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Max CS and min CS modules

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Abstract: In this work, we study max CS, min CS, max-min CS modules and their endomorphism rings. Under certain conditions (e.g., related to nonsingularity and duo-ness), we prove that a module is max CS if and only if it is min CS, and that direct sums of min (max) CS modules is again min (max) CS. Finally, symmetry of max-min CS property on the endomorphism rings of max-min CS modules is investigated.

Key words: Max CS modules, min CS modules, max-min CS rings, nonsingular modules, duo modules

1. Introduction

In this paper, we consider an associative ring R with identity, and a right $R-$module M with the endomorphism ring $S = \mathrm{End}(M_R)$. We denote $X \hookrightarrow M$ (resp. $X \overset{*}{\hookrightarrow} M$) for a submodule (resp. an essential submodule) X of M. We write $\mathbf{r}_X(Y)$ and $\mathbf{l}_X(Y)$ for the right annihilator and the left annihilator of Y in X, respectively. We denote the uniform dimension of the module M_R by u-dim(M_R). For a submodule X of M, we write $I_X := \{f \in S | f(M) \subseteq X\}$. For a subset K of S, we write $KM = K(M) := \sum_{f \in K} f(M)$. It is clear that I_X is a right ideal of S and KM is a submodule of M.

A *closed submodule* X of M means $X \overset{*}{\hookrightarrow} Y \Rightarrow X = Y$ for any submodule Y of M. A submodule $X \hookrightarrow M$ is *fully invariant* if $f(X) \subseteq X$ for every $f \in S$. M is called a *duo module* (resp. *weak duo module*) if every submodule (resp. every direct summand) is fully invariant. R is called a *right duo ring* (resp. *right weak duo ring*) if R_R is a duo module (resp. *weak duo module*), equivalently, every right ideal (resp. every right ideal generated by an idempotent) of R is 2-sided.

We adopt the notions of primeness and semiprimeness in module category introduced by Sanh et al. in [7]. Recall that a *min CS module* (or *uniform extending module*) provides that every minimal (i.e. uniform) closed submodule is a direct summand. M is called a *max CS module* if every maximal closed submodule with nonzero left annihilator in S is a direct summand. M is a *max-min CS module* if it is both max CS and min CS. A ring R is a right min CS (resp. right max CS, right max-min CS) ring if R_R is a min CS (resp. max CS, max-min CS) module. Left min CS, left max CS and left max-min CS rings are defined analogously. Readers can find more details in [2] and [8].

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The concepts of *nonsingular modules* and *nonsingular rings* are understood as usual. According to [5], M is a nonsingular module if and only if for any $X \hookrightarrow M$, $\mathbf{r}_R(X) \overset{*}{\hookrightarrow} R_R$ implies $X = 0$. M is said to be *cononsingular* if for any $X \hookrightarrow M$, $\mathbf{l}_S(X) \overset{*}{\hookrightarrow}_S S$ implies $X = 0$. Clearly, R is right (left) nonsingular if and only if every essential right (left) ideal of R has zero left (right) annihilator. Therefore, R is right (left) nonsingular if and only if R_R is a nonsingular (cononsingular) module. M is a *Utumi module* if every submodule X with $\mathbf{l}_S(X) = 0$ implies $X \overset{*}{\hookrightarrow} M$, where $S = \text{End}(M_R)$. It is easy to see that M is Utumi if and only if every proper closed submodule $X \subsetneq M$ has nonzero left annihilator in S.

Extending and its generalizations are important and attractive properties in ring and module theory which has been extensively investigated by many authors. In 2016, Tercan and Yücel published a beautiful monograph [10] that comprehensively presents their works and others in the topics: extending properties under certain classes of submodules, related concepts and generalizations, then addresses open problems for further research. Hadi and Majeed [3] (2012) studied max CS and min CS modules over commutative rings, especially equivalence of the max CS and min CS conditions. To enrich the field, in this paper, we investigate max CS and min CS modules over associative (generally not commutative) rings. Section 2 presents results on max CS and min CS modules. With the aid of duo-ness and retractability, Theorem 2.3 proves that a semiprime (or nonsingular and Utumi) module is max CS if and only if it is min CS. In Theorem 2.8 and Theorem 2.9, we take advantage of distribution, duo-ness and Utumi condition on modules to show that direct sums of min (max) CS modules are again min (max) CS. In subsequence, 2 examples are discussed to see strict necessity of those conditions. Section 3 studies symmetry of max-min CS property and uniform dimension on the endomorphism rings of max-min CS modules. There, we generalize the results of Jain et al. in [2].

2. Max CS modules and min CS modules

We observe that if M is a nonsingular module, then for any $f \in S, \mathbf{r}_M(f) \overset{*}{\hookrightarrow} M$ implies $f = 0$. Furthermore, any essential submodule of M has zero left annihilator in S. If M is a cononsingular module, then for any left ideal K of $S, K \overset{*}{\hookrightarrow}_S S$ implies $\mathbf{r}_M(K) = 0$. The following lemma on nonsingularity is elementary (also see [5, Examples]).

Lemma 2.1 *If M is a nonsingular CS module, then M is cononsingular. In particular, a right nonsingular right CS ring is left nonsingular.*

For any $X \hookrightarrow M$, there always exists a maximal submodule $Y \hookrightarrow M$ with respect to $Y \cap X = 0$. Such a submodule Y is called a *complement* of X in M. It is easy to verify that every complement is a closed submodule and every closed submodule is a complement of another.

Lemma 2.2 *[8, Lemma 3.2] Let X be a closed submodule of M, and Y be complement of X in M. Then X is a maximal (resp. minimal) closed if and only if Y is minimal (resp. maximal) closed.*

A module M is retractable if for any nonzero submodule X of $M, I_X \neq 0$. [8, Theorem 3.3] proved that for a finitely generated, quasi-projective self-generator M, if M is semiprime weak duo, then M is max CS if and only if it is min CS. We are going to study other classes of modules in which the max CS and min CS conditions are equivalent. Note that in the next results, we do not require every submodule of M to be fully invariant, although taking duo-ness into account.

Theorem 2.3 *Let M be a retractable, duo module with respect to all maximal and minimal closed submodules (i.e. if X is a minimal or maximal closed submodule, then X is fully invariant). Assume that M satisfies one of the following conditions:*

- *(i) M is a semiprime module;*
- *(ii) M is a nonsingular module.*

Then M is max CS if and only if M is min CS. In this case, M is a max-min CS module.

Proof For nonzero submodules A and B of M, since M is retractable, we have $I_A \neq 0$ and $I_B \neq 0$, and $I_A \cap I_B = 0 \Leftrightarrow A \cap B = 0$. We need some claims as elementary preparation.

- *Claim 1.* If X is fully invariant, then I_X is a 2-sided ideal of S. For any $s \in S$, since X is fully invariant, we have $sI_X(M) \subset s(X) \subset X$. Thus, $sI_X \subset I_X$ so I_X is 2-sided.
- *Claim 2.* For some $e = e^2 \in S$, if $Y = eM$ is fully invariant, then $I_Y = eS$ and $Se \subseteq eS$. It is clear that $eS \subseteq I_Y$. As a counterclaim, we assume that $eS \subsetneqq I_Y$, i.e. there exists $0 \neq f \in I_Y$ but $f \notin eS$. Since $f \in S = eS \oplus (1-e)S, f = eg + (1-e)h$ for some $h, g \in S$ and $(1-e)h \neq 0$. Then, we have $fM = eg(M) + (1-e)h(M)$ so $0 \neq (1-e)h(M) \subseteq (fM \cap (1-e)M) \subseteq (eM \cap (1-e)M)$. This is a contradiction because $fM \subseteq eM$ and $eM \cap (1-e)M = 0$. Therefore, it must hold $eS = I_Y$. It is clearly $Se \subseteq I_Y = eS$ because I_Y is 2-sided and $e \in I_Y$.

(i) M is a semiprime module.

Let M be a max CS module, X be a minimal closed submodule, and Y is a complement of X in M. Then Y is maximal closed by Lemma 2.2. By assumption, X and Y are fully invariant submodules.

- *Claim 3.* $\mathbf{r}_M(I_X)$ is the unique complement of X. We assume that $\mathbf{r}_M(I_X) \cap X = A$. Then $I_X(A) = 0$ so $I_X I_A(M) = 0$. This implies $I_X I_A = 0$. By [7, Theorem 2.9], S is a semiprime ring, and hence $I_X \cap I_A = 0$. Thus, by retractability of M, we have $X \cap A = 0$ so $A = 0$. Now we observe that $I_X(Y) \subset Y$ and $I_X(Y) \subset X$, hence $I_X(Y) \subset (Y \cap X) = 0$. This means $Y \subset \mathbf{r}_M(I_X)$. By maximality of Y, we obtain $Y = \mathbf{r}_M(I_X)$. This implies that $\mathbf{r}_M(I_X)$ is the unique complement of X.

Now we see that $\mathbf{l}_S(Y) = \mathbf{l}_S\mathbf{r}_M(I_X) \supset I_X \neq 0$. Therefore, we have $Y = eM$ for some $e = e^2 \in S$, since M is max CS. Arguing similarly to Claim 3, $\mathbf{r}_M(I_Y)$ is again the unique complement of Y in M, and hence $X = \mathbf{r}_M(I_Y)$. But $e \in I_Y$ so $\mathbf{r}_M(I_Y) \subseteq \mathbf{r}_M(e)$ and $X = \mathbf{r}_M(I_Y) \subset \mathbf{r}_M(e) = (1-e)M$. Seeing that $(1-e)M$ is also a complement of Y so it is minimal closed. By minimality, we induce $X = (1-e)M$, a direct summand of M. This shows that M is min CS.

Conversely, let M be min CS and X be a maximal closed submodule with nonzero left annihilator in S. Then Y, a complement of X in M, is minimal closed and $Y = fM$ for some $f = f^2 \in S$. Again, we see that $\mathbf{r}_M(I_Y)$ is again the unique complement of Y in M, and hence $X = \mathbf{r}_M(I_Y)$. Moreover, $X = \mathbf{r}_M(I_Y) \subset \mathbf{r}_M(e) = (1-e)M$ and hence $X = (1-f)M$ (because of maximality of X). This implies that X is a direct summand, and hence M is max CS.

(ii) M is a nonsingular module.

Let M be max CS and X be a minimal closed submodule of M. Then Y, a complement of X in M, is maximal closed by Lemma 2.2.

- *Claim 4.* $\mathbf{l}_S(Y) \neq 0$. We observe that $I_X(Y) \subset Y$ and $I_X(Y) \subset X$, hence $I_X(Y) \subset (Y \cap X) = 0$. This means $0 \neq I_X \subset \mathbf{l}_S(Y)$.

Since M is max CS, Y is a direct summand of $M, Y = eM$ for some $e = e^2 \in S$. Since M is weak duo, Y is a fully invariant submodule and hence $I_Y = eS$ is a 2-sided ideal. Therefore, $I_X I_Y \subset (I_X \cap I_Y) = 0$, and hence $I_X \subset \mathbf{l}_S(I_Y) \subseteq \mathbf{l}_S(e) = S(1-e) \subset (1-e)S$ (note that $(1-e)S$ is 2-sided as $(1-e)M$ is fully invariant). Therefore, $I_X(M) \subset (1-e)M$ and $0 \neq I_X(M) \subset (X \cap (1-e)M)$. Since M is nonsingular, $I_X(M)$ has a unique closure, namely U. Then we have $U \subset (1-e)M$ and $U \subset X$, so $X = (1-e)M = U$ because of minimality of X and $(1-e)M$. This implies that M is min CS.

Conversely, let M be min CS and X be a maximal closed submodule with nonzero left annihilator in S. Then Y, a complement of X in M, is minimal closed and hence $Y = fM$ for some $f = f^2 \in S$. Since M is weak duo, Y is a fully invariant submodule and hence $I_Y = fS$ is a 2-sided ideal. We observe that $I_X I_Y \subset (I_X \cap I_Y) = 0$. Thus, we have $I_X \subset \mathbf{l}_S(I_Y) = \mathbf{l}_S(f) = S(1-f) \subset (1-f)S$. Therefore, $I_X(M) \subset (1-f)M$ and $0 \neq I_X(M) \subset (X \cap (1-f)M)$. Since M is nonsingular, there is a unique essentially closed submodule of $(X \cap (1-f)M)$, namely U, that contains $I_X(M)$. We assume that V is a complement of U in X. Then $V \cap (1-f)M = 0$ and $I_V \cap (1-f)S = 0$, but $I_V(1-f)S \subset (I_V \cap (1-f)S)$ so $I_V(1-f)S = 0$ and $I_V \subset \mathbf{l}_S(1-f) = Sf \subset fS$. On the other hand, $I_V \subset I_X \subset (1-f)S$ so $I_V \subset (fS \cap (1-f)S) = 0$. Consequently, we have $I_V = 0$ so $V = 0$ by retractability of M. Therefore, $U = X$ and $X \subset (1-f)M$, and hence $X = (1-f)M$ because of maximality of X. This implies that M is max CS.

The proof is now complete. □

Thuat et al. [8, Theorem 3.2] implies that a semiprime, right weak duo ring is right max CS if and only if it is right min CS. Note that the class of right nonsingular rings includes reduced rings, right hereditary rings, right semihereditary rings, von Neumann regular rings, right Rickart rings, Baer rings, domains, and semisimple rings. Moreover, every prime ring is semiprime. Therefore, Theorem 2.3 and [8, Theorem 3.2] give us various broad classes of rings in which being right max CS and right min CS are equivalent.

Corollary 2.4 *Let R be a right weak duo ring. If R is either semiprime or right nonsingular, then R is right max CS if and only if R is right min CS. In this case, R is right max-min CS.*

[3, Theorem 1.33] (which proved that a commutative nonsingular ring is max CS if and only if it is min CS) is simply a special case of our corollary above. It is well-known that a commutative ring is semiprime if and only if it is nonsingular.

Example 2.5 *Let F be a field. Considering a right $F-$module $R = \{(x, y, z) | x, y, z \in F\}$. We do have:*

- *(i) R is a nonsingular but not weak duo module.*
- *(ii) R is a max CS but not min CS.*

Proof (i) It is easy to see that R_F is nonsingular. We observe that $A = \{(x, 0, 0) | x \in F\}$ is a direct summand of R. Considering the homomorphism $f : R \to R, f(x, y, z) \mapsto (0, 0, x)$. Clearly $f(A)$ is not contained in A so A is not a fully invariant of R. This shows that R is not weak duo.

(ii) Clearly $\{(x,y,0)|x,y\in F\},\{(0,y,z)|y,z\in F\}$ and $\{(x,0,z)|x,z\in F\}$ are the all maximal submodules of R. Therefore, R is a max CS module. However, R is not a min CS module. It is not hard to verify that $B=\{(x,x,0)|x\in F\}$ is a minimal closed submodule of R but it is not a direct summand.

This example shows that there exists a max CS module which is not min CS. Moreover, the duo (weak duo) condition in Theorem 2.3 cannot be dropped. □

[8, Proposition 3.8] proved that every closed submodule inherit the min CS property. However, we do not know whether a similar situation happens in the case of the max CS property. An affirmative answer is presented in the following proposition.

Proposition 2.6 *Let M be a max CS module with the endomorphism ring S.*

(1) If X is a direct summand of M. Then M/X and X are max CS modules.

(2) Assume that M is a duo module with respect to all closed submodules. If X is a closed submodule generated by M, then X is a max CS module.

Proof *(1)* Firstly, we will show that M/X is max CS. Let A/X be a maximal closed submodule of M/X. We will prove that A is maximal closed in M. Assume that A is essential in $B\hookrightarrow M$. By [1, Proposition 1.4], we have $A/X \overset{*}{\hookrightarrow} B/X$ so $A/X=B/X$. This implies that $A=B$ and A is closed in M. Let C be a closed submodule of M containing A. By [1, Exercise 16, p20], A/X is closed in C/X, and C/X is closed in M/X. By maximality of A/X, we obtain $A/X=C/X$ so $A=C$. Therefore, A is a maximal closed submodule of M.

Now assume that $\mathbf{l}_T(A/X)\neq 0$, where $T=\mathrm{End}_R(M/X)$. Then there is $0\neq f: M/X\to M/X$ such that $f(A/X)=0$. Let $p: M\to M/X$ be the canonical projection, i.e. $p(m)=m+X$ for every $m\in M$. Since X is a direct summand of M, we can decompose $M=X\oplus Y$, and define the $R-$homomorphism $g: M/X\to M, m+X\mapsto y$, where $m=x+y, x\in X, y\in Y$. Then we have $gfp(M)=gf(M/X)\neq 0$ and $gfp(A)=gf(A/X)=0$. Therefore, $0\neq gfp\in \mathbf{l}_S(A)\subseteq S$.

Since M is max CS, A is a direct summand of M, so $M=A\oplus B$. It is obvious to claim that $A/X\oplus(B+X)/X=M/X$. This shows that M/X is a max CS module.

Because $M=X\oplus Y$, it is clear that Y is isomorphic to M/X. This implies that Y is again a max CS module, so is X.

(2) Let X be a closed submodule of M. Assume that A is a maximal closed submodule of X with $\mathbf{l}_{S_X}(A)\neq 0$, where $S_X=\mathrm{End}_R(X)$. We will show that $A\oplus Y$ is maximal closed in M, where Y is a complement of X in M. Since A is closed in X, there is a complement of A in X, namely B, so that $A\oplus B\overset{*}{\hookrightarrow} X$. Thus, $Y\oplus A\oplus B\overset{*}{\hookrightarrow}$ is essential in M. This shows that $Y\oplus A$ is closed in M. By Lemma 2.2, B is minimal closed in X so is minimal closed in M, and hence $Y\oplus A$ is maximal closed in M.

Because $\mathbf{l}_{S_X}(A)\neq 0$, there exists a nonzero $R-$homomorphism $f: X\to X$ such that $f(A)=0$. Since X is generated by M, we have $X=\sum_\Lambda g(M)$, where $g\in\Lambda$ is a subset of S, hence $(f\sum_\Lambda g)\in S$. Since every closed submodule of M is fully invariant, $\sum_\Lambda g(A)\subseteq A$ so $f\sum_\Lambda g(A)\subseteq f(A)=0$. On the other hand, we have $f\sum_\Lambda g(M)=f(X)\neq 0$, and $\sum_\Lambda g(Y)\subseteq (Y\cap X)=0$. Therefore, $f\sum_\Lambda g(A\oplus Y)=0$ and $0\neq (f\sum_\Lambda g)\in \mathbf{l}_S(A\oplus Y)$. Since M is max CS, $A\oplus Y$ is a direct summand of M, hence $M=(A\oplus Y)\oplus B=A\oplus(Y\oplus B)$. Now we have $X=X\cap M=X\cap(A\oplus(Y\oplus B))=A\oplus(X\cap(Y\oplus B))$ by modular law. This means that A is a direct summand of X and X is max CS. □

Unlikely min CS modules, we require the condition of nonzero annihilator for max CS property. This makes trouble sometimes. In Proposition 2.6, it is not easy to construct an $R-$homomorphism from M to X to show that $\mathbf{l}_S(A) \neq 0$ in *(1)* or $\mathbf{l}_S(A \oplus Y) \neq 0$ in *(2)*. Note that a direct sum of min CS (max CS) modules may not be min CS (max CS), for instance $(\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/8\mathbb{Z})$. Next, we aim to determine when direct sums of min CS (max CS) modules are again min CS (max CS). In the 2 next theorems, we consider direct sums of 2 min CS (max CS) modules only. However, this can be generalized to any finite direct sum of min CS (max CS) modules. A module M is so called a distributive module if $A \cap (B + C) = (A \cap B) + (A \cap C)$ for submodules A, B, C of M. In the following theorem, we only need distribution with respect to a certain decomposition instead of any sum of submodules.

Lemma 2.7 *Let M_1 and M_2 be right $R-$modules and $M = M_1 \oplus M_2$. Then we have either $(X \cap M_1) = 0$ or $(X \cap M_2) = 0$ for every minimal closed submodule X of M.*

Proof It is clear since X is uniform. □

Theorem 2.8 *Let M_1 and M_2 be right $R-$modules and $M = M_1 \oplus M_2$. Assume that M is distributive with respect to $M_1 \oplus M_2$, i.e. $X \cap (M_1 \oplus M_2) = (X \cap M_1) \oplus (X \cap M_2)$ for every $X \hookrightarrow M$.*

(1) M is min CS if and only if M_1 and M_2 are min CS.

(2) If M_1 and M_2 are Utumi modules, then M is max CS if and only if M_1 and M_2 are max CS.

Proof *(1)* The necessity is induced by [8, Proposition 3.8]. We only need to prove the sufficiency. Let M_1 and M_2 be min CS. We will prove that M is also min CS.

Assume that X is a minimal closed submodule of M. Since M is distributive with respect to $M_1 \oplus M_2$, we have $X \cap (M_1 \oplus M_2) = (X \cap M_1) \oplus (X \cap M_2)$. By Lemma 2.7, we suppose $(X \cap M_2) = 0$ without loss of generality. Then we have $X = X \cap M_1$, hence X is a minimal closed submodule of M_1.

Since M_1 is min CS, we have decomposition $M_1 = X \oplus C$. Therefore, $M = M_1 \oplus M_2 = (X \oplus C) \oplus M_2$. We can re-arrange these terms to obtain $M = X \oplus (C \oplus M_2)$. This shows that X is a direct summand of M. Thus M is a min CS module.

(2) The necessity is induced by Proposition 2.6. We will prove the sufficiency for the sake of completeness. Let M_1 and M_2 be max CS. We will show that M is again max CS.

Let K be a maximal closed submodule of M with $\mathbf{l}_S(K) \neq 0$. Since M is distributive, we have $K = K \cap (M_1 \oplus M_2) = (K \cap M_1) \oplus (K \cap M_2)$. We will show that $(K \cap M_1)$ and $(K \cap M_2)$ are maximal closed submodules of M_1 and M_2, respectively. Let $K \cap M_1$ be essential in $X \hookrightarrow M_1$. Then $K = (K \cap M_1) \oplus (K \cap M_2)$ is essential in $X \oplus (K \cap M_2)$. Since K is closed in M, $(K \cap M_1) \oplus (K \cap M_2) = X \oplus (K \cap M_2)$, so $(K \cap M_1) = X$. This means that $(K \cap M_1)$ is closed in M_1. Suppose that $Y \hookrightarrow M_1$ is a closed submodule containing $(K \cap M_1)$. Then $Y \oplus (K \cap M_2)$ is a closed submodule of M containing K. By maximality of K, we have $K = Y \oplus (K \cap M_2)$. Consequently, $(K \cap M_1) = Y$ so $(K \cap M_1)$ is maximal closed in M_1. Similarly, we see that $(K \cap M_2)$ is maximal closed in M_2.

Now, since M_1 and M_2 are Utumi, the left annihilators $\mathbf{l}_{S_1}(K \cap M_1)$ and $\mathbf{l}_{S_2}(K \cap M_2)$ are all nonzero, where $S_1 = \mathrm{End}_R(M_1)$ and $S_2 = \mathrm{End}_R(M_2)$. Since M_1 and M_2 are max CS, we have decompositions $M_1 = (K \cap M_1) \oplus X$ and $M_2 = (K \cap M_2) \oplus Y$. Therefore, $M = ((K \cap M_1) \oplus X) \oplus ((K \cap M_2) \oplus Y) = ((K \cap M_1) \oplus (K \cap M_2)) \oplus (X \oplus Y) = K \oplus (X \oplus Y)$. This shows that K is a direct summand of M and M is max CS. □

Theorem 2.9 *Let M_1 and M_2 be right $R-$modules and $M = M_1 \oplus M_2$.*

(1) Assume that M is duo with respect to all minimal closed submodules. Then M is min CS if and only if M_1 and M_2 are min CS.

(2) Assume that M is duo with respect to all maximal closed submodules, and M_1 and M_2 are Utumi modules. Then M is max CS if and only if M_1 and M_2 are max CS.

Proof Let $\pi_i : M \to M_i$ be the natural projection, $i = 1, 2$. In particular, we have $\pi_1(M) = M_1, \pi_2(M) = M_2$. For any $X \hookrightarrow M$ and any $x \in X$, we have $x = m_1 + m_2$, where $m_i \in M_i, i \in \{1, 2\}$. It is obvious that $x = \pi_1(m_1) + \pi_2(m_2) = \pi_1(x) + \pi_2(x)$.

(1) The necessity follows from [8, Proposition 3.8]. For the sufficiency, let X be a minimal closed submodule of M. Since X is fully invariant, $\pi_i(x) \in X$ for $i \in \{1, 2\}$. Thus $\pi_i(x) = m_i \in (X \cap M_i)$ for $i \in \{1, 2\}$. By Lemma 2.7, we assume that $(X \cap M_2) = 0$ so $\pi_2(x) = 0$. This implies $x = \pi_1(x) = m_1 \in M_1$, hence $X \subseteq M_1$. It is clear that X is also minimal closed in M_1 so $M_1 = X \oplus Y$. Now we see that $M = M_1 \oplus M_2 = (X \oplus Y) \oplus M_2 = X \oplus (Y \oplus M_2)$. This means that X is a direct summand of M, and hence M is min CS.

(2) The necessity follows from Proposition 2.6. For the sufficiency, let X be a maximal closed submodule of M with $\mathbf{l}_S(X) \neq 0$. Since X is fully invariant, $\pi_i(x) \in X$ for $i \in \{1, 2\}$. Thus $\pi_i(x) = m_i \in (X \cap M_i)$ for $i \in \{1, 2\}$. Therefore, we have $X \subseteq ((X \cap M_1) \oplus (X \cap M_2)) \subseteq X$ so $X = (X \cap M_1) \oplus (X \cap M_2)$. Arguing similarly to the proof of Theorem 2.8, we claim that M is max CS. □

Example 2.10 *Let $\mathbb{Z}$ be the set of all integers. Consider $\mathbb{Z}-$module $M = (\mathbb{Z}/\mathbb{Z}2) \oplus (\mathbb{Z}/\mathbb{Z}8)$. We observe that M is a sum of max-min CS modules $A = \mathbb{Z}(1 + 2\mathbb{Z}, 0)$ and $B = \mathbb{Z}(0, 1 + 8\mathbb{Z})$. However, M is neither min CS nor max CS, since the minimal (also maximal) closed submodule $C = \mathbb{Z}(1+2\mathbb{Z}, 2+8\mathbb{Z})$ is not a direct summand. It is clear that M is not distributive with respect to $A \oplus B$. This shows that the condition of distribution in Theorem 2.8 is indispensable. We easily see that C is not fully invariant. Therefore, the requirement that M is duo with respect to the class of all maximal (resp. minimal) closed submodules of M in Theorem 2.9 cannot be dropped.*

3. On symmetry of the max-min CS condition

Jain et al. [2] studied the right-left symmetry of max-min CS condition on prime rings. In this section, we aim to generalize such the results to prime modules. We consider 2 conditions introduced in [6]:

- (I) For submodules X, Y of $M, X \overset{*}{\hookrightarrow} Y$ if and only if $I_X \overset{*}{\hookrightarrow} I_Y$;
- (II) For right ideals K, L of $S, K \overset{*}{\hookrightarrow} L$ if and only if $KM \overset{*}{\hookrightarrow} LM$.

Lemma 3.1 *For a module M, we have the following statements.*

(1) If M is nonsingular, then M is retractable if and only if (I) holds.

(2) Given (I), then (II) holds if and only if $K \overset{*}{\hookrightarrow} I_{KM}$ for every right ideal $K \hookrightarrow S_S$.

(3) Given (II), then (I) holds if and only if $I_X(M) \overset{*}{\hookrightarrow} X$ for every submodule $X \hookrightarrow M$.

(4) Given (I) and (II), then M is retractable.

(5) If M is a self-generator possessing (II), then (I) holds.

(6) Let M be a nonsingular retractable module with (II). Then X is a uniform submodule of M if and only if I_X is a uniform right ideal of S.

Proof We refer readers to [6, Corollary 2.3] for *(1)*, and to [6, Theorem 2.5] for *(2)* and *(3)*.

(4) Given (I) and (II). For every nonzero submodule $X \hookrightarrow M$, we have $I_X(M) \overset{*}{\hookrightarrow} X$ by *(3)*. This implies $I_X(M) \neq 0$ so $I_X \neq 0$. Thus M is retractable.

(5) It follows from *(3)* and $I_X(M) = X$ for every submodule $X \hookrightarrow M$.

(6) Let X be a uniform submodule of M. By retractability of M, we have $I_X \neq 0$. Let H be a nonzero right ideal contained in I_X. Then HM is a nonzero submodule contained in X, and hence $HM \overset{*}{\hookrightarrow} X$. Since M is a nonsingular retractable module, M has (I), hence $I_{HM} \overset{*}{\hookrightarrow} I_X$. By (II) and *(2)*, $H \overset{*}{\hookrightarrow} I_{HM}$ so $H \overset{*}{\hookrightarrow} I_X$. This shows that I_X is a right uniform ideal.

Conversely, let I_X be a uniform right ideal, and A be a nonzero submodule of X. Since M is retractable, I_A is nonzero, hence $I_A \overset{*}{\hookrightarrow} I_X$ and $A \overset{*}{\hookrightarrow} X$ by (I). This implies that X is a uniform submodule. □

Lemma 3.2 *[9, Theorem 12]*

Let M be a nonsingular and retractable module which possesses the condition (II). Then, the following statements hold.

(1) M is min CS if and only if S is right min CS.

(2) M is max CS if and only if S is right max CS.

(3) M is max-min CS if and only if S is right max-min CS.

Proposition 3.3 *Let M be a nonsingular prime retractable module with (II) and have a uniform submodule. Then the following statements hold.*

(1) If M is a min CS, then S is right min CS, right and left nonsingular.

(2) If M is a max CS cononsingular module, then S is right max CS and left min CS.

(3) If M is a CS, then S is right CS and left min CS.

(4) If R is a nondomain ring, and M is a min CS, then S is a right min CS ring with uniform right and left ideals.

Proof

By [4, Theorem 3.1], S is right nonsingular, and by [7, Theorem 2.4], S is a prime ring. Since M has a uniform submodule, S has a uniform right ideal by *(6)* of Lemma 3.1.

(1) By Lemma 3.2, S is right min CS. Since S is a right nonsingular, right min CS, prime ring with a uniform right ideal, S is left nonsingular by [2, Lemma 3.1].

(2) By Lemma 3.2, S is right max CS. By [5, Proposition 1], since M is cononsingular, S if left nonsingular so is nonsingular. Since S is a nonsingular, right max CS, prime ring with a uniform right ideal, S is left min CS by [2, Theorem 3.1].

(3) Obviously, M is a min CS module. Therefore, *(1)* implies that S is left nonsingular. Thus, M is cononsingular by [5, Proposition 1]. Now, it follows from *(2)* that S is left min CS. Clearly, S is right CS by [6, Theorem 3.2].

(4) It is clear that S is right min CS by Lemma 3.2, and S has a uniform right ideal by *(6)* of Lemma 3.1. Consequently, S has a uniform left ideal by [2, Lemma 3.4]. Note that in this statement, S is not a domain, especially S is not a commutative ring. □

Theorem 3.4 *Let R be a nondomain ring, and M be a nonsingular prime retractable module with (II). Then the following conditions are equivalent:*

(1) M is max-min CS with a uniform submodule;

(2) S is right max-min CS with a uniform right ideal;

(3) S is left max-min CS with a uniform left ideal.

Proof *(1)*⇔ *(2)* follows from Lemma 3.1 and Lemma 3.2. By [7, Theorem 2.4], S is a prime ring. By *(1)* of Proposition 3.3, S is right and left nonsingular. Now *(2)*⇔ *(3)* follows from [2, Theorem 3.4]. The proof is complete. □

Recall that a CS module is provided that every closed submodule is a direct summand. It is well-known that a module with finite uniform dimension is CS if and only if it is min CS.

Proposition 3.5 *Let M be a nonsingular prime retractable module possessing (II). If M is CS with a uniform submodule, then the following statements hold:*

(1) S is left CS if and only if $\text{u-dim}({}_S S) < \infty$;

(2) $\text{u-dim}(M_R) = n > 1$ *is equivalent to* $\text{u-dim}({}_S S) = n$.

In these cases, M is a Goldie module, S is a right and left Goldie, right and left CS ring, and $\text{u-dim}(M_R) = \text{u-dim}(S_S) = \text{u-dim}({}_S S)$.

Proof Since M is nonsingular and retractable, Lemma 3.1 says (I) held. By [7, Theorem 2.4], S is a prime ring, and by [4, Theorem 3.1], S is right nonsingular. In addition, S is right CS by [6, Theorem 3.2]. Since M has a uniform submodule, S has a uniform right ideal by Lemma 3.1. By *(1)* of Proposition 3.3, S is left nonsingular, so is nonsingular.

(1) Firstly, let S be left CS. Since S is right and left CS, every closed 1-sided ideal of S is a direct summand, hence is an annihilator. This, by [1, Theorem 2.38], implies that the maximal right and the maximal left quotient rings of S coincide, namely Q. Since S is a nonsingular prime ring with a minimal closed right ideal, Q is semisimple artinian. As a consequence, since ${}_S S$ is essential in ${}_S Q$, S has finite left uniform dimension. Conversely, we assume that S has finite left uniform dimension. By *(3)* of Proposition 3.3, S is left min CS. This implies that S is left CS.

(2) Note that S is prime so is semiprime. If $\text{u-dim}(M_R) = n$, by [9, Lemma 6], we have $\text{u-dim}(M_R) = n = \text{u-dim}(S_S)$. Since S is a nonsingular semiprime ring with finite right uniform dimension (greater than 1), the maximal right quotient ring of S (namely Q) is also the maximal left quotient ring of S. Moreover, Q is a semisimple artinian ring. Consequently, $\text{u-dim}({}_S S) = n$ because of ${}_S S \overset{*}{\hookrightarrow}_S Q$, indeed, $\text{u-dim}({}_S S) = \text{u-dim}({}_S Q) = \text{u-dim}(Q_S) = \text{u-dim}(S_S) = n$. The converse is argued similarly.

For the last claim, it is clear that $\text{u-dim}(M_R) = \text{u-dim}(S_S) = \text{u-dim}({}_S S) = n$. Recall that an $M-$annihilator X of M provides that $X = \mathbf{r}_M(K)$ for some $K \subseteq S$. Note that every nonsingular module with finite uniform dimension has the ACC on $M-$annihilators. In particular, every right (left) nonsingular ring with finite right (left) uniform dimension satisfies the ACC on right (left) annihilators. Therefore, M is a Goldie module, and S is a right and left Goldie. □

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