a_1\mathrm{e} - \ln \eta \right)^{-\frac{1}{3}} \frac{d\eta}{\eta} \right| \mathrm{e}^{-4(x-a_1)} \leq \frac{1}{2\sqrt{2}} \mathrm{e}^{x}.$$

Here $\psi(x)=\frac{1}{2\sqrt{2}}\mathrm{e}^{x}$. So, for all $x\in[a_1,a_1\mathrm{e}]$ we have

$$\left(\int_{a_1}^{x}\frac{1}{64}\mathrm{e}^{4\eta}d\eta\right)^{1/4}=\left(\frac{1}{256}\left(\mathrm{e}^{4x}-\mathrm{e}^{4a_1}\right)\right)^{1/4}\leq\frac{1}{4}\mathrm{e}^{4x}.$$

That is, we find $K^*=\frac{1}{4}$. Thus, $K^*L_1^*M_1^*=\frac{3\sqrt{3}}{8}$. Since Theorem 3.4, we say that there exists an unique continuous function $y_1\in C\left([a_1,a_1\mathrm{e}],\mathbb{R}\right)$ which satisfies

$$y_1(x)=c_1+\int_{a_1}^{x}3\eta^{\frac{1}{2}}\sin y_1(\eta)\left(\ln a_1\mathrm{e}-\ln\eta\right)^{-\frac{1}{3}}\frac{d\eta}{\eta}$$

and

$$\left|y(x)-y_1(x)\right|\mathrm{e}^{-4(x-a_1)}\leq\frac{1}{2\sqrt{2}}\frac{8}{8-3\sqrt{3}}\mathrm{e}^{x}.$$

for all $x\in[a_1,a_1\mathrm{e}]$. Thus by Theorem 3.4, Eq.(1.3) is HUR stable. However, in this example if we use Theorem 2.2 in [37], we get $K=\frac{1}{4}$, $M=\sqrt{3}$, $L=3$ and $KML=\frac{3\sqrt{3}}{4}>1$. Therefore, since the KLM is not smaller than one, this example cannot apply to Theorem 2.2 in [37].

5. Conclusion

Studies involving fractional calculus has played an important role in several areas of science and engineering. In recent years, HU stability of differential equations which have fractional derivative and integral in the various fields have been studied by many researchers. HU stability is one of the main topics in the theory of fractional equations. In this paper, we examine the HUR and HU stability of Hadamard type fractional integral equations on compact intervals. The stability conditions are developed by using a new GM definition and the fixed point technique by motivating Wang and Lin [37]. It is then shown that the results of this paper are better for some previous results. Finally, we investigate in detail two examples to show the reported results.

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