An extragradient algorithm for split generalized equilibrium problem and the set of fixed points of quasi-φ-nonexpansive mappings in Banach spaces

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Abstract: In this paper, we study the problem of finding a common solution to split generalized mixed equilibrium problem and fixed point problem for quasi-φ-nonexpansive mappings in 2-uniformly convex and uniformly smooth Banach space E_1 and a smooth, strictly convex, and reflexive Banach space E_2. An iterative algorithm with Armijo linesearch rule for solving the problem is presented and its strong convergence theorem is established. The convergence result is obtained without using the hybrid method which is mostly used when strong convergence is desired. Finally, numerical experiments are presented to demonstrate the practicability, efficiency, and performance of our algorithm in comparison with other existing algorithms in the literature. Our results extend and improve many recent results in this direction.

Key words: Split generalized mixed equilibrium problem, monotone mapping, strong convergence, Banach space, quasi-phi-nonexpansive mapping, linesearch rule

1. Introduction

Let E be a Banach space and C be a nonempty, closed, and convex subset of E. Let $T : C \to C$ be a nonlinear mapping, a point $x \in C$ is called a fixed point of T if $x = Tx$. We denote the set of fixed points of T by $F(T)$.

The split feasibility problem (SFP) in the sense of Censor and Elfving [10] is to find a point

$$\bar{x} \in C \ \text{ such that } \ \bar{y} = L\bar{x} \in Q,$$

where C and Q are nonempty, closed, and convex subsets of real Banach spaces E_1 and E_2 respectively and L is a bounded linear operator from E_1 to E_2. The SFP provides a unified framework for the study of many important real-life problems such as signal processing, medical image reconstruction, intensity modulated radiation therapy and so on ([3, 6, 22]).

For solving the SFP in finite dimensional Euclidean spaces, Bryne [6] proposed a CQ algorithm defined as follows:

$$x^{k+1} = P_C(x^k + \mu L^T (P_Q - I) L x^k),$$

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where I is the identity mapping, P_C and P_Q are projections onto C and Q respectively and L^T is the transpose of L. However, in the infinite dimensional Hilbert spaces, Xu [45] proposed a CQ algorithm given by:

$$x^{k+1} = P_C(x^k + \mu L^*(P_Q - I)Lv^k),$$

where $\mu \in (0, \frac{2}{||L||^2})$ and L^* is the adjoint of L. The SFP where C and Q are sets of fixed points or sets of common fixed points of mappings and solutions of variational inequality problems have been studied in some recent research papers, see for example [16].

Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction $\Psi : C \to E^*$ nonlinear mapping and $\varphi : C \to \mathbb{R}$ be a real-valued function, where C is a nonempty, closed and convex subset of a real Banach space E with E^* its dual. We consider the following generalized mixed equilibrium problem (GMEP):

Find $x \in C$ such that

$$\Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \quad y \in C.$$

The set of such $x \in C$ is denoted by $GMEP(\Theta, \Psi, \varphi)$ i.e.

$$GMEP(\Theta, \Psi, \varphi) = \{x \in C : \Theta(x, y) + \langle \Psi x, y - x \rangle + \varphi(y) \geq \varphi(x), \forall y \in C\}.$$

Similar problems have been extensively studied recently in different frameworks. In the case $\Psi = 0$, the problem (1) reduces to the mixed equilibrium problem (MEP) with solution set $MEP(\Theta, \varphi)$. In the case $\varphi = 0$, then (1) reduces to generalized equilibrium problem (GEP) with solution set $GEP(\Theta, \Psi)$. However, if $\Psi = \varphi = 0$, then problem (1) becomes the classical equilibrium problem in the sense of Blum and Oetlli [5].

The equilibrium problem (EP) is also referred to as Ky Fan inequality since the first result on the existence of its solution was proposed by Fan (see [17]). It has been extensively studied in recent years thanks to its vast applications (see for e.g., [5, 9, 19, 21, 24, 37]). The EP includes as special cases numerous problems in physics, economics, and optimization theory. For approximating the solution of EP and related optimization problems, many authors have proposed several iterative methods (see [11, 20, 23, 26, 33, 38]) and the references therein.

The split generalized mixed equilibrium problem (SGMEP) (see [25, 34]) consists of finding a point $\bar{x} \in C$ such that

$$g_1(\bar{x}, x) + h_1(\bar{x}, x) + \langle \Psi_1 x, x - \bar{x} \rangle + \varphi(x) \geq \varphi(\bar{x}), \quad \forall x \in C \text{ such that}$$

$$\bar{y} = L\bar{x} \text{ solves } g_2(\bar{y}, y) + h_2(\bar{y}, y) + \langle \Psi_2 y, y - \bar{y} \rangle + \varphi(y) \geq \varphi(\bar{y}), \quad \forall y \in Q, \tag{1.1}$$

where $g_1, h_1 : C \times C \to \mathbb{R}$, $g_2, h_2 : Q \times Q \to \mathbb{R}$, $\Psi_1 : C \to E_1^*$, $\Psi_2 : Q \to E_2^*$ are nonlinear mappings, $\varphi : C \to \mathbb{R} \cup \{+\infty\}$, $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ are proper lower semicontinuous and convex functions and $L : E_1 \to E_2$ a bounded linear operator with adjoint $L^* : E_2^* \to E_1^*$. We remark that the bifunctions g_1, g_2, h_1, and h_2 in the SGMEP (1.1) are monotone, and Θ has been written as a sum of two different bifunctions (i.e. $\Theta = g + h$).

In 1980, Cohen [13] introduced a useful tool for obtaining the solutions of some optimization problems, termed the auxiliary problem principle. This was later extended to variational inequality problem (see [14]). In auxiliary problem principle, a sequence $\{x^k\}$ is generated as follows: $x^{k+1} \in C$ is a unique solution of the strongly convex problem

$$\min\{\lambda_k f(x^k, y) + \frac{1}{2}||x^k - y||\}, \tag{1.2}$$

where $\lambda_k > 0$. Mastroeni [31], extended the auxiliary problem principle to EP under the assumptions that the bifuntion f is strongly monotone on $C \times C$ and that f satisfies the following Lipschitz type condition

$$f(x, y) + f(y, z) \geq f(x, z) - c_1||y - x||^2 - c_2||z - y||^2$$

for all x, y, z and $c_1, c_2 > 0$. To dispense with the monotone condition on f, Tran et al. [39], motivated by the research of Antipin [4] used an extrapolation step in each iteration for obtaining the solutions of (1.2). They assumed that f is pseudomonotone on $C \times C$ which is a weaker assumption. They also assumed that $\{y^k\}$ is a solution of (1.2) and the unique solution of the strongly convex problem

$$\min\{\lambda_k f(y^k, y) + \frac{1}{2}||y - x^k||\}$$

is denoted by $\{x^{k+1}\}$.

In recent years, many authors have proposed extragradient algorithms for solving EP in Hilbert spaces where convergence of the proposed algorithms required the bifunction f to satisfy a certain Lipschitz-type condition [32, 39, 40]. Lipschitz-type condition depends on parameters $c_1, c_2 > 0$, which in many cases are unknown or difficult to estimate. To avoid this requirement, several authors have proposed the use of linesearch technique for obtaining convergent algorithms for solving EP (see [32, 39, 40]).

The problem of finding a common element of the set of fixed points of a nonlinear mapping and the solution set of an equilibrium problem have been studied by many authors in the framework of the Hilbert spaces and Banach spaces (see [9, 18, 25, 29, 34]). In solving problems of this type when pseudomonotone bifunction is involved, there have been several works in the framework of Hilbert spaces where linesearch algorithms have been employed. Interest in the use of extragradient and linesearch algorithms for solving these problems is growing in the Banach spaces.

In 2016, Dinh et al. [15] studied the split equilibrium problem involving pseudomonotone and monotone equilibrium bifunctions and fixed point of nonexpansive mappings in real Hilbert spaces. The statement of the problem is given as follows:

Let $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction, $g : Q \times Q \to \mathbb{R}$ a monotone bifunction, $S : C \to C$ and $T : Q \to Q$ nonexpansive mappings. The split equilibrium problem for nonexpansive mappings (SEPNM) consists of finding a point $\bar{x} \in C$ such that

$$\bar{x} \in EP(C, f) \cap F(S) \quad \text{and} \quad L\bar{x} \in EP(Q, g) \cap F(T)$$

where $F(S)$ and $F(T)$ are the sets of fixed points of S and T respectively. In solving the (SEPNM), the authors in [15] proposed an extragradient method incorporated with the Armijo linesearch rule for solving the EP and the Mann method for finding a fixed point of the nonexpansive mapping.

Other results in these directions in real Hilbert space setting include the result of Rattanaseeha et al. (2017) (see [36]) and the references therein.

However, it is well known that most real life problems do not naturally live in Hilbert spaces. Jouymandi and Moradlou [27] extended this study to Banach space setting albeit to a single EP involving a pseudomonotone bifunction. They proposed an extragradient and linesearch algorithm for finding a common element of the set of solutions of an EP and the fixed point of a relatively nonexpansive mapping.

In this paper, inspired and motivated by the ongoing interest and research in this direction, we study a split generalized mixed equilibrium involving three bifunctions, one being a pseudomonotone bifunction and the

other two monotone bifunctions satisfying some mild conditions. We introduce a linesearch algorithm which does not require projection onto the set C_{k+1} for its strong convergence. Using this algorithm, we obtain a strong convergence result for finding a common element of the solutions of EP and the set of fixed points of quasi-φ-nonexpansive mappings in the framework of a 2-uniformly convex and uniformly smooth Banach space E_1 and a smooth, strictly convexü and reflexive Banach space E_2.

2. Preliminaries

In this section, we give some definitions and important results which will be useful in establishing our main results. We denote the weak and strong convergence of a sequence $\{x^k\}$ to a point x by $x^k \rightharpoonup x$ and $x^k \to x$ respectively. Let C be a nonempty, closed and convex subset of a real Banach space E with norm $||\cdot||$ and dual space E^*.

A continuous strictly increasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\varphi(0) = 0$ and $\lim_{t\to\infty} \varphi(t) = \infty$ is called a gauge function. Given a gauge function φ, the mapping $J_\varphi^E : E \to 2^{E^*}$ defined by

$$J_\varphi^E(x) = \{u^* \in E^* : \langle x, u^* \rangle = ||x|| ||u^*||_*, ||u^*||_* = \varphi(||x||)\}$$

is called the duality mapping with gauge function φ. It is known that $J_\varphi^E(x)$ is nonempty for any $x \in E$. In the particular case where $\varphi(t) = t^{p-1}$ where $p > 1$, the duality mapping $J_\varphi^E = J_p^E$ is called the generalized duality mapping from E to 2^{E^*}. For $\varphi(t) = t$, (i.e. $p = 2$), the duality map $J = J_2^E =$ is called the normalized duality map, (see [12, 35]) for more details. Alber [2], introduced a generalized projection operator Π_C which is an analogue of the metric projection $P_C : H \to C$ in the Hilbert space H. The generalized projection $\Pi_C : E \to C$ is defined by

$$\Pi_C(x) = \inf_{y\in C}\{\varphi(y, x), \quad \forall x \in E\}.$$

If H is a real Hilbert space, then $P_C(x) \equiv \Pi_C(x)$.

Consider the Lyapunov functional $\varphi : E \times E \to \mathbb{R}^+$ defined by

$$\varphi(x, y) = ||x||^2 - 2\langle x, Jy \rangle + ||y||^2, \quad \forall x, y \in E.$$

In the real Hilbert space, we observe that $\varphi(x, y) = ||x - y||^2$. It is obvious from the definition of the functional φ that

$$(||x|| - ||y||)^2 \leq \varphi(x, y) \leq (||x|| + ||y||)^2.$$

The functional φ also satisfies the following important properties:

$$\varphi(x, y) = \varphi(x, z) + \varphi(z, y) + 2\langle x - z, Jz - Jy \rangle \tag{2.1}$$

and

$$2\langle x - y, Jz - Jw \rangle = \varphi(x, w) + \varphi(y, z) - \varphi(x, z) - \varphi(y, w). \tag{2.2}$$

Note: If E is a reflexive, strictly convex, and smooth Banach space, then for $x, y \in E$, $\varphi(x, y) = 0$ if and only if $x = y$, see [12].

We also define the functional $V : E \times E^* \to \mathbb{R}$ by

$$V(x, x^*) = ||x||^2 - 2\langle x, x^* \rangle + ||x^*||^2 \tag{2.3}$$

for all $x \in E$ and $x^* \in E^*$. That is, $V(x, x^*) = \varphi(x, J^{-1}x^*)$ for all $x \in E$ and $x^* \in E^*$. It is well known that if E is a reflexive, strictly convex, and smooth Banach space, then

$$V(x, x^*) \leq V(x, x^* + y^*) - 2\langle J^{-1}x^* - x, y^* \rangle \tag{2.4}$$

for all $x \in E$ and all $x^*, y^* \in E^*$, see [42].

Let C be a nonempty, closed, and convex subset of E and $T : C \to C$ be a mapping. A point $p \in C$ is called an asymptotic fixed point of T if C contains a sequence $\{x^k\}$ such that $x^k \rightharpoonup p$ and $||x^k - Tx^k|| \to 0$ as $k \to \infty$. We denote by $\widehat{F(T)}$ the set of asymptotic fixed points of T. A mapping $T : C \to C$ is said to be relatively nonexpansive if $\widehat{F(T)} = F(T)$ and $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$ (see [7, 8, 10]). T is said to be φ-nonexpansive if $\varphi(Tx, Ty) \leq \varphi(x, y)$ for all $x, y \in C$ and quasi-φ-nonexpansive if $F(T) \neq \emptyset$ and $\varphi(p, Tx) \leq \varphi(p, x)$ for all $x \in C$ and $p \in F(T)$.

It is known that the class of quasi-φ-nonexpansive mappings is more general than the class of relatively nonexpansive mapping which requires the strict condition $F(T) = \widehat{F(T)}$, see ([7, 8, 10]).

Let E be a real Banach space. The modulus of convexity of E is the function $\delta_E : (0, 2] \to [0, 1]$ defined by

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, ||x - y|| \geq \epsilon\}. \tag{2.5}$$

Recall that E is said to be uniformly convex if $\delta_E(\epsilon) > 0$ for any $\epsilon \in (0, 2]$. E is said to be strictly convex if $\frac{||x + y||}{2} < 1$ for all $x, y \in E$ with $||x|| = ||y|| = 1$ and $x \neq y$. Also, E is p-uniformly convex if there exists a constant $c_p > 0$ such that $\delta_E(\epsilon) > c_p \epsilon^p$ for any $\epsilon \in (0, 2]$.

The modulus of smoothness of E is the function $\rho_E : \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$\rho_E(t) = \sup\{\frac{1}{2}(||x + ty|| - ||x - ty||) - 1 : ||x|| = ||y|| = 1\}. \tag{2.6}$$

E is said to be uniformly smooth if $\lim_{t \to 0} \frac{\rho_E(t)}{t} = 0$. Let $1 < q \leq 2$, then E is q-uniformly smooth if there exists $c_q > 0$ such that $\rho_E(t) \leq c_q t^q$ for $t > 0$. It is known that E is p-uniformly convex if and only if E^* is q-uniformly smooth, where $(p^{-1} + q^{-1} = 1)$. It is also known that every q-uniformly smooth Banach space is uniformly smooth.

It is widely known that if E is uniformly smooth, then the duality mapping J is norm-to-norm continuous on each bounded subset of E. For more properties of J, (see [1, 12, 35]). We now give the following useful and important lemmas that are needed in establishing our main results.

Lemma 2.1 *[44] Given a number $s > 0$. A real Banach space E is uniformly convex if and only if there exists a continuous strictly increasing function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||tx + (1 - t)y||^2 \leq t||x||^2 + (1 - t)||y||^2 - t(1 - t)g(||x - y||),$$

for all $x, y \in E$, $t \in [0, 1]$, with $||x|| < s$ and $||y|| < s$.

Lemma 2.2 *[28] Let E be a smooth and uniformly convex real Banach space and let $\{x^k\}$ and $\{y^k\}$ be two sequences in E. If either $\{x^k\}$ or $\{y^k\}$ is bounded and $\varphi(x^k, y^k) \to 0$ as $k \to \infty$, then $||x^k - y^k|| \to 0$ as $k \to \infty$.*

Lemma 2.3 *[2] Let C be a nonempty, closed, and convex subset of a reflexive, strictly convex, and smooth Banach space E. If $x \in E$ and $q \in C$, then*

$$q = \Pi_C x \iff \langle y - q, Jx - Jq \rangle \leq 0, \quad \forall y \in C \tag{2.7}$$

and

$$\varphi(y, \Pi_C x) + \varphi(\Pi_C x, x) \leq \varphi(y, x), \ \forall y \in C, \ x \in E. \tag{2.8}$$

Lemma 2.4 *[44] Let E be a 2-uniformly convex and smooth Banach space. Then, for all $x, y \in E$, we have*

$$||x - y|| \leq \frac{2}{c^2} ||Jx - Jy|| \tag{2.9}$$

where $\dfrac{1}{c}$, $c \in (0, 1]$ is the 2-uniformly convex constant of E.

Lemma 2.5 *[44] Let E be a 2-uniformly smooth Banach space with the best smoothness constant $d > 0$. Then, the following inequality holds:*

$$||x + y||^2 \leq ||x||^2 + \langle y, Jx \rangle + 2||dy||^2, \quad \forall x, y \in E.$$

Lemma 2.6 *[43] Let $\{a^k\}$ be a sequence of nonnegative real numbers satisfying the following relation*

$$a^{k+1} \leq (1 - b^k)a^k + b^k c^k, \quad k \geq 0$$

where $\{b^k\} \subset (0, 1)$ and $\{c^k\} \subset \mathbb{R}$ satisfy the conditions $\sum_{k=0}^{\infty} b^k = \infty$ and $\limsup_{k \to \infty} c^k \leq 0$. Then, $\lim_{k \to \infty} a^k = 0$.

Lemma 2.7 *[30] Let $\{a^k\}$ be a sequence of real numbers such that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that $a^{n_j} < a^{n_j+1}$ for all $j \in \mathbb{N}$. Then, there exists a nondecreasing subsequence $\{m^k\} \subset \mathbb{N}$ such that $m^k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$: $a^{m^k} < a^{m^k+1}$ and $a^k < a^{m^k+1}$. In fact, $m^k = \max\{i \leq k : a^i < a^{i+1}\}$.*

For solving the EPs, we will assume the bifunctions f, g and h satisfy the following:

Assumption A: The bifunction $f : C \times C \to \mathbb{R}$ satisfies the following:

A1. $f(x, x) = 0$ for all $x \in C$;

A2. f is pseudomonotone on C, that is, $f(x, y) \geq 0$ implies $f(y, x) \leq 0$ for all $x, y \in C$;

A3. $f(x, \cdot)$ is convex, lower semicontinuous and subdifferentiable on C for all $x \in C$;

A4. f is jointly weakly continuous on $C \times C$ in the sense that, if $x, y \in C$ and $\{x^k\}$ and $\{y^k\}$ in C converge weakly to x and y, respectively, then $f(x^k, y^k) \to f(x, y)$ as $k \to \infty$.

Assumption B: The bifunction $g : Q \times Q \to \mathbb{R}$ satisfies the following conditions:

B1. $g(x, x) = 0$ for all $x \in Q$;

B2. g is monotone, i.e. $g(x, y) + g(y, x) \leq 0$ for all $x, y \in Q$;

B3. $\limsup\limits_{t \downarrow 0} g(x + t(z - x), y) \leq g(x, y), \quad \forall x, y, z \in Q$;

B4. the function $y \mapsto g(x, y)$ is convex and lower semicontinuous.

Assumption C: The bifunction $h : Q \times Q \to \mathbb{R}$ satisfy the conditions

C1. $h(x, x) \geq 0$, for all $x \in Q$;

C2. for each fixed $y \in Q$, the function $x \mapsto h(x, y)$ is upper semicontinuous;

C3. for each fixed $x \in Q$, the function $y \mapsto h(x, y)$ is convex and lower semicontinuous.

Assumption D: For fixed $r > 0$ and $z \in C$, there exists a nonempty compact convex subset K of E_1 and $x \in C \cap K$ such that

$$f(x, y) + h(y, x) + \frac{1}{r}\langle y - x, Jx - Jz \rangle <, \quad \forall y \in C/K.$$

Lemma 2.8 *[41] Let E_2 be a smooth, strictly convex, and reflexive Banach space and Q be a nonempty, closed, and convex subset of E_2. Let $\Psi : C \to E_2^*$ be a continuous and monotone mapping, $\varphi : Q \to \mathbb{R}$ be a lower semicontinuous and convex function and $g : Q \times Q \to \mathbb{R}$ be a bifunction satisfying the conditions B1-B4. Let $r > 0$ be any given number and $x \in E_2$ be any given point. Then, the following hold:*

i there exists $z \in Q$, such that

$$g(z, y) + \langle \Psi z, y - z \rangle + \varphi(y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq \varphi(z), \quad \forall\, y \in Q;$$

ii the mapping $K_r^{g,h} : E_2 \to Q$ defined by

$$K_r^{g,h}(x) = \{z \in Q : g(z, y) + h(z, y) + \langle \Psi z, y - z \rangle + \varphi(y) + \frac{1}{r}\langle z - y, Jz - Jx \rangle \geq \varphi(z), \quad \forall y \in Q\},$$

$x \in E_2$, has the following properties:

(a) for all $x \in E_2$, $K_r^{g,h}(x) \neq \emptyset$.

(b) $K_r^{g,h}$ is single valued

(c) $K_r^{g,h}$ is firmly nonexpansive-type, i.e.

$$\langle K_r^{g,h}z - K_r^{g,h}y, JK_r^{g,h}z - JK_r^{g,h}y \rangle \leq \langle K_r^{g,h}z - K_r^{g,h}y, Jz - Jy \rangle, \quad \forall \, z, y \in E_2;$$

(d) $F(K_r^{g,h}) = GMEP(g,h,A,Q)$;

(e) $F(K_r^{g,h})$ is closed and convex;

(f) $\varphi(p, K_r^{g,h}z) + \varphi(K_r^{g,h}z, z) \leq \varphi(p, x), \quad \forall p \in F(K_r^{g,h}), \quad z \in E_2.$

Lemma 2.9 *[27] Let $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction satisfying Assumption A. Define $y^k := \min_{y \in C}\{f(x^k, y) + \dfrac{1}{2\rho_k}\varphi(x^k, y)\}$, where $\rho_k \subseteq (\rho, 1)$, $0 < \rho \leq 1$. Then, for every $p \in EP(C, f)$ and $k \in \mathbb{R}$, it holds that*

$$\langle Jx^k - Jy^k, y - y^k \rangle \leq \rho_k[f(x^k, y) - f(x^k, y^k)], \quad \forall y \in C.$$

Lemma 2.10 *Let $S : Q \to Q$ be a quasi-φ-nonexpansive mapping and let $K_r^{g,h} : E_2 \to Q$ be defined as in Lemma 2.8. Then, $F(SK_r^{g,h}) = F(S) \cap F(K_r^{g,h})$.*

Proof   Clearly, $F(S) \cap F(K_r^{g,h}) \subseteq F(SK_r^{g,h})$. We are left to show that $F(SK_r^{g,h}) \subseteq F(S) \cap F(K_r^{g,h})$. Indeed, let $\bar{x} \in F(SK_r^{g,h})$ and $\bar{y} \in F(S) \cap F(K_r^{g,h})$, then

$$
\begin{aligned}
\varphi(\bar{y}, \bar{x}) &= \varphi(\bar{y}, SK_r^{g,h}\bar{x}) \\
&\leq \varphi(\bar{y}, K_r^{g,h}\bar{x}). \tag{2.10}
\end{aligned}
$$

Now by Lemma 2.8(f) and (2.10), we have

$$
\begin{aligned}
\varphi(K_r^{g,h}\bar{x}, \bar{x}) &\leq \varphi(\bar{y}, \bar{x}) - \varphi(\bar{y}, K_r^{g,h}\bar{x}) \\
&\leq \varphi(\bar{y}, \bar{x}) - \varphi(\bar{y}, \bar{x}) \\
&= 0. \tag{2.11}
\end{aligned}
$$

Hence, $\varphi(K_r^{g,h}\bar{x}, \bar{x}) = 0$, and by the strict convexity of E, we obtain $\bar{x} \in F(K_r^{g,h})$. Next we show that $\bar{x} \in F(S)$. Since by assumption $\bar{x} \in F(SK_r^{g,h})$ and $\bar{x} \in F(K_r^{g,h})$, we have

$$
\begin{aligned}
\varphi(\bar{x}, S\bar{x}) &= \varphi(\bar{x}, SK_r^{g,h}\bar{x}) \\
&= \varphi(\bar{x}, \bar{x}) \\
&= 0.
\end{aligned}
$$

Hence, $\bar{x} \in F(S)$. This implies that $\bar{x} \in F(S) \cap F(K_r^{g,h})$. Therefore, $F(SK_r^{g,h}) = F(S) \cap F(K_r^{g,h})$.   □

3. Main result

In this section we prove our main result. Firstly, we explicitly state the problem considered in this paper, then we introduce a linesearch algorithm for obtaining the solution of this problem and finally discuss its convergence analysis.

Let C be a nonempty, closed, and convex subset of a 2-uniformly convex and uniformly smooth Banach space E_1, Q be a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E_2 and $L : E_1 \to E_2$ be a bounded linear operator with $L^* : E_2^* \to E_1^*$ its adjoint. Let $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction satisfying assumptions A. Let $g, h : Q \times Q \to \mathbb{R}$ be two monotone bifunctions satisfying assumptions B, C respectively and D. Let $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $\Psi : Q \to E_2^*$ be a continuous and monotone mapping. Let $T : C \to C$ and $S : Q \to Q$ be quasi-φ-nonexpansive mappings such that $I - T$ and $I - SK_r^{g,h}$ are demiclosed at zero, where $K_r^{g,h}$ is defined as in Lemma 2.8 (ii). We consider the problem of finding a point $p \in C$ such that

$$p \in EP(C, f) \cap F(T) : Lp \in GMEP(Q, g, h, \Psi, \varphi) \cap F(S). \tag{3.1}$$

Assume $\Gamma \neq \emptyset$, where Γ denotes the solution set of problem (3.1). To obtain the solution of (3.1), we consider the following iterative algorithm:

Algorithm 3.1. Pick $x^0, u \in C$ and choose the parameter $\eta, \theta \in (0, 1)$ and suppose that $\{\alpha_k\} \subset [a, e]$ for some $0 < a < e < 1$, $\{\beta_k\} \subseteq [d, b]$ for some $0 < d < b < 1$, $\{\rho_k\} \subseteq [\rho, 1]$, $0 < \rho \leq 1$, $r > 0$, $0 < \gamma_k < \dfrac{c^2}{2}$, where $\dfrac{1}{c}$, $c \in (0, 1]$ is the 2-uniformly convexity constant of E_1 and $\mu \in (0, \dfrac{1}{d^2 ||L||^2})$, d is the best smoothness constant of E_1.

We pick two sequences $\{\alpha_k\}$ and $\{\beta_k\}$ in $(0, 1)$ such that the following conditions are satisfied.

(i) $\lim\limits_{k \to \infty} \alpha_k = 0$, $\sum\limits_{k \to \infty}^{\infty} \alpha_k = \infty$;

(ii) $\liminf\limits_{k \to \infty} \beta_k (1 - \beta_k) > 0$.

For each $k > 0$. Having the k-iterate $\{x^k\}$, compute the following steps:

Step I: Solve the following strongly convex program

$$y^k = CP(x^k) = \arg\min\{f(x^k, y) + \frac{1}{2\rho_k}\varphi(x^k, y) : y \in C\}$$

to obtain its optimal solution y^k. If $y^k = x^k$, then set $u^k = x^k$ and go to Step IV. Otherwise, go to Step II.

Step II: (Armijo linesearch rule) Find the smallest nonnegative number m such that

$$\begin{cases} f(z^{k,m}, x^k) - f(z^{k,m}, y^k) \geq \dfrac{\theta}{2\rho_k}\varphi(y^k, x^k), \\ z^{k,m} = (1 - \eta^m)x^k + \eta^m y^k. \end{cases} \tag{3.2}$$

Set $\eta_k = \eta^{m_k}$, $z^k = z^{k,m}$.

Step III: Take $t^k \in \partial_2 f(z^k, x^k)$ and compute $u^k = \Pi_C J_{E_1}^{-1}(J_{E_1} x^k - \gamma_k \sigma_k t^k)$, where $\sigma_k = \dfrac{f(z^k, x^k)}{||t^k||^2}$.

Step IV: Compute

$$v^k = J_{E_1}^{-1}(\beta_k J_{E_1} T u^k + (1 - \beta_k) J_{E_1} u^k),$$

$$w^k = J_{E_1}^{-1}(J_{E_1} v^k + \mu L^* J_{E_2}(S K_r^{g,h} - I) L v^k),$$

$$x^{k+1} = J_{E_1}^{-1}(\alpha_k J_{E_1} u + (1 - \alpha_k) J_{E_1} w^k). \tag{3.3}$$

Step V: Set $k := k + 1$ and go to step I.

Remark 3.1 *Note that if $x^k = y^k = Tx^k$, $SLx^k = Lx^k$, and $K_r^{g,h} Lx^k = Lx^k$, we have arrived at the desired solution of our proposed problem. However, we will implicitly assume in our convergence analysis that this does not occur after finitely many iterations so that our algorithm generates an infinite sequence.*

Lemma 3.2 *[27] Suppose that $p \in EP(C, f)$, $f(x, \cdot)$ is convex and subdifferentiable on C for all $x \in C$ and that f is pseudomonotone on C. Assume that $y^k \neq x^k$ for some $k \in \mathbb{N}$. Then,*

(i) There exists a positive integer m such that the Armijo linesearch rule (3.2) is well defined;

(ii) $f(z^k, x^k) > 0$;

(iii) $0 \notin \partial_2 f(z^k, x^k)$.

Proposition 3.3 *For all $p \in EP(C, f)$ and all $k \in \mathbb{N}$, we get*

$$\varphi(p, u^k) \leq \varphi(p, x^k) - 2\gamma_k \left(1 - \frac{2}{c^2} \gamma_k\right) \sigma_k^2 ||t^k||^2.$$

Proof Using (2.3), inequality (2.4) and (2.9), we have

$$
\begin{aligned}
\varphi(p, u^k) &= \varphi(\Pi_C p, \Pi_C J_{E_1}^{-1}(J_1 x^k - \gamma_k \sigma_k t^k)) \\
&\leq \varphi(p, J_{E_1}^{-1}(J_{E_1} x^k - \gamma_k \sigma_k t^k)) \\
&= V(p, J_{E_1} x^k - \gamma_k \sigma_k t^k) \\
&\leq V(p, J_{E_1} x^k - \gamma_k \sigma_k t^k + \gamma_k \sigma_k t^k) - 2\langle J_{E_1}^{-1}(J_{E_1} x^k - \gamma_k \sigma_k t^k) - p, \gamma_k \sigma_k t^k \rangle \\
&= V(p, J_{E_1} x^k) - 2\langle J_{E_1}^{-1}(J_{E_1} x^k - \gamma_k \sigma_k t^k) - J_{E_1}^{-1}(J_{E_1} x^k), \gamma_k \sigma_k t^k \rangle - 2\langle x^k - p, \gamma_k \sigma_k t^k \rangle \\
&= \varphi(p, x^k) - 2\langle J_{E_1}^{-1}(J_{E_1} x^k - \gamma_k \sigma_k t^k) - J_{E_1}^{-1}(J_{E_1} x^k), \gamma_k \sigma_k t^k \rangle - 2\langle x^k - p, \gamma_k \sigma_k t^k \rangle. \tag{3.4}
\end{aligned}
$$

Using condition $A2$, we get from $t^k \in \partial_2 f(z^k, x^k)$, that

$$\langle t^k, x^k - p \rangle \geq f(z^k, x^k) - f(z^k, p) \geq f(z^k, x^k) = \sigma_k ||t^k||^2.$$

Therefore, we obtain

$$-2\gamma_k \sigma_k^2 ||t^k||^2 \geq -2\gamma_k \sigma_k \langle t^k, x^k - p \rangle.$$

On the other hand, from Lemma 2.4, we get

$$
2\langle \gamma_k\sigma_k t^k, J_{E_1}^{-1}(J_{E_1}x^k - \gamma_k\sigma_k t^k) - J_{E_1}^{-1}(J_{E_1}x^k)\rangle \;\;\leq\;\; 2||\gamma_k\sigma_k t^k||\,||J_{E_1}^{-1}(J_{E_1}x^k - \gamma_k\sigma_k t^k) - J_{E_1}^{-1}(J_{E_1}x^k)||
$$
$$
\leq \frac{4}{c^2}\gamma_k^2\sigma_k^2||t^k||^2, \tag{3.5}
$$

Thus, from (3.4), we get

$$
\begin{aligned}
\varphi(p,u^k) &= \varphi(x,x^k) - 2\gamma_k\sigma_k^2||t^k||^2 + \frac{4}{c^2}\gamma_k^2\sigma_k^2||t^k||^2 \\[2mm]
&\leq \varphi(p,x^k) - 2\gamma_k\left(1 - \frac{2}{c^2}\gamma_k\right)\sigma_k^2||t^k||^2. \tag{3.6}
\end{aligned}
$$

$\square$

Lemma 3.4 *Let $\{x^k\}$ be defined as in Algorithm 3 with the parameters on it. Then, the sequence $\{x^k\}$ is bounded. Consequently, the sequences $\{v^k\}$, $\{w^k\}$, and $\{z^k\}$ are bounded.*

Proof Let $p \in \Gamma$. Then $p \in EP(C,f) \cap F(T)$ and $Lp \in GMEP(Q,g,h,\Psi,\varphi) \cap F(S)$. From (3.3) and Lemma 2.1, we have

$$
\begin{aligned}
\varphi(p,v^k) &= \varphi(p, J_{E_1}^{-1}((1-\beta_k)J_{E_1}u^k + \beta_k J_{E_1}Tu^k)) \\[1mm]
&= ||p||^2 - 2\langle p, (1-\beta_k)J_{E_1}u^k + \beta_k J_{E_1}Tu^k\rangle + ||(1-\beta_k)J_{E_1}u^k + \beta_k J_{E_1}Tu^k||^2 \\[1mm]
&= ||p||^2 - 2\beta_k\langle p, J_{E_1}Tu^k\rangle - 2(1-\beta_k)\langle p, J_{E_1}u^k\rangle + (1-\beta_k)||u^k||^2 + \beta_k||Tu^k||^2 \\
&\quad -\beta_k(1-\beta_k)g(||J_{E_1}u^k - J_{E_1}Tu^k||) \\[1mm]
&= \beta_k\varphi(p,Tu^k) + (1-\beta_k)\varphi(p,u^k) - \beta_k(1-\beta_k)g(||J_{E_1}u^k - J_{E_1}Tu^k||) \\[1mm]
&\leq \beta_k\varphi(p,u^k) + (1-\beta_k)\varphi(p,u^k) - \beta_k(1-\beta_k)g(||J_{E_1}u^k - J_{E_1}Tu^k||) \tag{3.7} \\[1mm]
&\leq \varphi(p,u^k).
\end{aligned}
$$

Again, from (3.3) and Lemma 2.1, we have

$$
\begin{aligned}
\varphi(p,w^k) &= \varphi(p, J_{E_1}^{-1}(J_{E_1}v^k + \mu L^* J_{E_2}(SK_r^{g,h} - I)Lv^K)) \\[1mm]
&= ||p||^2 - 2\langle p, J_{E_1}v^k + \mu L^* J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle + ||J_{E_1}v^k + \mu L^* J_{E_2}(SK_r^{g,h} - I)Lv^k||^2 \\[1mm]
&= ||p||^2 - 2\langle p, J_{E_1}v^k\rangle - 2\mu\langle Lp, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle + ||v^k||^2 \\
&\quad +2\mu\langle Lv^k, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle + 2\mu^2 d^2||L||^2||(SK_r^{g,h} - I)Lv^k||^2 \\[1mm]
&= \varphi(p,v^k) - 2\mu\langle Lp - Lv^k, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle + 2\mu^2 d^2||L||^2||(SK_r^{g,h} - I)Lv^k||^2.
\end{aligned}
$$

We estimate the second term of the equation above as follows:

$$
\begin{aligned}
\langle Lp - Lv^k, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle &= \langle Lp - SK_r^{g,h}Lv^k, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle + ||(SK_r^{g,h} - I)Lv^k||^2 \\[1mm]
&= ||Lp - SK_r^{g,h}Lv^k||\,||(SK_r^{g,h} - I)Lv^k||^2 + ||(SK_r^{g,h} - I)Lv^k||^2 \\[1mm]
&\geq ||(SK_r^{g,h} - I)Lv^k||^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\varphi(p, w^k) &= \varphi(p, v^k) - 2\mu||(SK_r^{g,h} - I)Lv^k||^2 + 2\mu^2 c^2||L||^2||(SK_r^{g,h} - I)Lv^k||^2 \\
&= \varphi(p, v^k) - 2\mu(1 - d^2||L||^2\mu)||(SK_r^{g,h} - I)Lv^k||^2 \\
&\leq \varphi(p, v^k).
\end{aligned}
\tag{3.8}
$$

Furthermore,

$$
\begin{aligned}
\varphi(p, x^{k+1}) &= \varphi(p, J_{E_1}^{-1}(\alpha_k J_{E_1} u + (1 - \alpha_k)J_{E_1}w^k)) \\
&= ||p||^2 - 2\langle p, \alpha_k J_{E_1}u + (1 - \alpha_k)J_{E_1}w^k\rangle + ||\alpha_k J_{E_1}u + (1 - \alpha_k)J_{E_1}w^k||^2 \\
&= ||p||^2 - 2\alpha_k\langle p, J_{E_1}u\rangle - 2(1 - \alpha_k)\langle p, J_{E_1}w^k\rangle + \alpha_k||u||^2 + (1 - \alpha_k)||w^k||^2 \\
&\quad -\alpha_k(1 - \alpha_k)g(||J_{E_1}w^k - J_{E_1}u||) \\
&\leq \alpha_k\varphi(p, u) + (1 - \alpha_k)\varphi(p, w^k) \\
&\leq \alpha_k\varphi(p, u) + (1 - \alpha_k)\varphi(p, v^k) \\
&\leq \alpha_k\varphi(p, u) + (1 - \alpha_k)\varphi(p, x^k) \\
&\leq \max\{\varphi(p, u), \varphi(p, x^k)\} \\
&\vdots \\
&\leq \max\{\varphi(p, u), \varphi(p, x^0)\} \; k \geq 0,
\end{aligned}
\tag{3.9}
$$

which implies $\{\varphi(p, x^k)\}$ is bounded. Consequently, $\{x^k\}, \{u^k\}, \{v^k\}, \{y^k\}$, and $\{w^k\}$ are bounded. □

Lemma 3.5 *Assume $p \in EP(C, f)$ and $\{x^k\}$ be defined as in Algorithm 3 such that $\{\varphi(p, x^k)\}$ is monotonically nonincreasing for all $k \geq k_0$. Then $\lim_{k\to\infty} \sigma_k^2||t^k||^2 = 0$.*

Proof: By hypothesis, we have that $\{\varphi(p, x^k)\}$ converges as $k \to \infty$ and hence, $\varphi(p, x^k) - \varphi(p, x^{k+1}) \to 0$ as $k \to \infty$.

Observe from (3.7), (3.9), and Proposition 3.3 that

$$
\varphi(p, x^{k+1}) \leq \alpha_k\varphi(p, u) + (1 - \alpha_k)[\varphi(p, x^k) - 2\gamma_k\left(1 - \frac{2}{c^2}\gamma_k\right)\sigma_k^2||t^k||^2],
$$

which implies

$$
2\gamma_k(1 - \alpha_k)\left(1 - \frac{2}{c^2}\gamma_k\right)\sigma_k^2||t^k||^2 \leq \varphi(p, x^k) - \varphi(p, x^{k+1}) - \alpha_k\varphi(p, x^k) + \alpha_k\varphi(p, u).
$$

Therefore,

$$
\lim_{k\to\infty} \sigma_k^2||t^k||^2 = 0.
\tag{3.10}
$$

Lemma 3.6 *Let $\{x^k\}$ be the sequence given by Algorithm 3, given a subsequence of $\{x^{k_j}\}$ of $\{x^k\}$ such that $x^{k_j} \rightharpoonup p$. Then $p \in EP(C, f)$.*

Proof First, we show that $f(z^{k_j}, x^{k_j}) \to 0$ as $j \to \infty$. Indeed, if $x^{k_j} = y^{k_j}$, then we have $f(z^{k_j}, x^{k_j}) = 0$ and $\sigma_{k_j} = 0$. Suppose $x^{k_j} \neq y^{k_j}$, then by the definition of σ_{k_j} and Lemma 3.5, we have that

$$\lim_{j \to \infty} f(z^{k_j}, x^{k_j}) = \lim_{j \to \infty} \sigma_{k_j} ||t^{k_j}||^2 = 0. \tag{3.11}$$

Again, if $y^{k_j} = x^{k_j}$, then it follows from Lemma 2.9, that

$$\rho_{k_j} f(x^{k_j}, y) \geq 0. \tag{3.12}$$

By letting $j \to \infty$ in inequality 3.12 and using (A4), we get $f(p, y) \geq 0$ since $0 < \rho < \rho_{k_j} \leq 1$, $p \in EP(C, f)$. Now, Suppose $y^{k_j} \neq x^{k_j}$, since $f(z^{k_j}, \cdot)$ is convex, we obtain

$$\begin{aligned}
\eta_{k_j} f(z^{k_j}, y^{k_j}) + (1 - \eta_{k_j}) f(z^{k_j}, x^{k_j}) &\geq f(z^{k_j}, \eta_{k_j} y^{k_j} + (1 - \eta_{k_j}) x^{k_j}) \\
&= f(z^{k_j}, z^{k_j}) = 0.
\end{aligned}$$

Therefore,

$$\eta_{k_j} [f(z^{k_j}, x^{k_j}) - f(z^{k_j}, y^{k_j})] \leq f(z^{k_j}, x^{k_j}) \to 0 \quad \text{as } j \to \infty. \tag{3.13}$$

From (3.2) and (3.13), we have

$$\frac{\theta \eta_{k_j}}{2 \rho_{k_j}} \varphi(y^{k_j}, x^{k_j}) \leq \eta_{k_j} [f(z^{k_j}, x^{k_j}) - f(z^{k_j}, y^{k_j})] \leq f(z^{k_j}, x^{k_j}) \to 0. \tag{3.14}$$

Now we consider two cases:

Case 1: $\limsup_{j \to \infty} \eta_{k_j} > 0$. In this case there exists $\bar{\eta} > 0$ and a subsequence of $\{\eta_{k_j}\}$, say $\{\eta_{k_j}\}$ such that $\eta_{k_j} \to \bar{\eta}$, since $0 < \eta < \eta_{k_j} \leq 1$, from (3.14), we can conclude that $\varphi(y^{k_j}, x^{k_j}) \to 0$. Thus, by Lemma 2.2, we have $||y^{k_j} - x^{k_j}|| \to 0$. By this and $x^{k_j} \rightharpoonup p$, we obtain, $y^{k_j} \rightharpoonup p$.

Case 2: $\lim_{j \to \infty} \eta_{k_j} = 0$. From the boundedness of $\{y^{k_j}\}$, without loss of generality, we may assume that $y^{k_j} \rightharpoonup q$ as $j \to \infty$. Let m be the smallest nonnegative integer such that (3.2) is satisfied, i.e.

$$f(z^{k_j, m}, x^{k_j}) - f(z^{k_j, m}, y^{k_j}) \geq \frac{\theta}{2 \rho_{k_j}} \varphi(y^{k_j}, x^{k_j}),$$

where

$$z^{k_j, m} = (1 - \eta^m) x^{k_j} + \eta^m y^{k_j}. \tag{3.15}$$

Thus,

$$f(z^{k_j, m-1}, x^{k_j}) - f(z^{k_j, m-1}, y^{k_j}) < \frac{\theta}{2 \rho_{k_j}} \varphi(y^{k_j}, x^{k_j}). \tag{3.16}$$

If we set $y = x^{k_j}$ in Lemma 2.9, condition (A1) and equality (2.2), imply that

$$-\rho_{k_j} f(x^{k_j}, y^{k_j}) \geq \langle J_{E_1} y^{k_j} - J_{E_1} x^{k_j}, y^{k_j} - x^{k_j} \rangle = \frac{1}{2} \varphi(y^{k_j}, x^{k_j}) + \frac{1}{2} \varphi(x^{k_j}, y^{k_j}).$$

Therefore,

$$\frac{1}{2}\varphi(y^{k_j}, x^{k_j}) \le -\rho_{k_j} f(x^{k_j}, y^{k_j}). \tag{3.17}$$

From inequalities (3.16) and (3.17), we have

$$f(z^{k_j, m-1}, x^{k_j}) - f(z^{k_j, m-1}, y^{k_j}) < -\theta f(x^{k_j}, y^{k_j}). \tag{3.18}$$

From the algorithm, we have $z^{k_j, m_{k_j} - 1} = (1 - \eta^{m_{k_j} - 1}) x^{k_j} + \eta^{m_{k_j} - 1} y^{k_j}$. Since $\eta^{m_{k_j} - 1} = \eta_{k_j - 1} \to 0$, $\{x^{k_j}\} \rightharpoonup p$ and $y^{k_j} \rightharpoonup q$, we get, $z^{k_j, m-1} \rightharpoonup p$. Taking limit as $j \to \infty$ in (3.18) and using conditions (A1) and (A4), we get

$$-f(p, q) \le -\theta f(p, q).$$

Thus, since $\theta \in (0, 1)$, we have $f(p, q) \ge 0$. Hence, if we take limits as $j \to \infty$ in inequality (3.17), then we have $\varphi(y^{k_j}, x^{k_j}) \to 0$. Again, by Lemma 2.2, we have $||y^{k_j} - x^{k_j}|| \to 0$ which implies $y^{k_j} \rightharpoonup p$. By Lemma 2.9, we have

$$\rho_{k_j}[f(x^{k_j}, y) - f(x^{k_j}, y^{k_j})] \ge \langle J_{E_1} y^{k_j} - J_{E_1} x^{k_j}, y^{k_j} - x^{k_j} \rangle,$$

for all $y \in C$. By letting $j \to \infty$ in the inequality above, it follows that $f(p, y) \ge 0$, since $0 < \rho \le \rho_{k_j} < 1$. This implies that $p \in EP(C, f)$. $\qquad\square$

We now state our Main theorem.

Theorem 3.7 *Let C be nonempty, closed and convex subset of a 2-uniformly convex, uniformly smooth Banach space E_1, Q a nonempty, closed, and convex subset of a smooth, strictly convex, and reflexive Banach space E_2 and $L : E_1 \to E_2$ a bounded linear operator with $L^* : E_2^* \to E_1^*$ its adjoint. Let $f : C \times C \to \mathbb{R}$ be a pseudomonotone bifunction satisfying assumptions A. Let $g, h : Q \times Q \to \mathbb{R}$ be two monotone bifunctions satisfying assumptions B, C respectively and D. Let $\varphi : Q \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function and $\Psi : Q \to E_2^*$ be a continuous and monotone mapping. Assume that $T : C \to C$ and $S : Q \to Q$ are quasi-φ-nonexpansive mapping such that $I - T$ and $I - SK_r^{g,h}$ are demiclosed at zero. Let $\Gamma = \{p : EP(C, f) \cap F(T) : Lp \in GMEP(Q, g, h, \Psi, \varphi) \cap F(S)\} \ne \emptyset$. Then, the sequence $\{x^k\}$ generated by Algorithm 3 converges strongly to $p \in \Gamma$.*

Proof We shall divide the proof into two cases.

Case 1: Suppose there exists $k_0 \in \mathbb{N}$ such that $\{\varphi(p, x^k)\}$ is monotonically nonincreasing for all $k \ge k_0$. Then, $\{\varphi(p, x^k)\}$ converges as $k \to \infty$ and hence, $\varphi(p, x^k) - \varphi(p, x^{k+1}) \to 0$ as $k \to \infty$.

From (3.7), we have that

$$\beta_k(1 - \alpha_k)(1 - \beta_k)g(||J_{E_1} u^k - J_{E_1} T u^k||) \le \alpha_k \varphi(p, u) - \alpha_k \varphi(p, x^k) + \varphi(p, x^k) - \varphi(p, x^{k+1}).$$

Since $\lim_{k \to \infty} \beta_k(1 - \beta_k) > 0$, using the property of the function g and the uniform continuity of $J_{E_1}^{-1}$ on bounded subsets of E_1^*, we obtain

$$\lim_{k \to \infty} ||u^k - T u^k|| = 0. \tag{3.19}$$

Observe from (3.3), that

$$
\begin{aligned}
\varphi(u^k, v^k) &= \varphi(u^k, J_{E_1}^{-1}((1-\beta_k)J_{E_1}u^k + \beta_k J_{E_1}Tu^k)) \\
&= ||u^k||^2 - 2\langle u^k, (1-\beta_k)J_{E_1}u^k + \beta_k J_{E_1}Tu^k\rangle + ||(1-\beta_k)J_{E_1}u^k + \beta_k J_{E_1}Tu^k||^2 \\
&= ||u||^2 - 2(1-\beta_k)\langle u^k, J_{E_1}u^k\rangle - 2\beta_k\langle u^k, J_{E_1}Tu^k\rangle + (1-\beta_k)||J_{E_1}||^2 + \beta_k||J_{E_1}Tu^k||^2 \\
&\quad -\beta_k(1-\beta_k)g(||J_{E_1}u^k - J_{E_1}Tu^k||) \\
&= (1-\beta_k)\varphi(u^k, Tu^k)\beta_k\varphi(u^k, Tu^k) - \beta_k(1-\beta_k)g(||J_{E_1}u^k - J_{E_1}Tu^k||).
\end{aligned}
$$

Thus, by (3.19), we obtain

$$
\lim_{k\to\infty} \varphi(u^k, v^k) = 0.
$$

Lemma 2.2 implies

$$
\lim_{k\to\infty} ||u^k - v^k|| = 0.
$$

Further, set $\delta_k = J_{E_1}^{-1}(J_{E_1}x^k - \gamma_k\sigma_k t^k)$, then

$$
||J_{E_1}\delta_k - J_{E_1}x^k|| = ||\gamma_k\sigma_k t^k|| \to 0 \ \text{ as } \ k \to \infty.
$$

By the uniform continuity of $J_{E_1}^{-1}$ on bounded subsets of E_1^*, we obtain

$$
\lim_{k\to\infty} ||\delta_k - x^k|| = 0. \tag{3.20}
$$

Using the property of the generalized projection Π_C, we obtain

$$
\begin{aligned}
\varphi(p, u^k) &\leq \varphi(p, \delta_k) - \varphi(u^k, \delta_k) \\
&\leq \varphi(p, x^k) - \varphi(u^k, \delta_k). \tag{3.21}
\end{aligned}
$$

Substituting (3.21) into (3.9), we have

$$
\begin{aligned}
\varphi(p, x^{k+1}) &\leq \alpha_k\varphi(p, u) + (1-\alpha_k)\varphi(p, u^k) \\
&\leq \alpha_k\varphi(p, u) + (1-\alpha_k)[\varphi(p, x^k) - \varphi(u^k, \delta_k)], \tag{3.22}
\end{aligned}
$$

which implies

$$
(1-\alpha_k)\varphi(u^k, \delta_k) \leq \alpha_k\varphi(p, u) + (1-\alpha_k)\varphi(p, x^k) - \varphi(p, x^{k+1}).
$$

Therefore, $\varphi(u^k, \delta_k) \to 0$ as $k \to \infty$. This implies by Lemma 2.2 that

$$
\lim_{k\to\infty} ||u^k - \delta_k|| = 0. \tag{3.23}
$$

By (3.20) and (3.23), we have

$$
\lim_{k\to\infty} ||u^k - x^k|| = \lim_{k\to\infty} (||u^k - \delta_k|| + ||\delta_k - x^k||) = 0. \tag{3.24}
$$

Now, we show that $||(SK_r^{g,h} - I)Lv^k|| \to 0$ as $k \to \infty$. Using equation (3.8) in the expansion of $\varphi(p, x^{k+1})$, we obtain

$$
\begin{aligned}
\varphi(p, x^{k+1}) &\leq \alpha_k \varphi(p, u) + (1 - \alpha_k)\varphi(p, w^k) \\
&\leq \alpha_k \varphi(p, u) + (1 - \alpha_k)[\varphi(p, v^k) - 2\mu(1 - d^2||L||^2\mu)(||SK_r^{g,h} - I)Lv^k||^2],
\end{aligned}
$$

that is,

$$
2\mu(1 - \alpha_k)(1 - d^2||L||^2\mu)||(SK_r^{g,h} - I)Lv^k|| \leq \alpha_k \varphi(p, u) - \varphi(p, x^{k+1}) + (1 - \alpha_k)\varphi(p, x^k).
$$

Hence,

$$
||(SK_r^{g,h} - I)Lv^k|| \to 0 \text{ as } k \to \infty. \tag{3.25}
$$

Next, we show that $\lim\limits_{k\to\infty} ||v^k - w^k|| = 0$. Indeed,

$$
\begin{aligned}
\varphi(v^k, w^k) &= \varphi(v^k, J_{E_1}^{-1}(J_{E_1}v^k + \mu L^* J_{E_2}(SK_r^{g,h} - I)Lv^k)) \\
&= ||v^k||^2 - 2\langle v^k, J_{E_1}v^k + \mu L^* J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle \\
&\quad + ||J_{E_1}v^k + \mu L^* J_{E_2}(SK_r^{g,h} - I)Lv^k||^2 \\
&= ||v^k||^2 - 2\langle v^k, J_{E_1}v^k\rangle - 2\mu\langle Lv^k, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle + ||v^k||^2 \\
&\quad + 2\mu\langle Lv^k, J_{E_2}(SK_r^{g,h} - I)Lv^k\rangle \\
&\quad + 2d^2\mu^2||L||^2||(SK_r^{g,h} - I)Lv^k|| \\
&= \varphi(v^k, v^k) + 2\mu^2 d^2||L||^2||(SK_r^{g,h} - I)Lv^k||^2,
\end{aligned}
$$

which by (3.25) implies

$$
\lim_{k\to\infty} \varphi(v^k, w^k) = \lim_{k\to\infty} ||v^k - w^k|| = 0, \tag{3.26}
$$

where we have used Lemma 2.2.

Moreover, we show that $||x^{k+1} - x^k|| \to 0$ as $k \to \infty$. Observe that

$$
\lim_{k\to\infty} ||x^k - w^k|| = \lim_{k\to\infty} (||x^k - u^k|| + ||u^k - v^k|| + ||v^k - w^k||) = 0.
$$

Then, we obtain

$$
\begin{aligned}
\varphi(x^k, x^{k+1}) &= \varphi(x^k, J_{E_1}^{-1}(\alpha_k J_{E_1}u + (1 - \alpha_k)J_{E_1}w^k)) \\
&= ||x^k||^2 - 2\langle x^k, \alpha_k J_{E_1}u\rangle - 2(1 - \alpha_k)\langle x^k, J_{E_1}w^k\rangle + \alpha_k||u||^2 + (1 - \alpha_k)||w^k||^2 \\
&\quad - \alpha_k(1 - \alpha_k)g(||J_{E_1}w^k - J_{E_1}u||) \\
&= \alpha_k(x^k, u) + (1 - \alpha_k)\varphi(x^k, w^k) - \alpha_k(1 - \alpha_k)g(||J_{E_1}w^k - J_{E_1}u||). \tag{3.27}
\end{aligned}
$$

Thus, we obtain $\varphi(x^k, x^{k+1}) \to 0$ as $k \to \infty$ and Lemma 2.2 ensures

$$
\lim_{k\to\infty} ||x_{k+1} - x_k|| = 0.
$$

Since $\{x^k\}$ is bounded, there exists a subsequence $\{x^{k_i}\}$ of $\{x_k\}$ such that $x^{k_i} \rightharpoonup q$. We have by (3.24), (3.25), and (3.26) that $\{u^k\}$, $\{w^k\}$ and $\{v^k\}$ converge to q. Hence, by (3.19) and the demiclosedness assumption on $I - T$, we have that $q \in F(T)$. Thus, by this fact and Lemma 3.6, we have that $q \in EP(C, f) \cap F(T)$. On the other hand, by the linearity of L, we obtain $Lv^k \rightharpoonup Lq$. By Lemma 2.10 and the demiclosedness assumption on $I - SK_r^{g,h}$, we obtain that $Lq \in F(SK_r^{g,h}) = GMEP(Q, g, h, \Psi, \varphi) \cap F(S)$.

Next, we show that $\{x^k\}$ converges strongly to p. To do this, by applying (2.3) in (3.3) we have

$$\varphi(p, x^{k+1}) = \varphi(p, J_{E_1}^{-1}(\alpha_k J_{E_1} u + (1 - \alpha_k) J_{E_1} w^k))$$

$$= V(p, \alpha_k J_{E_1} u + (1 - \alpha_k) J_{E_1} w^k)$$

$$\leq V(p, \alpha_k J_{E_1} u + (1 - \alpha_k) J_{E_1} w^k - \alpha_k(J_{E_1} u - J_{E_1} p))$$

$$+ 2\langle J_{E_1}^{-1}(\alpha_k J_{E_1} u + (1 - \alpha_k) J_{E_1} w^k) - p, \alpha_k(J_{E_1} u - J_{E_1} p)\rangle$$

$$= \alpha_k V(p, J_{E_1} u) + (1 - \alpha_k) V(p, J_{E_1} w^k) + 2\alpha_k\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle$$

$$= (1 - \alpha_k)\varphi(p, w^k) + 2\alpha_k\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle$$

$$\leq (1 - \alpha_k)\varphi(p, x^k) + 2\alpha_k\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle.$$

Now, we need to show that $\limsup_{k\to\infty}\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle \leq 0$. To see this, choose a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x^{k_i} \rightharpoonup q$ and

$$\limsup_{k\to\infty}\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle = \lim_{i\to\infty}\langle x^{k_i+1} - p, J_{E_1} u - J_{E_1} p\rangle.$$

Since $||x^{k+1} - x^k|| \to 0$ as $k \to \infty$, we obtain $x^{k_i+1} \rightharpoonup q$. From (2.7) in Lemma 2.3, we have

$$\limsup_{k\to\infty}\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle = \lim_{i\to\infty}\langle x^{k_i+1} - p, J_{E_1} u - J_{E_1} p\rangle$$

$$= \langle q - p, J_{E_1} u - J_{E_1} p\rangle$$

$$\leq 0. \tag{3.28}$$

By using Lemma 2.3, 2.6 and (3.28), we obtain that $\{x^k\}$ converges strongly to p.

Case 2: Assume $\{\Phi_k = ||x^k - p||^2\}$ is monotonically nondecreasing. For some k_0 large enough, define a mapping

$$\tau(k) := \max\{j \in \mathbb{N} : j \leq k, \Phi_j \leq \Phi_{j+1}\}.$$

Clearly, τ is a nondecreasing sequence, $\tau(k) \to 0$ as $k \to \infty$ and

$$0 \leq \Phi_{\tau(k)} \leq \Phi_{\tau(k)+1}, \quad \forall k \geq k_0.$$

By the same argument as in **Case 1**, we have $||u^{\tau(k)} - Tu^{\tau(k)}|| \to 0$, $||(SK_r^{g,h} - I)Lv^{\tau(k)}|| \to 0$, and $||x^{\tau(k)+1} - x^{\tau(k)}|| \to 0$ as $k \to \infty$ and $\limsup_{k\to\infty}\langle x^{k+1} - p, J_{E_1} u - J_{E_1} p\rangle \leq 0$. Since $x^{\tau(k)}$ is bounded, there exists a subsequence $\{x^{\tau(k_j)}\}$ such that $\{x^{\tau(k_j)}\} \rightharpoonup \bar{q} \in C$. Also by the linearity of L, we have $Lv^{\tau(k_j)} \rightharpoonup L\bar{q} \in Q$. Following similar arguments as in the first case, we can conclude $\bar{q} \in \Gamma$. By applying Lemma 2.6, we have

$$\Phi_{\tau(k)+1} \leq (1 - b_{\tau(k)})\Phi_{\tau(k)} + b_{\tau(k)}c_{\tau(k)},$$

where $b_{\tau(k)} = \alpha_{\tau(k)}$, $c_{\tau(k)} = 2\langle x^{\tau(k)+1} - p, J_{E_1}u - J_{E_1}p\rangle$. Note that $b_{\tau(k)} \to 0$ as $k \to \infty$ and $\limsup\limits_{k\to\infty} c_{\tau(k)} \leq 0$.

Since $\Phi_{\tau(k)} \leq \Phi_{\tau(k+1)}$ and $b_{\tau(k)} > 0$, we have

$$||x^{\tau(k)} - p|| \leq c_{\tau(k)}.$$

This implies

$$\limsup_{k\to\infty} ||x^{\tau(k)} - p||^2 \leq 0;$$

hence

$$\lim_{k\to\infty} ||x^{\tau(k)} - p|| = 0. \tag{3.29}$$

By using $\lim\limits_{k\to\infty} ||x^{\tau(k)+1} - x^{\tau(k)}|| = 0$ and (3.29), we have that

$$\lim_{k\to\infty} ||x^{\tau(k)+1} - p|| \leq \lim_{k\to\infty} (||x^{\tau(k)+1} - x^{\tau(k)}|| + ||x^{\tau(k)} - p||) = 0. \tag{3.30}$$

Furthermore, for $k \geq k_0$, it is easy to see that $\Phi_{\tau(k)} \leq \Phi_{\tau(k)+1}$ if $k \neq \tau(k)$ (that is $\tau(k) < k$), because $\Phi_j \geq \Phi_{j+1}$ for $\tau(k) + 1 \leq j \leq k$. As a consequence, we obtain for all $k \geq k_0$, that

$$0 \leq \Phi_k \leq \max\{\Phi_{\tau(k)}, \Phi_{\tau(k)+1}\} = \Phi_{\tau(k)+1}.$$

By using (3.30), we can conclude that $\lim\limits_{k\to\infty} \Phi_k = 0$, that is $\{x^k\}$ converges strongly to p thereby completing the proof. $\qquad\square$

4. Numerical example

In this section, we present some numerical examples to illustrate the performance of our algorithm.

Example 4.1 *Let $E_1 = E_2 = \ell_2(\mathbb{R})$ be the linear spaces whose elements are all 2-summable sequences $\{x_i\}_{i=1}^\infty$ of scalars in $\mathbb{R}$, that is*

$$\ell_2(\mathbb{R}) := \left\{ x = (x_1, x_2 \cdots, x_i \cdots), \quad x_i \in \mathbb{R} \quad and \quad \sum_{i=1}^\infty |x_i|^2 < \infty \right\},$$

with the inner product $\langle \cdot, \cdot \rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x, y \rangle := \sum\limits_{i=1}^\infty x_i y_i$ and the norm $||\cdot|| : \ell_2 \to \mathbb{R}$ by

$$||x|| := \sqrt{\sum_{i=1}^\infty |x_i|^2}, \text{ where } x = \{x_i\}_{i=1}^\infty, \ y = \{y_i\}_{i=1}^\infty. \text{ Let } L : \ell_2 \to \ell_2 \text{ be given by } Lx = \left(\frac{x_1}{5}, \frac{x_2}{5}, \cdots, \frac{x_i}{5}, \cdots, \right)$$

for all $x = \{x_i\}_{i=1}^\infty \in \ell_2$. Then $L^*y = \left(\frac{y_1}{5}, \frac{y_2}{5}, \cdots, \frac{y_i}{5}, \cdots, \right)$ for each $y = \{y_i\}_\infty \in \ell_2$. Define the sets $C := \{x \in \ell_2 : ||x|| \leq 1\}$ and $Q := \{y \in \ell_2 : ||y|| \leq 1\}$. Let the mappings $g, h : Q \times Q \to \mathbb{R}$ be defined by $g(w, u) = uw - w^2$, $h(w, u) = 5u - 5w$. Let $\varphi(w) := w$, for all $u = \{u_i\}_{i=1}^\infty \in \ell_2$ and $w = \{w_i\}_{i=1}^\infty \in \ell_2$. It is easy to check that

$$K_r^{g,h}L(v) = \frac{v - 30r}{5(r+1)}.$$

Now define $f : C \times C \to \mathbb{R}$ by $f(x,y) = 2x^2 + 5xy - 7y^2$, for $x = \{x_i\}_{i=1}^{\infty}$, $y = \{y_i\}_{i=1}^{\infty} \in \ell_2$. It is easy to see that f satisfies conditions $(A1)-(A4)$. Now define $T : C \to C$ and $S : Q \to Q$ by $Tx = \left(\dfrac{2x_1}{3}, \dfrac{2x_2}{3}, \cdots, \dfrac{2x_i}{3}, \cdots, \right)$ for all $x = \{x_i\}_{i=1}^{\infty} \in \ell_2$ and $Sy = \left(\dfrac{3y_1}{5}, \dfrac{3y_2}{5}, \cdots \dfrac{3y_i}{5}, \cdots, \right)$ for all $y = \{y_i\}_{i=1}^{\infty} \in \ell_2$. It is easy to show that $F(T) = F(S) = \{0\}$.

We choose $\alpha_k = \frac{1}{k+1}$, $\beta_k = \frac{1}{3} + \frac{2k-1}{3k+5}$, $\rho_k = \frac{1}{2k}$, $\gamma_k = \frac{1}{k}$, $\eta = 0.03$, $\theta = 0.03$, $u = (-0.2345, 0.8943, 0, \ldots, 0, \ldots)'$. Using $\frac{||x^{k+1} - x^k||_{\ell_2}}{||x^2 - x^1||_{\ell_2}} < 10^{-5}$ as the stopping criterion, we test our algorithm 3 for different values of x_1 as follows:

Case (i) $x_1 = (3.2158, -5.8091, 0, \ldots, 0, \ldots)^T$,

Case (ii) $x_1 = (1.7601, -2.6457, 0, \ldots, 0, \ldots)^T$.

We then plot the graphs of error '$||x^{k+1} - x^k||_{\ell_2}$' against the number of iteration in each case. The computational results can be found for Case (i) in Table 1, Figure 1a and those for Case (ii) in Table 2 and Figure 1b. These show that the change in the initial value does not have a significant effect on the performance of the algorithm.

Table 1. Computational results for Example 4.1, Case (i); Time: 0.0537 s.

Iteration	x^{k+1}	$||x^{k+1} - x^k||_{\ell_2}$
1	$(-0.3385, 0.2259, 0, \ldots, 0, \ldots)$	4.0957
2	$(-0.3386, 0.2258, 0, \ldots, 0, \ldots)$	$1.1491e^{-4}$
3	$(-0.3387, 0.2257, 0, \ldots, 0, \ldots)$	$9.1926e^{-5}$
4	$(-0.3388, 0.2256, 0, \ldots, 0, \ldots)$	$7.5212e^{-5}$
5	$(-0.3389, 0.2255, 0, \ldots, 0, \ldots)$	$6.2677e^{-5}$
10	$(-0.3389, 0.2255, 0, \ldots, 0, \ldots)$	$3.0417e^{-5}$
15	$(-0.3390, 0.2255, 0, \ldots, 0, \ldots)$	$1.7708e^{-5}$
20	$(-0.3390, 0.2254, 0, \ldots, 0, \ldots)$	$1.1785e^{-5}$
$\vdots$	$\vdots$	$\vdots$
23	$(-0.3390, 0.2254, 0, \ldots, 0, \ldots)$	$9.5096e^{-6}$

Next, we give another numerical example and compare the performance of our Algorithm 3 with Algorithm 3 in [15] and Algorithm 2 in [27].

Example 4.2 *Let $E_1 = \mathbb{R}^2$ and $E_2 = \mathbb{R}^5$. Let $L : \mathbb{R}^2 \to \mathbb{R}^5$ be defined by*

$$L(x) = (2x_1 + x_2, x_1 + 2x_2, -x_2, 0, -3x_1 - x_2) \quad x = (x_1, x_2).$$

Also let $C := [-5, 5] \times [-5, 5]$ and $Q = [-10, 10] \times [-10, 10] \times [-10, 10] \times [-10, 10] \times [-10, 10]$. Define the mappings $g, h : Q \times Q \to \mathbb{R}$ by $g(x,y) = x^2 + 2xy + y^2$, $h(x,y) = 2(xy - x^2)$, and $\varphi : Q \to Q$ by $\varphi(x) = x^2$. It

Table 2. Computational results for Example 4.1, Case (ii); Time: 0.0479 s.

Iteration	x^{k+1}	$||x^{k+1} - x^k||_{\ell_2}$
1	$(-1.0025, -1.4691, 0, \ldots, 0, \ldots)$	3.0027
2	$(-1.0026, -1.4691, 0, \ldots, 0, \ldots)$	$6.7432e^{-5}$
3	$(-1.0026, -1.4691, 0, \ldots, 0, \ldots)$	$5.6321e^{-5}$
4	$(-1.0026, -1.4692, 0, \ldots, 0, \ldots)$	$4.5212e^{-5}$
5	$(-1.0027, -1.4692, 0, \ldots, 0, \ldots)$	$3.6715e^{-5}$
10	$(-1.0027, -1.4693, 0, \ldots, 0, \ldots)$	$1.7818e^{-5}$
12	$(-1.0028, -1.4693, 0, \ldots, 0, \ldots)$	$1.0489e^{-5}$
$\vdots$	$\vdots$	$\vdots$
16	$(-1.0028, -1.4693, 0, \ldots, 0, \ldots)$	$9.5096e^{-6}$

is easy to see that conditions B and C are satisfied and $K_r^{g,h}(w) = \dfrac{w}{6r+1} \quad \forall \ x \in Q$. Now let $f : C \times C \to \mathbb{R}$ be defined by $f(x,y) = x^2 + 4xy - 4y^2$. It is easy to see that f satisfies conditions $(A1) - (A4)$.

We define $T : C \to C$ and $S : Q \to Q$ by $Tx = \dfrac{2x}{3}$ and $Sy = \dfrac{3y}{5}$ respectively, for all $x \in C$ and $y \in Q$ respectively. It is easy to see that $\Gamma \neq \emptyset$. For each $k = 0, 1, \cdots$, choose the sequences $\alpha_k = \dfrac{3k}{2k^2 + 3}$, $\beta_k = \dfrac{2k}{3k+1}$, $u = 1$, $\eta = 0.3$, $\theta = 0.5, \gamma_k = 0.95$, $\mu = 0.25$ and $\rho_k = \dfrac{1}{\log(k+1)}$. We use the optimization tool box in MATLAB to find the argmin element and we let $\dfrac{||x^{k+1} - x^k||}{||x^2 - x^1||} < 5 \times 10^{-4}$ be our stopping criterion. We compare the performance of our Algorithm 3 with Algorithm 3 in [15] and Algorithm 2 in [27] using different initial values as follows:

1. *Case (i)* $x^1 = (-3, 2)^T$, $u = (1, 5)^T$,

2. *Case (ii)* $x^1 = (-4, 0)^T$, $u = (-3, -2)^T$,

3. *Case (iii)* $x^1 = (-5, 1)^T$, $u = (0, 5)^T$.

The numerical results for these cases can be seen in Table 3, Figure 2a–2c respectively below.

Table 3. Comparison between Algorithm 3, Algorithm 3 in [15] and Algorithm 2 in [27], for Example 4.2.

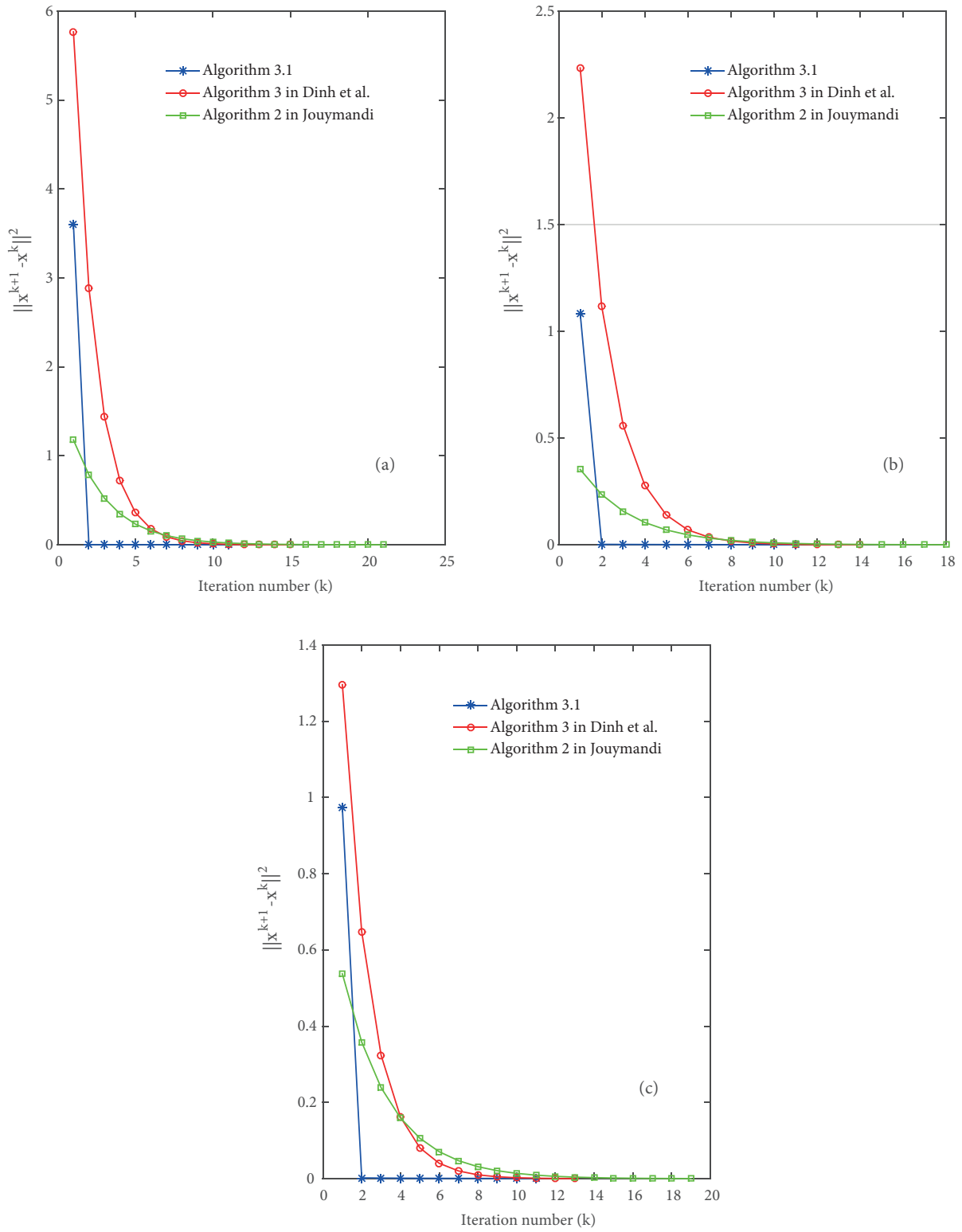|		Algorithm 3	Algorithm 3 in [15]	Algorithm 2 in [27]
Case (i)	CPU time (s)	0.0073	0.0118	0.3725
	No. of iter.	9	13	21
Case (ii)	CPU time (s)	0.0043	0.0133	0.0114
	No. of iter.	11	12	15
Case (iii)	CPU time (s)	0.0057	0.0330	0.0263
	No. of iter.	11	13	19

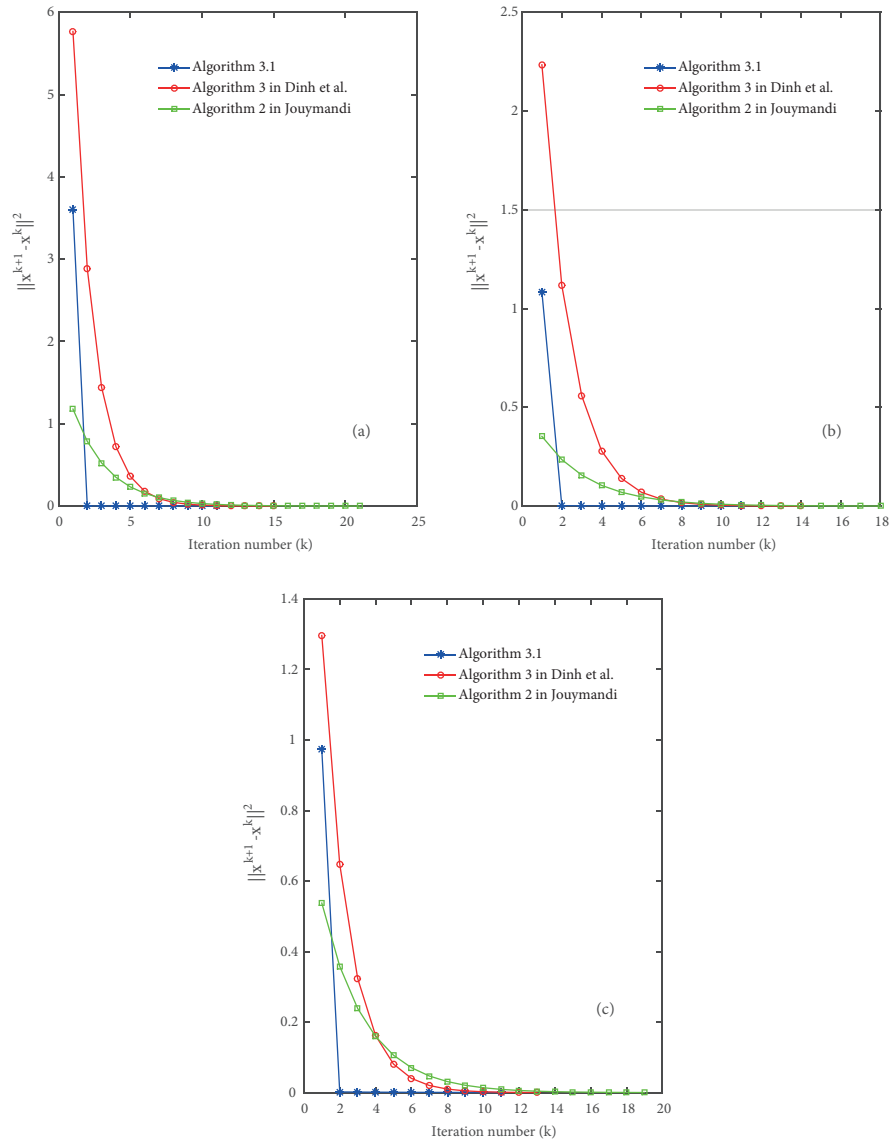**Figure 1**. Example 4.1, a: Case (i); b: Case (ii).

Figure 2. Example 4.2, a: Case (i); b: Case (ii); c: Case (iii).

5. Conclusion

In this article, we considered the problem of approximating a common element in the solution set of split generalized mixed equilibrium problem involving both monotone and pseudomonotone bifunctions which is also a fixed point of a quasi-φ-nonexpansive mapping. We introduced an extragradient algorithm based on the Armijo line search rule and established a strong convergence theorem in the framework of 2-uniformly convex and uniformly smooth Banach space E_1 and a smooth, strictly convex, and reflexive Banach space E_2. Furthermore, we gave some numerical examples in both finite and infinite dimensional spaces to illustrate the performance and behavior of our method as well as comparing it with some related methods in the literature. The results presented in this paper generalizes many recent and related results in the literature.

Acknowledgments

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