Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1137 – 1145
© TÜBİTAK
doi:10.3906/mat-1906-63

Gradient Weyl-Ricci Soliton

Cornelia-Livia BEJAN[1,*], **Şemsi EKEN MERİÇ**[2], **Erol KILIÇ**[3]
[1]Department of Mathematics, "Gh. Asachi" Technical University of Iasi, Iasi, Romania
[2]Department of Mathematics, Faculty of Sciences and Arts, Mersin University, Mersin, Turkey
[3]Department of Mathematics, Faculty of Sciences and Arts, İnönü University, Malatya, Turkey

Received: 19.06.2019 • **Accepted/Published Online:** 04.05.2020 • **Final Version:** 08.07.2020

Abstract: The classical notion of gradient Ricci soliton is extended here to the gradient Weyl-Ricci soliton. A Weyl structure of the base manifold M is lifted to its tangent bundle TM, by using the Sasaki metric. We give some necessary and sufficient conditions such that the Weyl structure on TM to be a gradient Weyl-Ricci soliton.

Key words: Weyl structure, Ricci soliton, Sasaki metric

1. Introduction

Over the years, the notion of conformality has shown its importance in both mathematics and theoretical physics. A Weyl structure on a manifold M consists of a class of conformal Riemannian metrics $[g]$ on M, and the Weyl connection, which is a special torsion-free connection that preserves the conformal class $[g]$. The Weyl connection is an example of a non-Riemannian connection. The role played by the Weyl connection for the Weyl structure is, in a sense, similar with the role played by the Levi-Civita connection for a Riemannian metric. Roughly speaking, a Weyl manifold is a conformal manifold equipped with a Weyl connection which is a torsion-free connection preserving the conformal structure. The physical motivation and some historical notes for which H. Weyl introduced Weyl's structure (mainly as a generalization of Riemannian geometry), are described in [8].

In the present paper we focus on the tangent bundle TM of a manifold M, which proves to be rich in geometrical structures. One of the most used Riemannian metric on the total space of TM is the Sasaki metric, introduced by Sasaki in [14]. Since the Sasaki metric is rather rigid, several extensions of the Sasaki metric were constructed on TM. We recall here only some, including those obtained by Abbassi and Sarih in [1, 2], Janyska [10], Kowalski and Sekizawa [11], Oproiu and Papaghiuc [13], Bejan and Druta-Romaniuc [5].

A Weyl manifold is said to be Einstein-Weyl if the symmetric part of the Ricci tensor is proportional to the conformal metric. In particular, Einstein-Weyl manifolds appear as the natural background for static Yang-Mills-Higgs theory. In [6], Bejan and Gül first obtain the behavior of the Sasaki metric on TM under the gauge transformations of the metrics in the conformal class $[g]$ and then the authors characterize (in terms of the Sasaki metric) both Weyl structures on M and on TM to be simultaneously Einstein-Weyl. The Weyl structures were previously studied by the first author in [3].

*Correspondence: bejanliv@yahoo.com
2010 *AMS Mathematics Subject Classification:* 53C25, 53C05
Dedicated to the memory of Professor Vasile Oproiu (1941-2020)

The notion of Ricci solitons generalizes the one of Einstein manifolds. In the present paper, we first introduce a new notion, namely the gradient Weyl-Ricci soliton (see Definition 2.1). Then we deal with the vertical and horizontal lifts from the base manifold M to its tangent bundle TM. As it was noticed in [15], an interesting research task is to compare the geometric structures between two manifolds. In our work, starting with a Weyl structure on the base manifold M, we construct a Weyl structure on the total space of the tangent bundle whose conformal class of metrics contains the Sasaki metric on TM. We provide here the expression of the Ricci tensor field of the Weyl structure on TM and also that of its symmetrical part. Our main result (see Theorem 3.9) characterizes (in terms of Sasaki metric) the Weyl structure on TM to be a gradient Weyl-Ricci soliton.

2. Preliminaries

Let $(M,[g])$ be an m-dimensional manifold endowed with a conformal class of Riemannian metrics. A Weyl connection is defined as a torsion-free connection D preserving the conformal class $[g]$. If we fix a Riemannian metric $g\in[g]$, then D determines a 1-form ω by $Dg=-2w\otimes g$ and conversely, D is determined by a 1-form ω as follows:

$$D_XY=\nabla_XY+\omega(Y)X+\omega(X)Y-g(X,Y)\xi, \forall X,Y\in\Gamma(TM), \tag{2.1}$$

where ∇ is the Levi-Civita connection of g and ξ is the dual vector field of ω with respect to g (i.e. $\omega=g(\xi,.)$).

It follows that the squared length $\|\xi\|^2$ of ξ with respect to g is given by $\|\xi\|^2=g(\xi,\xi)=\omega(\xi)$.

The Weyl connection D is called closed (resp. exact) according as ω is closed (resp. exact). This definition is independent of the conformal change of the Riemannian metric g, since any conformal change $e\to e^{2\lambda}g$ determines the following transformation $x\to\omega-d\lambda$, which shows that ω is closed (resp. exact) if and only if $\omega-d\lambda$ is so.

Let $R_g=[\nabla,\nabla]-\nabla_{[,]}$ and $R_{[g]}=[D,D]-D_{[,]}$ be the curvature tensor fields of the Levi-Civita connection of ∇ and the Weyl connection D, respectively. Hence, they are related by:

$$\begin{aligned} R_{[g]}(X,Y) \;=\;& R_g(X,Y)Z+d\omega(X,Y)Z-((\nabla_Y\omega)(Z))X+((\nabla_X\omega)(Z))Y\\ &+\omega(Y)\omega(Z)X-g(Y,Z)\nabla_X\xi-g(Y,Z)\omega(\xi)X\\ &+g(Y,Z)\omega(X)\xi-\omega(X)\omega(Z)Y+g(X,Z)\nabla_Y\xi\\ &+g(X,Z)\omega(\xi)Y-g(X,Z)\omega(Y)\xi, \quad \forall X,Y,Z\in\Gamma(TM). \end{aligned}$$

Consequently, the relation between the Ricci tensor fields Ric_g and $Ric_{[g]}$ of the Levi-Civita connection ∇ and respectively the Weyl connection D is given by

$$\begin{aligned} Ric_{[g]}(X,Y) \;=\;& Ric_g(X,Y)+d\omega(X,Y)+(\delta\omega-(m-2)\|\xi\|^2)g(X,Y)\\ &-(m-2)(\nabla_X\omega)Y+(m-2)\omega(X)\omega(Y), \forall X,Y,Z\in\Gamma(TM). \end{aligned}$$

where the co-differential $\delta\omega$ of ω is defined by

$$\delta\omega=-trace_g\{(U,V)\to(\nabla_U\omega)V\}.$$

If $Ric_{[g]}^{sym}$ denotes the symmetrical part of $Ric_{[g]}$, then

$$\begin{aligned} Ric_{[g]}^{sym}(X,Y) &= Ric_g(X,Y) + (\delta\omega - (m-2)\|\xi\|^2)g(X,Y) \\ &-\frac{1}{2}(m-2)[(\nabla_X\omega)Y + (\nabla_Y\omega)X] \\ &+(m-2)\omega(X)\omega(Y), \quad \forall X,Y \in \Gamma(TM). \end{aligned} \tag{2.2}$$

To introduce our main notion, we denote as usually the Hessian (with respect to the metric g) of any smooth function f on M, by

$$(Hess_g f)(X,Y) = XYf - (\nabla_X Y)f, \quad \forall X,Y \in \Gamma(TM) \tag{2.3}$$

Definition 2.1 *We define a gradient Weyl-Ricci soliton to be a manifold M endowed with a triple (g,ω,f), where (g,ω) is a Weyl structure with the Weyl connection D and f is a smooth function satisfying*

$$Ric_{[g]}^{sym} + Hess_g(f) = \alpha g, \tag{2.4}$$

for some real function α.

The gradient Weyl-Ricci soliton equation we defined above is a conformally invariant generalization of the gradient Ricci soliton introduced by R. S. Hamilton in [9].

Remark 2.2

(i) If in the above definiton f vanishes identically, then one obtains the notion of Einstein-Weyl manifold (which provides a natural generalization of Einstein geometry), see [7].

(ii) Since not every Weyl connection is Levi-Civita, it follows that gradient Weyl-Ricci solitons provide a natural generalization of gradient Ricci solitons.

3. The Tangent Bundle Carrying the Sasaki Metric

To fix notations, let (x^i) be the local coordinates on any Riemannian manifold (M,g) and let (x^i,y^i) be the induced local coordinates on its tangent bundle $\pi : TM \to M$. By using vertical and horizontal lifts, one can lift some geometric objects from the base manifold M to its tangent bundle TM (see [16]), as follows:

(i) For any smooth function f on M, the vertical lift f^v of f is defined by $f^v = f \circ \pi$.

(ii) For any vector field X on M, (given locally by $X = X^i \frac{\partial}{\partial x^i}$), its vertical and horizontal lifts X^v and X^h are given locally on TM, respectively by $X^v = X^i \frac{\partial}{\partial y^i}$ and $X^h = X^i \frac{\partial}{\partial x^i} - y^j \Gamma^i_{jk} X^k \frac{\partial}{\partial y^k}$, where Γ^i_{jk} are the Christoffel symbols of the Levi-Civita connection of g.

Under the notation (i) and (ii), one has

$$X^v f^v = 0, \qquad X^h f^v = (Xf)^v, \tag{3.1}$$

(see [16]).

Convention: Note that if not otherwise stated, all functions on M are identified with their vertical lift on TM.

From [14], if (M,g) is a Riemannian manifold, then the Sasaki metric G on TM is defined at any point $(x,u)\in TM$ by

$$\begin{aligned} G_{(x,u)}(X^h,Y^h) &= (g_x(X,Y))^v = G_{(x,u)}(X^v,Y^v), \\ G_{(x,u)}(X^h,Y^v) &= 0, \quad \forall X,Y\in\Gamma(TM). \end{aligned} \tag{3.2}$$

Hence, at any point $(x,u)\in TM$, the curvature tensor field $\bar{R}_G$ of the metric G is related to the curvature tensor field R_g of the metric g on M (see [4]) by

$$\begin{aligned}
\bar{R}_G(X^h,Y^h)Z^h &= (R_g(X,Y)Z)^h + \frac{1}{4}\Big[R_g(u,R_g(X,Z)u)Y \\
&\quad - R_g(u,R_g(Y,Z)u)X + 2R_g(u,R_g(X,Y)u)Z\Big]^h \\
&\quad + \frac{1}{2}\Big[(\nabla_Z R_g)(X,Y)u\Big]^v, \\
\bar{R}_G(X^h,Y^h)Z^v &= \Big[R_g(X,Y)Z + \frac{1}{4}R_g(Y,R_g(u,Z)X)u \\
&\quad - \frac{1}{4}R_g(X,R_g(u,Z)Y)u\Big]^v + \frac{1}{2}\Big[(\nabla_X R_g)(u,Z)Y \\
&\quad - (\nabla_Y R_g)(u,Z)X\Big]^h, \\
\bar{R}_G(X^h,Y^v)Z^h &= \frac{1}{2}\big[(\nabla_X R_g)(u,Y)Z\big]^h + \frac{1}{2}\big[R_g(X,Z)Y \\
&\quad - \frac{1}{2}R_g(X,R_g(u,Y)Z)u\big]^v, \\
\bar{R}_G(X^h,Y^v)Z^v &= -\frac{1}{2}\big[R_g(Y,Z)X\big]^h - \frac{1}{4}\big[R_g(u,Y)R_g(u,Z)X\big]^h, \\
\bar{R}_G(X^v,Y^v)Z^h &= (R_g(X,Y)Z)^h + \frac{1}{4}\Big[R_g(u,X)R_g(u,Y)Z \\
&\quad - R_g(u,Y)R_g(u,X)Z\Big]^h, \\
\bar{R}_G(X^v,Y^v)Z^v &= 0, \quad \forall X,Y,Z\in\Gamma(TM).
\end{aligned} \tag{3.3}$$

Let $Ric_g(X,Y)=\sum_{i=1}^m g(R_g(X,e_i)e_i,Y)$ be the Ricci tensor field of (M,g) and similarly, let $\overline{Ric}_G$ be the Ricci tensor field of (TM,G), where $\{e_i\}_{i=\overline{1,m}}$ is an orthonormal frame around an arbitrary point $x\in M$. Since $\{E_1=e_1^h,...,E_m=e_m^h,E_{m+1}=e_1^v,...,E_{2m}=e_m^v\}$ is an orthonormal frame around $(x,u)\in TM$, one has

$$\begin{aligned} \overline{Ric}_G(U,V) &= \sum_{i=1}^{2m} G(\bar{R}_G(U,E_i)E_i,V) \\ &= \sum_{i=1}^{m} G(\bar{R}_G(U,e_i^h)e_i^h,V) + \sum_{i=1}^{m} G(\bar{R}_G(U,e_i^v)e_i^v,V), \forall U,V\in\Gamma(TM). \end{aligned}$$

Hence, from (3.2) and (3.3), it follows that at any point $(x,u) \in TM$, the Ricci curvature $\overline{Ric}_G$ of G is related by the Ricci curvature Ric_g of g on M by

$$\begin{aligned} \overline{Ric}_G(X^h,Y^h) &= \Big[Ric_g(X,Y) - \frac{3}{4}\sum_{i=1}^{m} g(R_g(X,e_i)u, R_g(Y,e_i)u) \\ &\quad -\frac{1}{4}\sum_{i=1}^{m} R_g(u,e_i,R_g(u,e_i)X,Y)\Big]^v \\ &= \Big[Ric_g(X,Y) - \frac{3}{4}\sum_{i=1}^{m} g(R_g(X,e_i)u, R_g(Y,e_i)u) \\ &\quad +\frac{1}{4}\sum_{i=1}^{m} g(R_g(u,e_i)X, R_g(u,e_i)Y)\Big]^v \end{aligned} \tag{3.4}$$

$$\overline{Ric}_G(X^v,Y^v) = \Big[\frac{1}{4}\sum_{i=1}^{m} g(R_g(u,X)e_i, R_g(u,Y)e_i)\Big]^v, \forall X,Y \in \Gamma(TM). \tag{3.5}$$

Proposition 3.1 *[6] The Sasaki metrics on TM, corresponding to any representative of the conformal class $[g]$ on M, form a class which is invariant under the vertical conformal change. That is, if g is a metric on the manifold M and G is its corresponding Sasaki metric on TM, then to any conformal change $g \to e^{\lambda}g$ on M, will correspond the change of the Sasaki metric $G \to (e^{\lambda})^v G$ on TM.*

Lemma 3.2 *[6] Let M be an $m-$dimensional manifold $(m>2)$ endowed with the Weyl structure (g,ω) and let G be the Sasaki metric on TM induced by g. Then:*

(i) (G,ω^v) is the induced Weyl structure on TM;

(ii) The symmetric part $\overline{Ric}^{sym}_{[G]}$ of the Ricci tensor field of the Weyl structure (G,ω^v) on TM satisfies:

$$\begin{aligned} \overline{Ric}^{sym}_{[G]}(X^h,Y^h) &= \overline{Ric}_G(X^h,Y^h) - \frac{m}{m-2}\delta\omega g(X,Y) \\ &\quad + \frac{2(m-1)}{m-2}(Ric^{sym}_{[g]}(X,Y) - Ric_g(X,Y)) \end{aligned} \tag{3.6}$$

$$\overline{Ric}^{sym}_{[G]}(X^v,Y^h) = \frac{1}{2}\sum_{i=1}^{m} g((\nabla_{e_i}R)(X,e_i)u,Y) + \frac{1}{2}\omega(R(u,X)Y), \tag{3.7}$$

$$\begin{aligned} \overline{Ric}^{sym}_{[G]}(X^v,Y^v) &= \frac{1}{4}\sum_{i=1}^{m} g(R(u,X)e_i, R(u,Y)e_i) \\ &\quad +(\delta\omega - 2(m-1)\|\xi\|^2)g(X,Y), \forall X,Y \in \Gamma(TM), \end{aligned} \tag{3.8}$$

where $\{e_i\}_{i=\overline{1,m}}$*,* $\nabla, R, Ric_g, \overline{Ric}_G, Ric^{sym}_{[g]}$ *are respectively an orthonormal frame with respect to* g*, the Levi-Civita connection of* g*, the curvature of* g*, the Ricci tensor field of* g *and* G*, and the symmetric part of* $Ric_{[g]}$*, (here we used the above convention).*

Lemma 3.3 *Let* f *be a smooth function on a Riemannian manifold* (M,g)*. Then, the Hessian (with respect to the Sasaki metric* G*) of its vertical lift is expressed by:*

$$\begin{aligned} Hess_G f^v(X^h,Y^h) &= X^hY^hf^v - (\nabla_{X^h}Y^h)f^v \\ &= X^hY^hf^v - (\nabla_XY)^hf^v + \frac{1}{2}(R_g(X,Y)u)^vf^v \\ &= (XYf)^v - ((\nabla_XY)f)^v + \frac{1}{2}(R_g(X,Y)u)^vf^v \\ &= (Hess_gf(X,Y))^v \end{aligned}$$

$$\begin{aligned} Hess_G f^v(X^h,Y^v) &= X^hY^vf^v - (\nabla_{X^h}Y^v)f^v \\ &= -(\nabla_XY)^vf^v - \frac{1}{2}(R_g(u,Y)X)^hf^v \\ &= -\frac{1}{2}((R_g(u,Y)X)f)^v \end{aligned} \tag{3.9}$$

$$\begin{aligned} Hess_G f^v(X^v,Y^h) &= X^vY^hf^v - (\nabla_{X^v}Y^h)f^v \\ &= X^v(Yf)^v - \frac{1}{2}(R_g(u,X)Y)^hf^v \\ &= -\frac{1}{2}((R_g(u,X)Y)f)^v \end{aligned}$$

$$Hess_G f^v(X^v,Y^v) = X^vY^vf^v - (\nabla_{X^v}Y^v)f^v = 0.$$

Now we obtain our main result:

Theorem 3.4 *Let* M *be an* m*-dimensional manifold endowed with a Weyl structure* (g,ω) *whose induced Weyl structure on* TM *is* (G,ω^v)*. For any smooth function* f *on* M*, the triple* (G,ω^v,f^v) *is a gradient Weyl-Ricci soliton on* TM *if and only if* (M,g) *is flat and*

$$\begin{aligned} Hess_gf(X,Y) &= (m-1)\{(\nabla_X\omega)Y + (\nabla_Y\omega)X - 2\omega(X)\omega(Y)\}, \\ & \quad \forall X,Y \in \Gamma(TM). \end{aligned} \tag{3.10}$$

Proof Suppose that the triple (G,ω^v,f^v) is a gradient Weyl-Ricci soliton, that is

$$\overline{Ric}^{sym}_{[G]} + Hess_Gf^v = \bar{\alpha}G. \tag{3.11}$$

If in (3.8), (3.6), (3.4) and (3.5) we take $X = Y = e_j, j \in \{1, ..., m\}$, where $\{e_i\}_{i=\overline{1,m}}$ is an orthonormal basis on (M, g) around any point $x \in M$, then at any point $(x, u) \in TM$ we obtain:

$$\begin{aligned}
\bar{\alpha} &= \frac{1}{4}\sum_{i=1}^{m} \|R_g(u, e_j)e_i\|^2 + \delta\omega - 2(m-1)\|\xi\|^2 \\
&= \overline{Ric}_G(e_j^h, e_j^h) - \frac{m}{m-2}\delta\omega + \frac{2(m-1)}{m-2}\Big[Ric^{sym}_{[g]}(e_j, e_j) - Ric_g(e_j, e_j)\Big] \\
&\quad + Hess_G f^v(e_j^h, e_j^h) \\
&= \frac{-m}{m-2}Ric_g(e_j, e_j) - \frac{3}{4}\sum_{i=1}^{m} \|R_g(e_j, e_i)u\|^2 + \frac{1}{4}\sum_{i=1}^{m} \|R_g(u, e_i)e_j\|^2 \\
&\quad - \frac{m}{m-2}\delta\omega + \frac{2(m-1)}{m-2}Ric^{sym}_{[g]}(e_j, e_j) + Hess_g f(e_j, e_j), \\
&\quad \text{(no summation over j)}, \quad j \in \{1, ..., m\}.
\end{aligned}$$

From the last equalities, by restricting to the zero section of TM, it follows:

$$\begin{aligned}
Ric^{sym}_{[g]}(e_j, e_j) &= \frac{m-2}{2(m-1)}\Big\{\delta\omega - 2(m-1)\|\xi\|^2 + \frac{m}{m-2}Ric_g(e_j, e_j) \\
&\quad + \frac{m}{m-2}\delta\omega - Hess_g f(e_j, e_j)\Big\}, \\
&\quad \text{(no summation over j)}, \quad j \in \{1, ..., m\}.
\end{aligned}$$

Hence

$$\begin{aligned}
Ric^{sym}_{[g]}(e_j, e_j) &= \delta\omega - (m-2)\|\xi\|^2 + \frac{m}{2(m-1)}Ric_g(e_j, e_j) \\
&\quad - \frac{m-2}{2(m-1)}Hess_g f(e_j, e_j), \\
&\quad \text{(no summation over j)}, \quad j \in \{1, ..., m\}.
\end{aligned} \tag{3.12}$$

From the last equality and the above expressions of $\bar{\alpha}$, it follows that

$$\sum_{i,j=1}^{m} \|R_g(e_i, e_j)u\|^2 = 0,$$

for any $(x, u) \in TM$. If we replace u by e_k and summing over k, we obtain

$$\sum_{i,j,k=1}^{m} \|R_g(e_i, e_j)e_k\|^2 = 0,$$

which gives us $R_g(e_i, e_j)e_k = 0$, $\forall i, j, k \in \{1, 2, ..., m\}$. Therefore (M, g) is flat.

If we replace (3.12) in (2.2), we obtain (3.10).

Conversely, if we suppose (3.10), then from (2.2) we deduce:

$$\begin{aligned} Ric^{sym}_{[g]}(X,Y) &= Ric_g(X,Y) + (\delta w - (m-2)\|\xi\|^2)g(X,Y) \\ &\quad - \frac{m-2}{2(m-1)} Hess_g f(X,Y), \forall X,Y \in \Gamma(TM). \end{aligned} \tag{3.13}$$

Assuming that (M,g) is flat, then from (3.13), the relations (3.6)-(3.8) reduce to

$$\begin{aligned} &\overline{Ric}^{sym}_{[G]}(X^h,Y^h) = (\delta w - 2(m-1)\|\xi\|^2)g(X,Y) - Hess_g f(X,Y); \\ &\overline{Ric}^{sym}_{[G]}(X^v,Y^h) = 0; \\ &\overline{Ric}^{sym}_{[G]}(X^v,Y^v) = (\delta w - 2(m-1)\|\xi\|^2)g(X,Y), \forall X,Y \in \Gamma(TM). \end{aligned} \tag{3.14}$$

From (3.9) and (3.14) we obtain (3.11), where

$$\bar{\alpha} = \delta w - 2(m-1)\|\xi\|^2,$$

which complete the proof. □

Acknowledgment

This work is supported by 1001-Scientific and Technological Research Projects Funding Program of TUBITAK project number 117F434.

References

[1] Abbassi MTK, Sarih M. On Riemannian g-natural metrics of the form $a.gs + b.gh + c.gv$ on the tangent bundle of a Riemannian manifold (M,g). Mediterranean Journal of Mathematics 2005; 2 (1): 19-43. doi:10.1007/s00009-005-0028-8

[2] Abbassi MTK, Sarih M. On some herditary properties of Riemannian g-natural metrics on tangent bundles of Riemannian manifolds. Differential Geometry and its Applications 2005; 22 (1): 19-47. doi.org/10.1016/j.difgeo.2004.07.003

[3] Bejan CL, Chiriac NC. Weyl structures on almost paracontact manifolds. International Journal of Geometric Methods and Modern Physics 2013; 10 (1): 1220019 (9 pages). doi.org/10.1142/S0219887812200198

[4] Bejan CL, Crasmareanu M. Weakly-symmetry of the Sasakian lifts on tangent bundles. Publicationes Mathematicae Debrecen, 2013; 83, 1-2 (4): 63-69.

[5] Bejan CL, Druta-Romaniuc SL. Connections which are harmonic with respect to general natural metrics. Differential Geometry and its Applications 2012; 30 (4): 306-317. doi.org/10.1016/j.difgeo.2012.05.004

[6] Bejan CL, Gül İ. Sasaki metric on the tangent bundle of a Weyl manifold. Publications de I'Institut Mathematique (Beograd) (N.S.) 2018; 103 (117): 25-32. doi.org/10.2298/PIM1817025B

[7] Calderbank D, Pedersen H. Einstein-Weyl geometry; in: Essays on Einstein manifolds (LeBrun and Wang editors). Surveys in Differential Geometry 2001; 6: 387-423. doi: 10.4310/SDG.2001.v6.n1.a14

[8] Folland G. Weyl manifold. Journal of Differential Geometry 1970; 4 (2): 145-153. doi:10.4310/jdg/1214429379

[9] Hamilton RS. The Formation of Singularities in the Ricci Flow. Surveys in Differential Geometry, Vol. II (Cambridge, MA, 1993) 7–136, International Press, Cambridge, MA, 1995.

[10] Janyska J. Natural vector fields and 2-vector fields on the tangent bundle of a pseudo-Riemannian manifold. Archivum Mathematicum (Brno) 2001; 37 (2): 143-160. doi: 10338.dmlcz/107794

[11] Kowalski O, Sekizawa M. Natural transformations of Riemannian metrics on manifolds to metrics on the tangent bundles-a classification. Bulletin of Tokyo Gakugei University 1988; 40 (4): 1-29.

[12] Lee SD, Kim BH, Choi JH. Warped product spaces with Ricci conditions. Turkish Journal of Mathematics 2017; 41 (6): 1365-1375. doi: 10.3906/mat-1606-49

[13] Oproiu V, Papaghiuc N. General natural Einstein Kähler structures on tangent bundles. Differential Geometry and its Applications 2009; 27 (3): 384-392. doi.org/10.1016/j.difgeo.2008.10.017

[14] Sasaki S. On the differential geometry of tangent bundles of Riemannian manifolds. Tôhoku Mathematical Journal 1958; 10 (3): 338-354. doi:10.2748/tmj/1178244668

[15] Şahin B. Biharmonic Riemannian maps, Annales Polonici Mathematici; 2011: 102 (1): 39-49. doi: 10.4064/ap102-1-4

[16] Yano K, Ishihara S. Tangent and Cotangent Bundles, Marcel Dekker, Inc. New York, 1973.