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Analytic functions associated with cardioid domain

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Abstract: In this article, we define and study new domain for analytic functions which is named as cardioid domain for being of cardioid structure. Analytic functions producing cardioid domain are defined and studied to some extent. The Fekete–Szegö inequality is also investigated for such analytic functions.

Key words: Analytic functions, shell-like curve, Fibonacci numbers, cardioid domain

1. Introduction and definitions

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. The function f is said to be subordinate to the function g, written symbolically as $f \prec g$, if there exists a function w such that

$$f(z) = g(w(z)), \quad z \in \mathcal{U},$$

where $w(0) = 0$, $|w(z)| < 1$ for $z \in \mathcal{U}$. Using this concept of subordination, several subclasses of analytic functions are defined on the basis of geometrical interpretation of their image domains. It is worthwhile here to consider some following classes of analytic functions having renowned and interesting geometrical structures as their image domains and their causal leading analytic functions.

1. The domain $p(\mathcal{U}) = \{w \in \mathbb{C} : \Re(w) > 0\}$ is the right half plane due to analytic function $p(z) = \frac{1+z}{1-z}$, for details, see [4].

2. The domain $p(\mathcal{U}) = \{w \in \mathbb{C} : \Re(w) > \alpha,\ 0 \le \alpha < 1\}$ is a plane, to the right of line $\Re(w) = \alpha$, due to analytic function $p(z) = \frac{1+(1-2\alpha)z}{1-z}$, for details, see [4].

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3. The domain $p(\mathcal{U}) = \left\{ w \in \mathbb{C} : \left| w - \frac{1-AB}{1-B^2} \right| < \frac{A-B}{1-B^2},\ -1 < B < A \leq 1 \right\}$ is a disk due to analytic function $p(z) = \frac{1+Az}{1+Bz}$, for details, see [5].

4. The domain $p(\mathcal{U}) = \{ w \in \mathbb{C} : \Re(w) > k\,|w-1|\,,\ k \geq 0 \}$ represents conic regions, like right half plane for $k = 0$, hyperbolic regions for $0 < k < 1$, parabolic region for $k = 1$ and elliptic regions when $k > 1$, due to the analytic function

$$p_k(z) = \begin{cases} \frac{1+z}{1-z}, \quad k = 0, \\ 1 + \frac{2}{\pi^2}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, \quad k = 1, \\ 1 + \frac{2}{1-k^2}\sinh^2\left[\left(\frac{2}{\pi}\arccos k\right)\operatorname{arctan} h\sqrt{z}\right], \quad 0 < k < 1, \\ 1 + \frac{1}{k^2-1}\sin\left(\frac{\pi}{2R(t)}\int\limits_0^{\frac{u(z)}{\sqrt{t}}} \frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}dx\right) + \frac{1}{k^2-1}, \quad k > 1, \end{cases} \tag{1.2}$$

where $u(z) = \frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t \in (0,1)$, $z \in E$ and z is chosen such that $k = \cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is complementary integral of $R(t)$, for more detail, see [6, 7]. These conic regions are fixed in size.

5. The domain $p(\mathcal{U}) = \left\{ u + iv : (u-a)^2 > k^2\left[(u-a+b-1)^2 + v^2 + 2b(1-b)\right]\right\}$ gives a number of conic regions of any size by assigning suitable values to parameters a and b, due to the analytic function $p_k(a,b;z) = a + b + (1-b)\,p_k(z)$, where $p_k(z)$ is defined by (1.2) and a, b must be chosen accordingly, as:

$$\left.\begin{array}{ll} (i) & \text{For } k = 0, \text{ we take } b = 0, \\ (ii) & \text{For } k \in \left(0, \frac{1}{\sqrt{2}}\right), \text{ we take } b \in \left[\frac{1}{2k^2-1}, 1\right), \\ (iii) & \text{For } k \in \left[\frac{1}{\sqrt{2}}, 1\right], \text{ we take } b \in (-\infty, 1), \\ (iv) & \text{For } k \in (1, \infty), \text{ we take } b \in \left(-\infty, \frac{1}{2k^2-1}\right]. \end{array}\right\}$$

and

$$\left.\begin{array}{ll} \frac{k^2(1-b)}{1-k^2} - \sigma \leq a < 1 - \frac{k^2(1-b)}{k^2-1} + \sigma, & 0 \leq k < 1, \\ -\frac{1+b}{2} \leq a < \frac{1-b}{2}, & k = 1, \\ \max\left(\frac{k^2(1-b)}{1-k^2} - \sigma, 1 - \frac{k^2(1-b)}{k^2-1} - \sigma\right) \leq a < 1 - \frac{k^2(1-b)}{k^2-1} + \sigma, & k > 1, \end{array}\right\}$$

where $\sigma = \frac{k\sqrt{k^2(1-b)^2 + (1-k^2)(1-b^2)}}{k^2-1}$. For more details, see [9].

6. The domain

$$\begin{aligned} \Omega_k[A,B] = \Big\{ u + iv : &\left[\left(B^2-1\right)\left(u^2+v^2\right) - 2(AB-1)u + \left(A^2-1\right)\right]^2 \\ &> k^2\left[\left(-2(B+1)\left(u^2+v^2\right) + 2(A+B+2)u - 2(A+1)\right)^2 \right. \\ &\left. + 4(A-B)^2 v^2\right]\Big\}. \end{aligned}$$

gives oval and petal type regions due to the analytic function $p(z) = \frac{(A+1)p_k(z)-(A-1)}{(B+1)p_k(z)-(B-1)}$, where $p_k(z)$ is defined by (1.2) and $-1 \leq B < A \leq 1$. For further details, see [10].

7. The domain $p(\mathcal{U}) = \left\{w \in \mathbb{C} : |w^\alpha - \beta| < \beta,\ Argw \leq \frac{\pi}{2\alpha},\ \alpha \geq 1,\ \beta \geq \frac{1}{2}\right\}$ is a leaf-like domain due to the analytic function $p(z) = \left(\frac{1+z}{1+\frac{1-\beta}{\beta}z}\right)^{1/\alpha}$, for details, see [11].

8. The motivational geometrical structure is shell-like curves, upon which our present work is based. The shell-like curve is caused by the function $p(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$, where $\tau = \frac{1-\sqrt{5}}{2}$. The image of unit circle under the function $p(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$ gives the conchoid of Meclaurin, also named as shell-like curve. That is,

$$p\left(e^{i\varphi}\right) = \frac{\sqrt{5}}{2(3-2\cos\varphi)} + i\frac{\sin\varphi(4\cos\varphi-1)}{2(3-2\cos\varphi)(1+\cos\varphi)},\ 0 \leq \varphi < 2\pi.$$

The function $p(z) = \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2}$ has the following series representation

$$\begin{aligned} p(z) &= \frac{1+\tau^2z^2}{1-\tau z-\tau^2z^2} \\ &= 1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n, \quad \text{where} \quad u_n = \frac{(1-\tau)^n-\tau^n}{\sqrt{5}},\ \tau = \frac{1-\sqrt{5}}{2}. \end{aligned}$$

This generates a Fibonacci series of coefficient constants which made it closer to Fibonacci numbers. For more details, we refer the readers to [1–3, 12].

Getting inspiration from the concept of shell-like curves and the Janowski functions, we define and consider a new geometrical structure as image domain. Before that, first we state the following lemma that is useful in our main results.

Lemma 1.1 *[8] If $p(z) = 1+\sum_{n=1}^{\infty} h_n z^n$ is a function with positive real parts in $\mathcal{U}$, then for v, a complex number*

$$\left|h_2 - vh_1^2\right| \leq 2\max\{1, |2v-1|\}.$$

2. Main Results

We define a class of analytic functions as follows.

Definition 2.1 *Let $CP[A,B]$ be the class of functions $p(z)$ which are defined by the subordination relation*

$$p(z) \prec \widetilde{p}(A,B;z),$$

where $\widetilde{p}(A,B;z)$ is defined by

$$\widetilde{p}(A,B;z) = \frac{2A\tau^2z^2+(A-1)\tau z+2}{2B\tau^2z^2+(B-1)\tau z+2}, \tag{2.1}$$

with $-1 < B < A \leq 1$ and $\tau = \frac{1-\sqrt{5}}{2}$, $z \in \mathcal{U}$.

For in-depth understanding of the class $CP[A,B]$, it would be worthwhile here to have a geometrical description of the function $\widetilde{p}(A,B;z)$ defined by (2.1). If we denote $\Re\widetilde{p}\left(A,B;e^{i\theta}\right)=u$ and $\Im\widetilde{p}\left(A,B;e^{i\theta}\right)=v$, then the image $\widetilde{p}\left(A,B;e^{i\theta}\right)$ of the unit circle is a cardioid like curve defined by the following parametric form as

$$\begin{aligned} u &= \frac{4+(A-1)(B-1)\tau^2+4AB\tau^4+2\lambda\cos\theta+4(A+B)\tau^2\cos 2\theta}{4+(B-1)^2\tau^2+4B^2\tau^4+4(B-1)(\tau+B\tau^3)\cos\theta+8B\tau^2\cos 2\theta}, \\ v &= (A-B)\frac{\left(\tau-\tau^3\right)\sin\theta+2\tau^2\sin 2\theta}{4+(B-1)^2\tau^2+4B^2\tau^4+4(B-1)(\tau+B\tau^3)\cos\theta+8B\tau^2\cos 2\theta}, \end{aligned} \tag{2.2}$$

where $\lambda=(A+B-2)\tau+(2AB-A-B)\tau^3$, $-1<B<A\le 1$, $\tau=\frac{1-\sqrt{5}}{2}$ and $0\le\theta<2\pi$.

Furthermore, we note that

$$\widetilde{p}(A,B;0)=1 \quad \text{and} \quad \widetilde{p}(A,B;1)=\frac{AB+9(A+B)+1+4(B-A)\sqrt{5}}{B^2+18B+1}.$$

The cusp of the cardioid-like curve, defined by (2.2), is given by

$$\gamma(A,B)=\widetilde{p}\left(A,B;e^{\pm i\arccos(1/4)}\right)=\frac{2AB-3(A+B)+2+(A-B)\sqrt{5}}{2\left(B^2-3B+1\right)}.$$

The above discussed cardioid-like curve with different values of parameters can be seen in Figure 1.

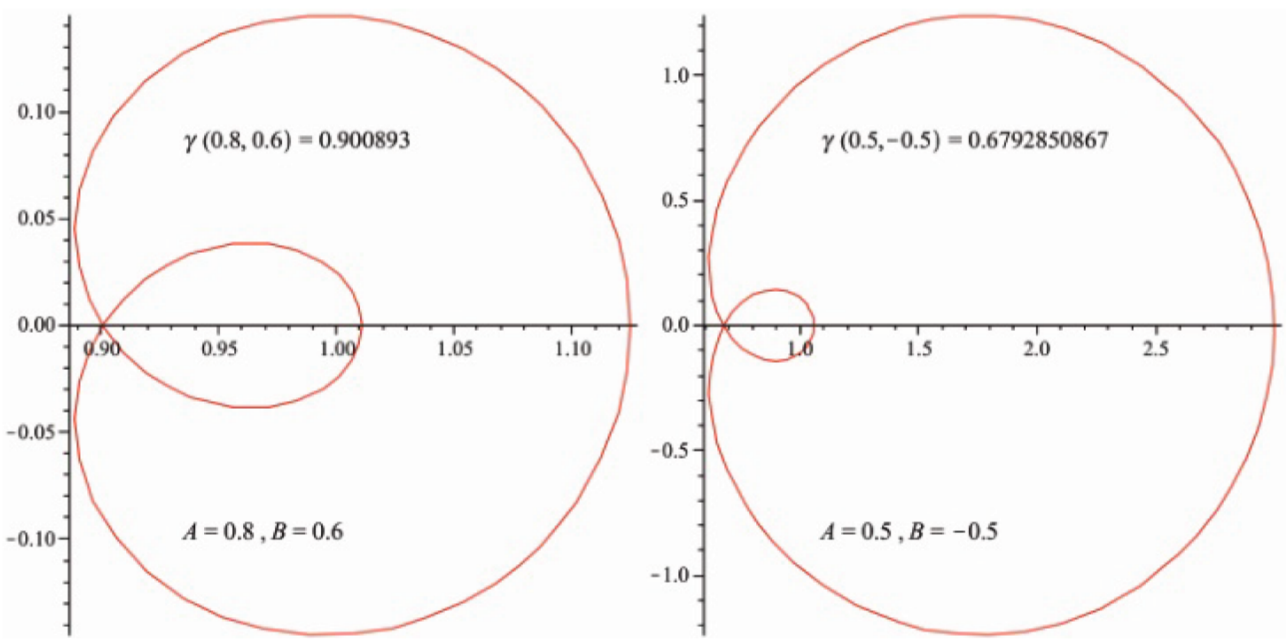

Figure 1. The curve (1.7) with A = 0.8; B = 0.6 and curve (1.7) with A = 0.5; B=-0.5

The parameters A, B are related by the relation $B<A$. Its violation flips over the cardioid curve as shown in Figure 2.

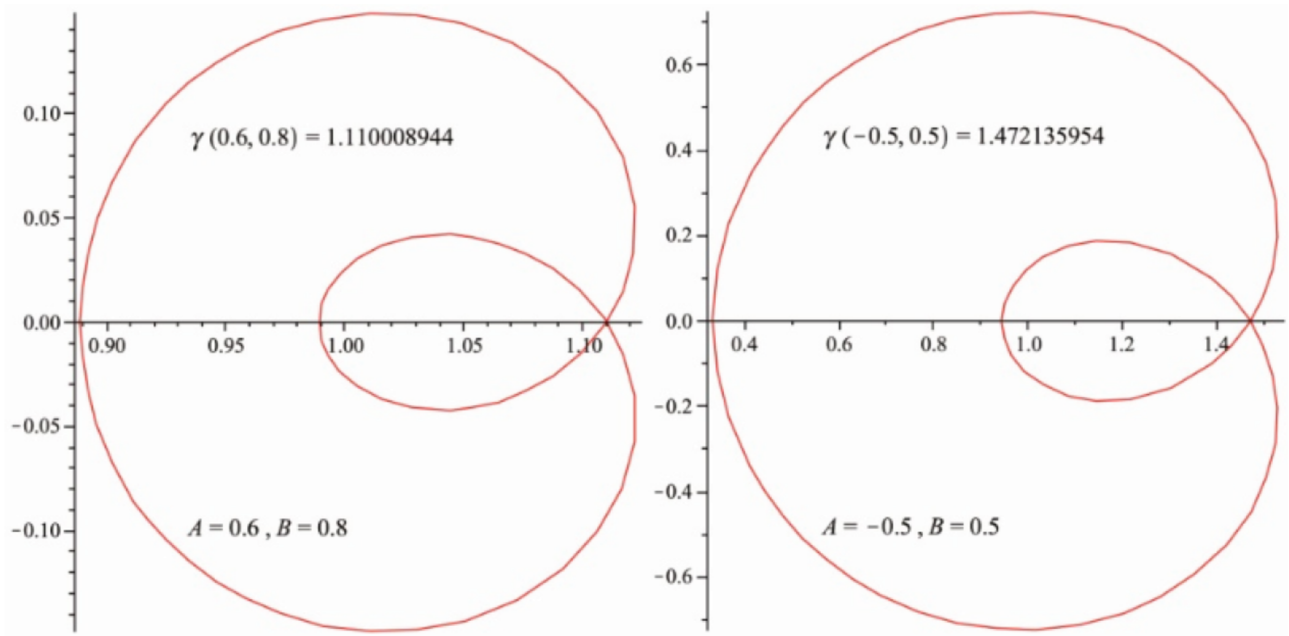

Figure 2. The curve (1.7) with A = 0.6; B = 0.8, and curve (1.7) with A=-0.5; B = 0.5

The parameter B is bounded below by relation $-1 < B$. Its violation does not result in the cardioid curve. Figure 3 can better explain this fact.

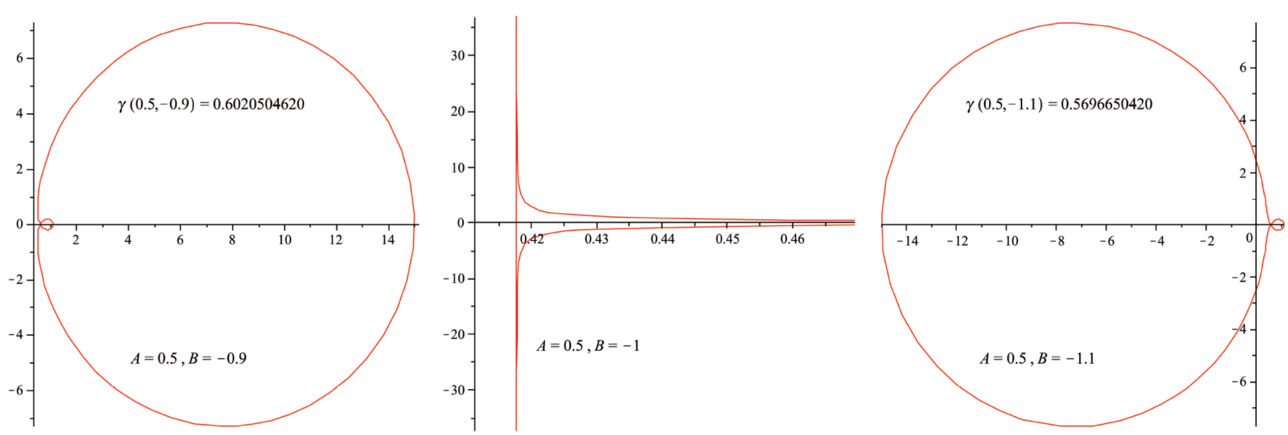

Figure 3. The curve (1.7) with A = 0.6; B = 0.8 and the curve (1.7) with A =-0.5; B = 0.5

If we consider the open unit disk $\mathcal{U}$ as the collection of concentric circles having origin as center, then we have the following image of open unit disk $\mathcal{U}$, shown in Figure 4.

Figure 4 shows the images of certain concentric circles. The image of each inner circle is a nested cardioid-like curve. Therefore, the function $\widetilde{p}(A, B; z)$ maps the open unit disk $\mathcal{U}$ onto a cardioid region. That is, $\widetilde{p}(A, B; \mathcal{U})$ is a cardioid domain.

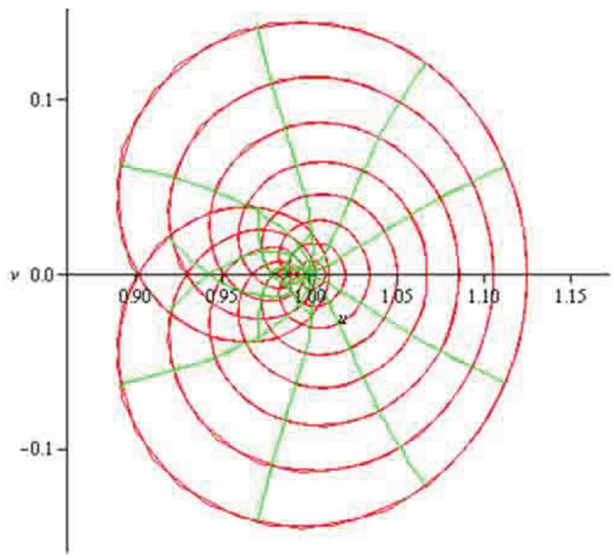

Figure 4. The curve (1.7) with A = 0.6; B = 0.8 and the curve (1.7) with A =-0.5; B = 0.5

If we set $\widetilde{p}(A,B;z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then it can be found that

$$p_n = \begin{cases} (A-B)\frac{\tau}{2}, & \text{for } \ n=1 \\ (A-B)(5-B)\frac{\tau^2}{2^2}, & \text{for } \ n=2 \\ \frac{1-B}{2}\tau p_{n-1} - B\tau^2 p_{n-2}, & \text{for } \ n=3,4,5,... \end{cases}$$

Theorem 2.2 *Let $p(z) \in CP[A,B]$. Then $p(z) \in P(\alpha)$, with*

$$\alpha = \frac{2(A+B-2)\tau + 2(2AB-A-B)\tau^3 + 16(A+B)\tau^2\eta}{4(B-1)(\tau+B\tau^3) + 32B\tau^2\eta}, \tag{2.3}$$

where $\eta = \frac{4+\tau^2-B^2\tau^2-4B^2\tau^4-(1-B\tau^2)\sqrt{5(2B\tau^2-(B-1)\tau+2)(2B\tau^2+(B-1)\tau+2)}}{4\tau(1+B^2t^2)}$, $-1 < B < A \leq 1$ *and* $\tau = \frac{1-\sqrt{5}}{2}$.

Proof As we know that

$$\begin{aligned} \alpha &= \min\left\{\Re\widetilde{p}\left(A,B;e^{i\theta}\right)\right\} \\ &= \min u(\theta), \end{aligned}$$

where $u(\theta)$ is defined by the relation (2.2). The $\min u(\theta)$ is attained at $\theta = \varphi$, where φ is one of the roots of $\frac{d}{d\theta}u(\theta) = 0$. A little simplification leads us to the value of φ, which is

$$\varphi = \arccos\left(\frac{4+\tau^2-B^2\tau^2-4B^2\tau^4-(1-B\tau^2)\sqrt{5(2B\tau^2-(B-1)\tau+2)(2B\tau^2+(B-1)\tau+2)}}{4\tau(1+B^2t^2)}\right).$$

That is,

$$\begin{aligned}
\alpha &= \lim_{\theta\to\varphi} u(\theta) \\
&= \lim_{\theta\to\varphi} \frac{4+(A-1)(B-1)\tau^2+4AB\tau^4+2\lambda\cos\theta+4(A+B)\tau^2(2\cos^2\theta-1)}{4+(B-1)^2\tau^2+4B^2\tau^4+4(B-1)(\tau+B\tau^3)\cos\theta+8B\tau^2(2\cos^2\theta-1)} \\
&= \lim_{\cos\theta\to\eta} \frac{4+(A-1)(B-1)\tau^2+4AB\tau^4+2\lambda\cos\theta+4(A+B)\tau^2(2\cos^2\theta-1)}{4+(B-1)^2\tau^2+4B^2\tau^4+4(B-1)(\tau+B\tau^3)\cos\theta+8B\tau^2(2\cos^2\theta-1)},
\end{aligned}$$

where

$$\eta = \frac{4+\tau^2-B^2\tau^2-4B^2\tau^4-(1-B\tau^2)\sqrt{5(2B\tau^2-(B-1)\tau+2)(2B\tau^2+(B-1)\tau+2)}}{4\tau(1+B^2t^2)}$$

and

$$\lambda = (A+B-2)\tau + (2AB-A-B)\tau^3.$$

This limit gets the form of $\frac{0}{0}$ when parameters A and B are set 1 and -1 respectively. That is, this limit expression is not stable. Applying L.Hopital's rule, we have

$$\begin{aligned}
\alpha &= \lim_{\cos\theta\to\eta} \frac{\frac{d}{d(\cos\theta)}\left(4+(A-1)(B-1)\tau^2+4AB\tau^4+2\lambda\cos\theta+4(A+B)\tau^2(2\cos^2\theta-1)\right)}{\frac{d}{d(\cos\theta)}\left(4+(B-1)^2\tau^2+4B^2\tau^4+4(B-1)(\tau+B\tau^3)\cos\theta+8B\tau^2(2\cos^2\theta-1)\right)} \\
&= \lim_{\cos\theta\to\eta} \frac{2\lambda+16(A+B)\tau^2\cos\theta}{4(B-1)(\tau+B\tau^3)+32B\tau^2\cos\theta} \\
&= \frac{2(A+B-2)\tau+2(2AB-A-B)\tau^3+16(A+B)\tau^2\eta}{4(B-1)(\tau+B\tau^3)+32B\tau^2\eta}.
\end{aligned}$$

□

Corollary 2.3 *When $A=1$, $B=-1$. Then, the order α defined by (2.3) reduces to*

$$\alpha = \frac{1}{2}\frac{1+\tau^2}{1-\tau^2+4\tau\eta},$$

where $\eta = -\frac{1}{2}\frac{2\tau^2-2+\sqrt{5}\sqrt{(\tau^2-\tau-1)(\tau^2+\tau-1)}}{\tau}$. Taking $\tau=\frac{1-\sqrt{5}}{2}$, we get $\alpha=\frac{\sqrt{5}}{10}$.

This result is proved in [3].

Theorem 2.4 *The function $\widetilde{p}(A,B;z)$ defined by (2.1) is univalent in the disk $|z|<\tau^2$, where $\tau=\frac{1-\sqrt{5}}{2}$.*

Proof For $z,w\in\mathcal{U}$, we consider that

$$\widetilde{p}(A,B;z) = \widetilde{p}(A,B;w).$$

This implies that

$$\frac{2+(A-1)\tau z+2A\tau^2 z^2}{2+(B-1)\tau z+2B\tau^2 z^2}=\frac{2+(A-1)\tau w+2A\tau^2 w^2}{2+(B-1)\tau w+2B\tau^2 w^2}.$$

A little simplification leads us to

$$\tau(B-A)(z-w)\left(w-\frac{2\tau z+1}{\tau^2 z^2-2\tau}\right)=0.$$

Now using similar argument as discussed in [3], we can conclude the required result. □

Theorem 2.5 *Let $p(z)\in CP[A,B]$ and of the form $p(z)=1+\sum_{n=1}^{\infty}p_n z^n$. Then, for a complex number μ*

$$\left|p_2-\mu p_1^2\right|\leq\frac{(A-B)\tau}{2}\max\left\{1,\left|\frac{\tau}{2}(\mu A+(1-\mu)B-5)\right|\right\}.$$

Proof For $h(z)\in P$ and of the form $h(z)=1+\sum_{n=1}^{\infty}c_n z^n$, consider

$$h(z)=\frac{1+w(z)}{1-w(z)},$$

where $w(z)$ is such that $w(0)=0$ and $|w(z)|<1$. It follows easily that

$$\begin{aligned}w(z) &= \frac{h(z)-1}{h(z)+1}\\ &= \frac{(1+c_1z+c_2z^2+c_3z^3+...)-1}{(1+c_1z+c_2z^2+c_3z^3+...)+1}\\ &= =\frac{1}{2}\left(c_1z+c_2z^2+c_3z^3+...\right)\left(1+\left(\frac{c_1}{2}z+\frac{c_2}{2}z^2+\frac{c_3}{2}z^3+...\right)\right)^{-1}.\end{aligned}$$

A little simplification reduces the above expression to

$$w(z)=\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+.... \tag{2.4}$$

Since $p(z)\in CP[A,B]$; therefore,

$$p(z)=\frac{2A\tau^2w^2+(A-1)\tau w+2}{2B\tau^2w^2+(B-1)\tau w+2}.$$

This implies that

$$1+\sum_{n=1}^{\infty}p_nz^n=\frac{2A\tau^2\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+....\right)^2+(A-1)\tau\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+....\right)+2}{2B\tau^2\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+....\right)^2+(B-1)\tau\left(\frac{1}{2}c_1z+\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)z^2+....\right)+2}$$

which reduces to

$$1+\sum_{n=1}^{\infty}p_nz^n=\frac{1+\frac{1}{4}(A-1)\tau c_1z+\frac{1}{4}\left\{A\tau^2c_1^2+2\tau(A-1)\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)\right\}z^2+...}{1+\frac{1}{4}(B-1)\tau c_1z+\frac{1}{4}\left\{B\tau^2c_1^2+2\tau(B-1)\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)\right\}z^2+...}.$$

After reducing right hand side of above equation to its series form and then comparing corresponding coefficients, we have the following relations.

$$\begin{aligned} p_1 &= \frac{1}{4}(A-B)\tau c_1 \\ p_2 &= \frac{1}{2}(A-B)\tau\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)+\frac{1}{16}\tau^2c_1^2(A-B)(5-B). \end{aligned}$$

Now, for complex number μ, consider

$$\begin{aligned} p_2-\mu p_1^2 &= \frac{1}{2}(A-B)\tau\left(\frac{1}{2}c_2-\frac{1}{4}c_1^2\right)+\frac{1}{16}\tau^2c_1^2(A-B)(5-B)-\frac{\mu}{16}(A-B)^2\tau^2c_1^2 \\ &= \frac{1}{4}(A-B)\tau\left\{c_2-\left(\frac{1}{2}-\frac{(5-B)}{4}\tau+\frac{\mu}{4}(A-B)\tau\right)c_1^2\right\} \\ &\leq \frac{1}{4}(A-B)\tau\left(2\max\left(1,|2v-1|\right)\right), \end{aligned}$$

where

$$v=\frac{1}{2}-\frac{(5-B)}{4}\tau+\frac{\mu}{4}(A-B)\tau.$$

Therefore,

$$\left|p_2-\mu p_1^2\right| \leq \frac{(A-B)\tau}{2}\max\left\{1,\left|\frac{\tau}{2}\left(\mu A+(1-\mu)B-5\right)\right|\right\}.$$

$\square$

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