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Asymptotic socle behaviors for cones over curves in positive characteristic

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Abstract: This paper studies the distribution of socle degrees of $R/I^{[p^e]}$ when e is large, for a homogeneous ideal I in a two-dimensional standard-graded normal domain R in positive characteristic p. We prove that the distribution is very much related to the asymptotic slopes of the syzygy bundle $\mathrm{Syz}(I)$, which have been known to determine the Hilbert-Kunz multiplicity of I.

Key words: Frobenius power, socle, diagonal F-threshold, semistability, strong semistability, syzygy bundle, Hilbert-Kunz multiplicity

1. Introduction

Let $(R, \mathfrak{m}, k)$ be a local ring of dimension d in characteristic $p > 0$. Let $q = p^e$, where e is some positive integer. For an ideal I in R, the q-th Frobenius power of I is the ideal $I^{[q]} = \langle a^q | a \in I \rangle$ (Eventhough we will restrict ourself mainly to I being $\mathfrak{m}$-primary, this definition of Frobenius power $I^{[q]}$ is valid for an arbitrary ideal I). Let $\lambda(M)$ denote the length of an R-module M. Following Kunz's lead in [8, 9], Monsky [14] introduced the Hilbert-Kunz multiplicity $e_{\mathrm{HK}}(I)$ of R with respect to an $\mathfrak{m}$-primary ideal I, defined as follows:

$$e_{\mathrm{HK}}(I) = \lim_{q \to \infty} \frac{\lambda(R/I^{[q]})}{q^d}.$$

The above limit always exists. This invariant is a characteristic p analog of the more classical Hilbert-Samuel multiplicity, which measures the complexity of the singularity of R. On the other hand, the computation of this invariant is subtle in general. Our result in this paper is related to a remarkable result independently due to Brenner and Trivedi, regarding an explicit computation of this invariant and showing that it is rational for any two-dimensional graded ring (see Theorem 3.1). In this case, the Hilbert-Kunz multiplicity is essentially determined by the data coming from the strong Harder-Narasimhan filtration of the syzygy bundle of $I^{[q]}$ on $\mathrm{Proj}\, R$. In particular, it takes a quadratic form on the asymptotic slopes ν_k's which will be defined in (2.1). In this paper, we investigate some other roles the invariants ν_k's play under the same setting, from a more algebraic point of view. Our main result shows that these rational numbers also control the distribution of the socle degrees of $R/I^{[q]}$. Recall that the socle degrees of $R/I^{[q]}$ are the degrees of the generators (as a vector space over the residue field k) of $(I^{[q]} : \mathfrak{m})/I^{[q]}$. In general, it is subtle how these degrees grow as q grows,

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except for some special cases (see, for example [10]). With the extra hypothesis that the syzygy bundle of $\mathfrak{m}$ is strongly semistable, we are able to show that the socle degrees of $R/I^{[q]}$ only live in some small ranges near $q\nu_k$ for q large enough (Theorem 3.2). The key idea in the proof is to consider, on the smooth projective curve $Y = \operatorname{Proj} R$, the double complex obtained by tensoring the short exact sequence of locally free sheaves (here we assume $I = (f_1, \dots, f_s)$, and $\mathcal{O} = \mathcal{O}_Y$, the structure sheaf of Y)

$$0 \longrightarrow \operatorname{Syz}(f_1^q, \dots, f_s^q)(m) \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}(m - qd_i) \overset{f_1^q,\dots,f_s^q}{\longrightarrow} \mathcal{O}(m) \longrightarrow 0 \tag{1.1}$$

where $d_i = \deg(f_i)$, with the dual of another short exact sequence (set $\mathfrak{m} = (x_1, \dots, x_n)$)

$$0 \longrightarrow \operatorname{Syz}(\mathfrak{m}) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(-1) \overset{x_1,\dots,x_n}{\longrightarrow} \mathcal{O} \longrightarrow 0. \tag{1.2}$$

After taking global sections of the locally free sheaves in the double complex, one can use some elementary diagram-chasing arguments to show that no socle element of $R/I^{[q]}$ lives in certain degrees provided a numerical condition on the global sections of syzygy bundles can be verified. Fortunately, a result of Brenner [1, Lemma 3.1] makes such a verification possible.

2. Background

In order to introduce the aforementioned result of Brenner and Trivedi, and also to formulate our main result, we briefly recall some basics on vector bundle theory in this section. We refer to [5] for more details. Let Y be a smooth projective curve over an algebraically closed field. For any vector bundle $\mathcal{V}$ on Y of rank r, the slope of $\mathcal{V}$, denoted $\mu(\mathcal{V})$, is defined as the fraction $\deg(\mathcal{V})/r$, where $\deg \mathcal{V}$ is the degree of the line bundle $\wedge^r \mathcal{V}$. The slopes is additive on tensor products of bundles: $\mu(\mathcal{V} \otimes \mathcal{W}) = \mu(\mathcal{V}) + \mu(\mathcal{W})$. If $f : Y' \longrightarrow Y$ is a finite map of degree q, then $\deg(f^*(\mathcal{V})) = q \deg(\mathcal{V})$ and so $\mu(f^*(\mathcal{V})) = q\mu(\mathcal{V})$.

A bundle $\mathcal{V}$ is called *semistable* if for every locally free subsheaf $\mathcal{W} \subseteq \mathcal{V}$ one has $\mu(\mathcal{W}) \leq \mu(\mathcal{V})$. Clearly, bundles of rank 1 are always semistable, and duals and twists of semistable bundles are semistable.

Any bundle $\mathcal{V}$ has a filtration by locally free subsheaves

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_t = \mathcal{V}$$

such that $\mathcal{V}_k/\mathcal{V}_{k-1}$ is semistable and $\mu(\mathcal{V}_k/\mathcal{V}_{k-1}) > \mu(\mathcal{V}_{k+1}/\mathcal{V}_k)$ for each k. This filtration is unique, and it is called the *Harder-Narasimhan (or HN) filtration* of $\mathcal{V}$.

In positive characteristic, we use F to denote the absolute Frobenius morphism $F : Y \longrightarrow Y$. A semistable bundle is called *strongly semistable* if the semistability is preserved under the Frobenius pull-back. Not every semistable bundle is strongly semistable. Therefore, the pull-back under F^e of the HN filtration of $\mathcal{V}$ does not always give the HN filtration of $(F^*)^e(\mathcal{V})$. However, by the work of Langer [11, Theorem 2.7], there always exists a so called strong NH filtration, i.e. for some e_0, the HN filtration of $(F^*)^{e_0}(\mathcal{V})$ has the property that all its Frobenius pull-backs are the HN filtrations of $(F^*)^e(\mathcal{V})$, for all $e \geq e_0$.

Using the strong HN filtration, one defines rational numbers $\nu_1 < \cdots < \nu_t$ for the bundle $\mathcal{V}$ by

$$\nu_k = -\frac{\mu(F^{*(e-e_0)}(\mathcal{V}_k)/F^{*(e-e_0)}(\mathcal{V}_{k-1}))}{p^e \deg Y}, e \geq e_0, \text{ for } k = 1, \dots, t, \tag{2.1}$$

where (by abuse of notations) $0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_t = (F^*)^{e_0}(\mathcal{V})$ is the HN filtration of $F^{*e_0}(\mathcal{V})$ which is strong. For the sake of convenience, we call these rational numbers the *asymptotic slopes associated with the vector bundle* $\mathcal{V}$. Notice that if $\mathcal{V}$ is strongly semistable, then all these ν_k's degenerate to just one ν.

3. Main result

Let k be an algebraically closed field in prime characteristic p. Let R be a standard-graded algebra over k generated by degree one elements $x_1, \ldots, x_n$. Let $\mathfrak{m}$ denote the maximal ideal $(x_1, \ldots, x_n)$ of R. Let $I = (f_1, \ldots, f_s)$ where f_i is homogeneous of degree d_i for $1 \le i \le s$ and assume $\sqrt{I} = \mathfrak{m}$. In this section, we use tools developed in [1] to study the distribution of the degrees of the socle generators of $R/I^{[q]}$, in the case that R is a two-dimensional normal domain. Recall that the socle of an R-module M is the submodule $\operatorname{Hom}_R(k, M)$, which is a vector space over k. When M is graded, the basis elements of the socle of M are homogeneous elements. We simply refer to the degrees of these elements as the *socle degrees* of M. We use $\operatorname{t.s.d}(M)$ to denote the *top socle degree* of M. When R is $\mathbb{Z}$-graded and M is Artinian, the top socle degree of M is an integer.

Let $Y = \operatorname{Proj} R$. Consider the syzygy bundle $\mathcal{S} = \operatorname{Syz}(I)$ on Y given by the exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}(-d_i) \overset{f_1,\ldots,f_s}{\longrightarrow} \mathcal{O} \longrightarrow 0 \tag{3.1}$$

and the pull-back of this exact sequence along F^e (with a subsequent twist by $m \in \mathbb{Z}$)

$$0 \longrightarrow \mathcal{S}^q(m) \longrightarrow \bigoplus_{i=1}^{s} \mathcal{O}(m - qd_i) \overset{f_1^q,\ldots,f_s^q}{\longrightarrow} \mathcal{O}(m) \longrightarrow 0 \tag{3.2}$$

where $\mathcal{S}^q$ denotes the pull-back $(F^*)^e(\mathcal{S}) = \operatorname{Syz}(f_1^q, \ldots, f_s^q)$.

Assume R is a two-dimensional normal domain. Let $\nu_1 < \cdots < \nu_t$ be the asymptotic slopes associated with $\mathcal{S}$ (as defined in (2.1)). Let r_k $(1 \le k \le t)$ denote the rank of the k-th successive quotient in the strong HN filtration of $\mathcal{S}^q$. Recall that Brenner [1, Theorem 3.6] (independently, Trivedi [19, Theorem 4.12]) proved

Theorem 3.1 *Adopt the above notations. Then*

$$e_{HK}(I) = \frac{\deg Y}{2}\left(\sum_{k=1}^{t} r_k \nu_k^2 - \sum_{i=1}^{s} d_i^2\right).$$

In particular, $e_{HK}(I)$ is a rational number in such a case.

Note that these ν_k's are particularly positive because of the relation $\min_i(d_i) \le \nu_1 < \cdots < \nu_t \le \max_{i \ne j}(d_i + d_j)$, which follows easily from (3.2).

Throughout the rest, we make the same assumptions on R, i.e. a two-dimensional standard-graded normal domain. We also let ω_Y be the canonical bundle on $Y = \operatorname{Proj} R$, and make the additional convention that $\nu_0 = -\infty$ and $\nu_{t+1} = \infty$. Let $\mathcal{E}$ be the syzygy bundle of $\mathfrak{m}$. In the following theorem which is the main result of this paper, we show that the socle degrees of $R/I^{[q]}$ are asymptotically controlled by the asymptotic slopes of the syzygy bundle of I, in the case that $\mathcal{E}$ is strongly semistable. Specifically, there are only small ranges around $q\nu_k$ where these socle degrees may live.

Theorem 3.2 *Suppose that $\mathcal{E}$ is strongly semistable. Then for $q \gg 0$, all socle degrees of $R/I^{[q]}$ live in the range*

$$\Big[\lceil q\nu_k - \frac{1}{n-1}\rceil - 1, \lfloor q\nu_k + \frac{\deg \omega_Y}{\deg Y}\rfloor\Big],$$

for $k = 1, \ldots, t$.

Proof We prove an alternative (but equivalent) statement that no socle degree of $R/I^{[q]}$ lives in the range

$$\Big[\lfloor q\nu_k + \frac{\deg \omega_Y}{\deg Y}\rfloor + 1, \lceil q\nu_{k+1} - \frac{1}{n-1}\rceil - 2\Big],$$

for $k = 0, \ldots, t$ and $q \gg 0$.

There is a short exact sequence of bundles

$$0 \longrightarrow \mathcal{E} \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}(-1) \xrightarrow{x_1,\ldots,x_n} \mathcal{O} \longrightarrow 0. \tag{3.3}$$

Twist the above exact sequence by $\mathcal{O}(1)$ and then dualize. One has

$$0 \longrightarrow \mathcal{O}(-1) \xrightarrow{\psi} \bigoplus_{i=1}^{n} \mathcal{O} \longrightarrow (\mathcal{E}(1))^{\vee} \longrightarrow 0. \tag{3.4}$$

Use $\mathcal{E}'$ to denote $(\mathcal{E}(1))^{\vee}$. The degree of $\mathcal{E}'$ equals $\deg Y$ and the rank equals $n-1$. So $\mathcal{E}'$ has slope $\mu(\mathcal{E}') = \frac{\deg Y}{n-1}$.

Let $\mathcal{S}^q$ denote the syzygy bundle of $I^{[q]} = (f_1^q, \ldots, f_s^q)$. Then one has the following diagram with exact rows and columns, in which the second and third rows come from the first row by tensoring with $\mathcal{O}(1)^n$ and with $\mathcal{E}'(1)$ respectively. Moreover, the rightmost column is just a twist of (3.4).

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{S}^q(m) & \longrightarrow & \bigoplus_{i=1}^{s}\mathcal{O}(m-qd_i) & \longrightarrow & \mathcal{O}(m) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \oplus\psi} & & \downarrow{\scriptstyle \psi} & & \\
0 & \longrightarrow & (\mathcal{S}^q(m+1))^n & \longrightarrow & \bigoplus_{i=1}^{s}\mathcal{O}^n(m+1-qd_i) & \longrightarrow & \mathcal{O}^n(m+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{S}^q \otimes \mathcal{E}'(m+1) & \longrightarrow & \bigoplus_{i=1}^{s}\mathcal{E}'(m+1-qd_i) & \longrightarrow & \mathcal{E}'(m+1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{3.5}$$

Taking global sections, one then obtains

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & H^0(Y, \mathcal{S}^q(m)) & \to & \bigoplus_{i=1}^{s} R_{m-qd_i} & \to & R_m & \to & \left(\frac{R}{I^{[q]}}\right)_m \to 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \oplus\psi_{m-qd_i}} & & \downarrow{\scriptstyle \psi_m} & & \downarrow{\scriptstyle \pi_m} \\
0 \to & & (H^0(Y, \mathcal{S}^q(m+1)))^n & \to & \bigoplus_{i=1}^{s} R^n_{m+1-qd_i} & \to & R^n_{m+1} & \to & \left(\frac{R}{I^{[q]}}\right)^n_{m+1} \to 0 \\
 & & \downarrow{\scriptstyle \delta_m} & & \downarrow & & \downarrow & & \\
0 \to & & H^0(Y, \mathcal{S}^q \otimes \mathcal{E}'(m+1)) & \to & \bigoplus_{i=1}^{s} H^0(Y, \mathcal{E}'(m+1-qd_i)) & \to & H^0(Y, \mathcal{E}'(m+1)) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \bigoplus_{i=1}^{s} H^1(Y, \mathcal{O}(m-qd_i)) & \to & H^1(Y, \mathcal{O}(m)) & &
\end{array}
\tag{3.6}
$$

In the above diagram, we use M_m to denote the m-th graded component of the $\mathbb{Z}$-graded R-module M. Notice the m-th graded component of the socle of $R/I^{[q]}$ is exactly the kernel of π_m.

We need the following simple lemma. The proof of this lemma only requires some standard diagram-chasing arguments and we leave it for interested readers to verify.

Lemma 3.3 *Given the following commutative diagram (with exact rows and columns) of modules over any commutative ring*

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \gamma} & & \\
E & \longrightarrow & F & \longrightarrow & G & \longrightarrow & H & & \\
\downarrow{\scriptstyle \delta} & & \downarrow & & \downarrow & & & & \\
I & \longrightarrow & J & \longrightarrow & K & & & &
\end{array}
\tag{3.7}
$$

If δ is surjective, then γ is injective.

In order to prove our theorem, by Lemma 3.3, it suffices to show that for

$$
m \in \bigcup_{k=0}^{t} \Big[\lfloor q\nu_k + \frac{\deg \omega_Y}{\deg Y} \rfloor + 1, \lceil q\nu_{k+1} - \frac{1}{n-1} \rceil - 2 \Big],
$$

the map δ_m in diagram (3.6) is surjective. In fact, we claim that for m in the above range, the following holds

$$
h^0(\mathcal{S}^q(m)) - nh^0(\mathcal{S}^q(m+1)) + h^0(\mathcal{S}^q \otimes \mathcal{E}'(m+1)) = 0. \tag{3.8}
$$

We next prove this claim. Since $\mathcal{E}$ is assumed to be strongly semistable, so is $\mathcal{E}'(m)$. Since the tensor product of strongly semistable bundles is still strongly semistable [15, Theorem 7.3], tensoring $\mathcal{E}'(m)$ with the strong HN filtration of $\mathcal{S}^q$ yields the HN filtration of $\mathcal{S}^q \otimes \mathcal{E}'(m)$. Thus we can use the following lemma [1, Lemma 3.1] to compute the dimensions of those global sections involved in (3.8).

Lemma 3.4 *Let $\mathcal{S}$ denote a locally free sheaf on a smooth projective curve Y of genus g over an algebraically closed field. Let $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_t = \mathcal{S}$ be the HN filtration of $\mathcal{S}$. Set $\mu_k = \mu_k(\mathcal{S}) = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$ and $r_k = \mathrm{rank}(\mathcal{S}_k/\mathcal{S}_{k-1})$ for $k = 1, \dots, t$. Then*

(i) For m such that $\mu_1(\mathcal{S}(m)) < 0$, we have $h^0(\mathcal{S}(m)) = 0$.

(ii) Fix k. Let m be such that $\mu_k(\mathcal{S}(m)) > \deg(\omega_Y)$ and $\mu_{k+1}(\mathcal{S}(m)) < 0$. Then

$$h^0(\mathcal{S}(m)) = h^0(\mathcal{S}_k(m)) = \deg(\mathcal{S}_k) + \mathrm{rank}(\mathcal{S}_k)(m \deg Y + 1 - g).$$

(iii) For m such that $\mu_t(\mathcal{S}(m)) > \deg(\omega_Y)$, $h^1(\mathcal{S}(m)) = 0.$

See [1, Lemma 3.1]. □

Now we apply Lemma 3.4 to the bundles $\mathcal{S}^q$ and $\mathcal{S}^q \otimes \mathcal{E}'$ respectively. Suppose q is sufficiently large, let

$$0 \subset \mathcal{S}_1^q \subset \cdots \subset \mathcal{S}_t^q = \mathcal{S}^q$$

be the (strong) HN filtration of $\mathcal{S}^q$. By Lemma 3.4 (ii), for m such that

$$\deg \omega_Y - \mu_k(\mathcal{S}^q) < m \deg Y < -\mu_{k+1}(\mathcal{S}^q), \tag{3.9}$$

one has

$$h^0(\mathcal{S}^q(m)) = \deg(\mathcal{S}_k^q) + \mathrm{rank}(\mathcal{S}_k^q)(m \deg Y + 1 - g).$$

Reapplying the same computation to $h^0(\mathcal{S}^q(m+1))$, one gets that for m such that

$$\deg \omega_Y - \mu_k(\mathcal{S}^q) < (m+1) \deg Y < -\mu_{k+1}(\mathcal{S}^q)$$

i.e.

$$\deg \omega_Y - \mu_k(\mathcal{S}^q) - \deg Y < m \deg Y < -\mu_{k+1}(\mathcal{S}^q) - \deg Y, \tag{3.10}$$

$$h^0(\mathcal{S}^q(m+1)) = \deg(\mathcal{S}_k^q) + \mathrm{rank}(\mathcal{S}_k^q)((m+1) \deg Y + 1 - g).$$

Since $\mathcal{E}'$ is assumed to be strongly semistable, tensoring $\mathcal{E}'$ with any strongly semistable bundle remains strongly semistable. Hence the HN filtration of $\mathcal{S}^q \otimes \mathcal{E}'$ is of the form:

$$0 \subset \mathcal{S}_1^q \otimes \mathcal{E}' \subset \cdots \subset \mathcal{S}_t^q \otimes \mathcal{E}' = \mathcal{S}^q \otimes \mathcal{E}'.$$

Therefore, one obtains

$$\mu_k(\mathcal{S}^q \otimes \mathcal{E}') = \mu_k(\mathcal{S}^q) + \mu(\mathcal{E}'). \tag{3.11}$$

Notice

$$\mu_k(\mathcal{S}^q \otimes \mathcal{E}'(m)) > \deg(\omega_Y) \Leftrightarrow \mu_k(S^q) + \mu(\mathcal{E}') + m \deg Y > \deg(\omega_Y)$$

and

$$\mu_{k+1}(\mathcal{S}^q \otimes \mathcal{E}'(m)) < 0 \Leftrightarrow \mu_{k+1}(S^q) + \mu(\mathcal{E}') + m \deg Y < 0.$$

Applying Lemma 3.4 (ii) and noticing $\mu(\mathcal{E}') = \frac{\deg Y}{n-1}$, one immediately gets that for values of m such that

$$\deg \omega_Y - \mu_k(\mathcal{S}^q) - \frac{\deg Y}{n-1} < (m+1)\deg Y < -\mu_{k+1}(\mathcal{S}^q) - \frac{\deg Y}{n-1},$$

i.e.

$$\deg \omega_Y - \mu_k(\mathcal{S}^q) - \frac{\deg Y}{n-1} - \deg Y < m \deg Y < -\mu_{k+1}(\mathcal{S}^q) - \frac{\deg Y}{n-1} - \deg Y, \tag{3.12}$$

$$h^0(\mathcal{S}^q \otimes \mathcal{E}'(m+1)) = \deg(\mathcal{S}^q_k \otimes \mathcal{E}') + \operatorname{rank}(\mathcal{S}^q_k \otimes \mathcal{E}')((m+1)\deg Y + 1 - g).$$

Notice that

$$\operatorname{rank}(\mathcal{S}^q_k \otimes \mathcal{E}') = (n-1)\operatorname{rank}(\mathcal{S}^q_k)$$

and

$$\deg(\mathcal{S}^q_k \otimes \mathcal{E}') = (n-1)\deg(\mathcal{S}^q_k) + \operatorname{rank}(\mathcal{S}^q_k)\deg Y.$$

It then follows trivially that, for values of m such the $m \deg Y$ is in the intersection of the intervals (3.9), (3.10) and (3.12), the equation (3.8) holds.

Since for q large enough, one has (by definition of ν_k in (2.1))

$$-\mu_k(\mathcal{S}^q) = q\nu_k \deg Y.$$

It follows that equation (3.8) holds for m in the range

$$q\nu_k + \frac{\deg \omega_Y}{\deg Y} < m < q\nu_{k+1} - 1 - \frac{1}{n-1}, k = 1, 2, \ldots, t.$$

For $m < q\nu_1 - 1 - \frac{1}{n-1}$, it also follows immediately from Lemma 3.4 (i) that

$$h^0(\mathcal{S}^q(m)) = h^0(\mathcal{S}^q(m+1)) = h^0(\mathcal{S}^q \otimes \mathcal{E}'(m+1)) = 0.$$

Thus equation (3.8) holds trivially in this case and our proof is complete.

□

Remark 3.5 *The strong semistability condition on $\mathcal{E}$ is only used to derive equation (3.11). The theorem might still be valid with a weaker hypothesis that $\mathcal{E}$ is semistable, provided one can show an estimate on $\mu_k(\mathcal{S}^q \otimes \mathcal{E}')$ similar to (3.11), namely, $\mu_k(\mathcal{S}^q \otimes \mathcal{E}') = -q\nu_k \deg Y + O(1)$. However, we do not know if this is true if $\mathcal{E}$ is only assumed to be semistable.*

4. Applications of the main theorem and the special case of elliptic curves

Theorem 3.2 implies the following result immediately.

Corollary 4.1 *Adopt the same assumptions as in Theorem 3.2. Let $\nu_1 < \cdots < \nu_t$ be the asymptotic slopes of* $\operatorname{Syz} I$*. Then for any $\epsilon > 0$, one can take q large enough such that for every socle degree α of $R/I^{[q]}$, there exists an $k \in \{1, \ldots, t\}$, which depends on α, such that*

$$\left| \frac{\alpha}{q} - \nu_k \right| < \epsilon.$$

In particular, if the syzygy bundle of I is strongly semistable, then the difference between the top and lowest socle degrees of $R/I^{[q]}$ is bounded as $q \to \infty$.

For another application of Theorem 3.2, we need to recall the definition of diagonal F-threshold in [13, Definition 2.2], which is a special case of the concept of F-threshold in a more general setting, see [2, 6, 7, 16–18]. Let q be a power of p, for any $\mathfrak{m}$-primary ideal I and q, define

$$\nu_{\mathfrak{m}}^{I}(q) := \max\{r \in \mathbb{N} | \mathfrak{m}^r \not\subseteq I^{[q]}\} = \mathrm{t.s.d}(R/I^{[q]}).$$

The limit of $\{\nu_{\mathfrak{m}}^{I}(q)/q\}$ as $q \to \infty$ is called the *diagonal F-threshold* of R with respect to I, denoted $c^I(R)$. The existence of such a limit was recently proved by De Stefani, Núñez-Betancourt and Pérez [18] (They proved a much more general result that F-threshold always exists). We also refer to [3, 4, 12] for some interesting computations especially related to diagonal F-thresholds.

Corollary 4.2 *Let $(R, \mathfrak{m})$ be a standard-graded two-dimensional normal domain and $I = (f_1, \ldots, f_s)$ a homogeneous $\mathfrak{m}$-primary ideal. Let d_i's, ν_k's be as before. Then the diagonal F-threshold $c^I(R)$ is a rational number in the following two cases:*

Case I, *$\nu_t > \max_i\{d_i\}$. In this case, $c^I(R) = \nu_t$.*

Case II, $\operatorname{Syz}(\mathfrak{m})$ *is strongly semistable. In this case, $c^I(R) \in \{\nu_1, \ldots, \nu_t\}$.*

Proof Case I is proved in [13] (see Theorem 2.6 in [13]). Case II follows immediately from Corollary 4.1, since the limit of $\{\nu_{\mathfrak{m}}^{I}(q)/q\}$ (as q goes to infinity) must be one of the ν_k's. $\square$

The following example shows that the diagonal F-threshold $c^I(R)$ might not be equal to ν_t. It also shows the difference between the top and lowest socle degrees of $R/I^{[q]}$ being bounded does not imply that $\operatorname{Syz}(I)$ is strongly semistable. Hence the converse of the last sentence of Corollary 4.1 is not necessarily true.

Example 4.3 *Suppose* $\operatorname{char} k = 7$*. Let R be $k[x,y,z]/(x^4 + y^4 + z^4)$ and $I = (x, y, z^l)$ where $l \geq 3$. One computes the Hilbert-Kunz multiplicity $e_{HK}(I) = 4$. Hence by Theorem 3.1, $\nu_1 = 2$ and $\nu_2 = l$. Macaulay 2 computes that the top socle degree of $R/I^{[q]}$ is approximately $q\nu_1$ (instead of $q\nu_2$). In fact, in this case $R/I^{[q]}$ is Gorenstein for all q.*

In this last part, we briefly discuss the consequences of applying our calculations on distribution of socle degrees to the special case of elliptic curves. In such a case, we have $\deg \omega_Y = 0$. Furthermore, $\operatorname{Syz}(I)$ is semistable if and only if it is strongly semistable, and its unique asymptotic slope is $\nu = (\sum d_i)/(s-1)$ [1, Corollary 2.4]. Suppose $m > q\nu$. Applying Lemma 3.4 (iii), we have $h^1(\mathcal{S}^q(m)) = 0$. Hence the m-th graded component $(R/I^{[q]})_m = 0$ and no socle element lives in degree $> q\nu$. Thus the top socle degree of $R/I^{[q]}$ is at most $\lfloor q\nu \rfloor$. On the other hand, by the last part of the proof of Theorem 3.2, the lowest socle degree of $R/I^{[q]}$ is at least

$$\lceil q\nu - \frac{1}{n-1} \rceil - 1.$$

If we let $\{q\nu\}$ to denote $q\nu - \lfloor q\nu \rfloor$, then the above gives us the following:

Corollary 4.4 *Suppose that Y is an elliptic curve and both $\mathrm{Syz}(I)$ and $\mathrm{Syz}(\mathfrak{m})$ are semistable. Then the difference between the top and the lowest socle degrees of $R/I^{[q]}$ is*

$$\begin{cases} 0, & \textit{if } \{q\nu\} > \frac{1}{n-1}; \\ \textit{at most } 1, & \textit{otherwise} \end{cases}.$$

Here is an example: if we take the elliptic curve $Y = \mathrm{Spec}(k[x,y,z]/(x^3+y^3+z^3)), \mathrm{char}\, k = 5$ and $I = (xy, x^2, y^2, z^2)$. Then both $\mathrm{Syz}(I)$ and $\mathrm{Syz}(\mathfrak{m})$ are semistable. The asymptotic slope of $\mathrm{Syz}(I)$ is $\nu = \frac{8}{3}$. The difference between the top and lowest socle degrees of $R/I^{[q]}$ is 0 or 1 according as $q \equiv 1$ or $2 \mod 3$.

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