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Dual and canonical dual of controlled K-g-frames in Hilbert spaces

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Abstract: This paper is devoted to studying the controlled dual K-g-Bessel sequences of controlled K-g-frames. In fact, we introduce the concept of dual K-g-Bessel sequences of controlled K-g-frames and then, we present some necessary and/or sufficient conditions under which a controlled g-Bessel sequence is a controlled dual K-g-frame of a given controlled K-g-frame. Subsequently, we pay attention to investigating the structure of the canonical controlled dual K-g-Bessel sequence of a Parseval controlled K-g-frame and some other related results.

Key words: Controlled frame, K-frame, canonical dual

1. Introduction

The notion of frame dates back to Gabor [14] and Duffin and Schaeffer [12]. However, the frame theory had not attracted much attention until the celebrated work by Daubechies et al. [10]. Frames have been used as a powerful alternative to Hilbert bases because of their redundancy and flexibility. They are also very important for applications, e.g., in physics [1, 8], signal processing [3, 5, 6], numerical treatment of operator equations [9, 24], and acoustics [4, 22].

Over the years, various extensions of frame theory have been investigated. Several of these are contained as special cases of the elegant theory for g-frames that was introduced by Sun in [25]. For example, fusion frames, bounded quasiprojectors, outer frames, oblique frames, pseudoframes, and a class of time-frequency localization operators.

Atomic systems for subspaces were first introduced by Feichtinger and Werther in [13] based on examples arising in sampling theory. In [15], Găvruţa introduced K-frames in Hilbert spaces to study atomic decomposition systems, and discussed some properties of them. K-g-frames, which are more general than g-frames, were put forward by Zhou et al. in [27].

Weighted and controlled frames, as one of the newest generalizations of frames, have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator on abstract Hilbert spaces in [2]; however, they are used earlier in [18] for spherical wavelets. Since then, controlled frames have been generalized to other kinds of frames [17, 19, 21, 23].

Since the frame operator of a controlled K-g-frame may not be invertible, there is no classical canonical dual for a controlled K-g-frame. Thus, it is interesting to find or even define the canonical dual of a controlled

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K-g-frame. Recently, Guo in [16] proposed the concept of canonical dual K-Bessel sequences from the operator theoretic point of view. This idea has been developed to K-g-frame in [26]. In this paper, we generalize this concept to the case of controlled K-g-frames. Indeed, we define the concept of dual and canonical dual of controlled K-g-frames and then we give several equivalent characterizations of them.

2. Notation and definitions

In this section, we collect the basic notation and some preliminary results. Throughout the paper, $\mathcal{H}$ is a separable Hilbert space and $\{\mathcal{H}_i : i \in \mathbb{I}\}$ is a sequence of Hilbert spaces, where $\mathbb{I}$ is an at most countable index set. We denote by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the set of all bounded linear operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. For $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, we use the notations U^*, $R(U)$ and $N(U)$ to denote respectively the adjoint operator, the range, and the null space of U. We define $GL(\mathcal{H}_1, \mathcal{H}_2)$ as the set of all bounded linear operators with a bounded inverse, and similarly for $GL(\mathcal{H})$. It is easy to see that if $U, V \in GL(\mathcal{H})$, then U^* and UV are also in $GL(\mathcal{H})$. A bounded operator T is called positive (respectively nonnegative), if $\langle Tf, f\rangle > 0$, for all $0 \neq f \in \mathcal{H}$, (respectively $\langle Tf, f\rangle \geq 0$, for all $f \in \mathcal{H}$). Every nonnegative operator is clearly self-adjoint. If $U \in \mathcal{B}(\mathcal{H})$ is nonnegative, then there exists a unique nonnegative operator V such that $V^2 = U$. This will be denoted by $V = U^{\frac{1}{2}}$. The operator V commutes with every operator that commutes with U. The set of positive operators in $GL(\mathcal{H})$ will be denoted by $GL^+(\mathcal{H})$. Notice that $U \in GL^+(\mathcal{H})$ if and only if U is positive and $U^{\frac{1}{2}} \in GL(\mathcal{H})$ [2, Proposition 2.4].

Later we will need the following important result from operator theory.

Lemma 2.1 *[7] Suppose that $U \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ has closed range. Then there exists a unique operator $U^\dagger \in \mathcal{B}(\mathcal{K}, \mathcal{H})$, called the pseudoinverse of U, satisfying*

$$UU^\dagger U = U, \quad U^\dagger UU^\dagger = U^\dagger, \quad \left(UU^\dagger\right)^* = UU^\dagger, \quad \left(U^\dagger U\right)^* = U^\dagger U, \quad (K^*)^\dagger = \left(K^\dagger\right)^*,$$

$$N\left(U^\dagger\right) = (R(U))^\perp = N\left(U^*\right), \quad R\left(U^\dagger\right) = (N(U))^\perp = R\left(U^*\right).$$

The following lemma is a key tool for the proofs of our main results.

Lemma 2.2 *[11] Let $T \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H}_2, \mathcal{H})$. The following statements are equivalent:*

1. *$R(T) \subset R(S)$.*
2. *There exists $\lambda > 0$ such that $TT^* \leq \lambda SS^*$.*
3. *There exists $U \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ such that $T = SU$.*

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator U such that

(a) $\|U\|^2 = \inf\{\mu, TT^* \leq \mu SS^*\}$.

(b) $N(T) = N(U)$.

(c) $R(U^*) \subset \overline{R(S^*)}$.

Definition 2.3 *[25] A sequence $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ is called a generalized frame, or simply a g-frame, for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ if there exist constants $0 < A \leq B < \infty$, such that*

$$A\|f\|^2 \leq \sum_{i\in\mathbb{I}} \|\Lambda_i f\|^2 \leq B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.1}$$

The numbers A and B are called g-frame bounds. The family $\{\Lambda_i\}_{i\in\mathbb{I}}$ is called a tight g-frame if $A = B$ and Parseval g-frame if $A = B = 1$. If in (2.1), only the second inequality holds, then the sequence is called a g-Bessel sequence.

For each sequence $\{\mathcal{H}_i : i \in \mathbb{I}\}$, we define the space

$$\left(\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i\right)_{\ell^2} = \left\{\{f_i\}_{i\in\mathbb{I}} : f_i \in \mathcal{H}_i, i \in \mathbb{I}, \text{ and} \sum_{i\in\mathbb{I}} \|f_i\|^2 < \infty\right\},$$

with the inner product defined by

$$\langle \{f_i\}_{i\in\mathbb{I}}, \{g_i\}_{i\in\mathbb{I}}\rangle = \sum_{i\in\mathbb{I}} \langle f_i, g_i\rangle.$$

It is clear that $(\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i)_{\ell^2}$ is a Hilbert space.

We define the synthesis operator for a g-Bessel sequence $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}$ as follows,

$$T_\Lambda : \left(\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i\right)_{\ell^2} \to \mathcal{H}, \quad T_\Lambda\left(\{f_i\}_{i\in\mathbb{I}}\right) = \sum_{i\in\mathbb{I}} \Lambda_i^*(f_i).$$

It is easy to show that the adjoint operator of T_Λ is achieved as follows,

$$T_\Lambda^* : \mathcal{H} \to \left(\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i\right)_{\ell^2}, \quad T_\Lambda^*(f) = \{\Lambda_i f\}_{i\in\mathbb{I}}.$$

The operator T_Λ^* is called the analysis operator for $\{\Lambda_i\}_{i\in\mathbb{I}}$. Composing T_Λ and T_Λ^*, the g-frame operator is obtained as follows,

$$S_\Lambda = T_\Lambda T_\Lambda^* : \mathcal{H} \to \mathcal{H}, \quad S_\Lambda f = \sum_{i\in\mathbb{I}} \Lambda_i^* \Lambda_i f.$$

If $\Lambda = \{\Lambda_i\}_{i\in\mathbb{I}}$ is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ with bounds A and B, then the g-frame operator $S_\Lambda : \mathcal{H} \to \mathcal{H}$ is a bounded, positive, and invertible operator. Moreover, for every $f \in \mathcal{H}$, we have

$$f = S_\Lambda S_\Lambda^{-1} = S_\Lambda^{-1} S_\Lambda f = \sum_{i\in\mathbb{I}} \Lambda_i^* \Lambda_i S_\Lambda^{-1} f = \sum_{i\in\mathbb{I}} S_\Lambda^{-1} \Lambda_i^* \Lambda_i f. \tag{2.2}$$

Let $\tilde{\Lambda}_i = \Lambda_i S_\Lambda^{-1}$. Then the above equalities become

$$f = \sum_{i\in\mathbb{I}} \Lambda_i^* \tilde{\Lambda}_i f = \sum_{i\in\mathbb{I}} \tilde{\Lambda}_i^* \Lambda_i f. \tag{2.3}$$

The family $\{\tilde{\Lambda}_i\}_{i\in\mathbb{I}}$, which is a g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$, is called the canonical dual g-frame of $\{\Lambda_i : i \in \mathbb{I}\}$. Recall that a g-Bessel sequence $\{\Gamma_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$, is called an alternate dual g-frame of $\{\Lambda_i\}_{i\in\mathbb{I}}$ if for every $f \in \mathcal{H}$,

$$f = \sum_{i\in\mathbb{I}} \Lambda_i^* \Gamma_i f.$$

Definition 2.4 *[2] Let $C \in GL(\mathcal{H})$. A frame controlled by the operator C or C-controlled frame is a family of vectors $\{f_i\}_{i\in\mathbb{I}}$ in $\mathcal{H}$ such that there exist two constants $0 < A \le B < \infty$, such that*

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} \langle f, f_i\rangle \langle Cf_i, f\rangle \le B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.4}$$

Definition 2.5 *[23] Let $C, C \in GL^+(\mathcal{H})$. The family $\Lambda = \{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i), i \in \mathbb{I}\}$ is called a (C, C')-controlled g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ if Λ is a g-Bessel sequence and there exist constants $0 < A \le B < \infty$, such that*

$$A\|f\|^2 \le \sum_{i\in\mathbb{I}} \langle \Lambda_i C f, \Lambda_i C' f\rangle \le B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.5}$$

In the first section of [17], it was claimed that any two bounded positive operators can commute with each other. It seems that this is not true in general. Indeed, it is easy to check that if $T, S \in \mathcal{B}(\mathcal{H})$ are two positive operators, then $TS = ST$ if and only if TS is positive. Thus, in the following definition, we assume that two operators $C, C' \in GL^+(\mathcal{H})$ commute with each other.

Definition 2.6 *[17] Suppose that $K \in \mathcal{B}(\mathcal{H})$ and $C, C' \in GL^+(\mathcal{H})$, which commute with each other. The family $\{\Lambda_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ is called a (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ if there exist constants $0 < A \le B < \infty$, such that*

$$A\|K^* f\|^2 \le \sum_{i\in\mathbb{I}} \langle \Lambda_i C f, \Lambda_i C' f\rangle \le B\|f\|^2, \quad (f \in \mathcal{H}). \tag{2.6}$$

The numbers A, B are called the lower and upper frame bounds for (C, C')-controlled K-g-frame, respectively. Particularly, if

$$A\|K^* f\|^2 = \sum_{i\in\mathbb{I}} \langle \Lambda_i C f, \Lambda_i C' f\rangle, \quad (f \in \mathcal{H}),$$

then we call $\{\Lambda_i\}_{i\in\mathbb{I}}$ a (C, C')-controlled tight K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. The (C, C')-controlled tight K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$ is said to be Parseval if $A = 1$.

If the right-hand side of (2.6) holds, then $\{\Lambda_i\}_{i\in\mathbb{I}}$ is called a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$.

If $C' = I$, then $\{\Lambda_i\}_{i\in\mathbb{I}}$ is called a C-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ and if $C = C'$, then $\{\Lambda_i\}_{i\in\mathbb{I}}$ is called a C^2-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$.

Suppose that $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i\in\mathbb{I}\}$. The bounded linear operator $T_{C\Lambda C'} : \left(\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i\right)_{\ell^2}\to\mathcal{H}$ defined as

$$T_{C\Lambda C'}(\{f_i\}_{i\in\mathbb{I}}) = \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*(f_i), \quad (\{f_i\}_{i\in\mathbb{I}}\in(\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2}), \tag{2.7}$$

is called the synthesis operator. The adjoint operator $T^*_{C\Lambda C'} : \mathcal{H}\to\left(\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i\right)_{\ell^2}$ that is obtained as

$$T^*_{C\Lambda C'}(f) = \left\{\Lambda_i(CC')^{\frac{1}{2}}f\right\}_{i\in\mathbb{I}}, \quad (f\in\mathcal{H}), \tag{2.8}$$

is called the analysis operator. Composing $T_{C\Lambda C'}$ and $T^*_{C\Lambda C'}$, the operator $S_{C\Lambda C'} : \mathcal{H}\to\mathcal{H}$ given by

$$S_{C\Lambda C'}f = T_{C\Lambda C'}T^*_{C\Lambda C'}f = \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(CC')^{\frac{1}{2}}f, \quad (f\in\mathcal{H}), \tag{2.9}$$

is called the (C,C')-controlled g-Bessel sequence operator. If $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-g-frame, then $S_{C\Lambda C'}$ is called the (C,C')-controlled K-g-frame operator.

Remark 2.7 *Assume that $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i\in\mathbb{I}\}$, $S_\Lambda : \mathcal{H}\to\mathcal{H}$ is the g-frame operator*

$$S_\Lambda f = \sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i f,$$

and C and C' commute with S_Λ. Then, the (C,C')-controlled g-Bessel sequence operator $S_{C\Lambda C'}$ can be represented in the form

$$\begin{aligned} S_{C\Lambda C'}f &= \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(CC')^{\frac{1}{2}}f \\ &= (CC')^{\frac{1}{2}}\sum_{i\in\mathbb{I}}\Lambda_i^*\Lambda_i(CC')^{\frac{1}{2}}f \\ &= (CC')^{\frac{1}{2}}S_\Lambda(CC')^{\frac{1}{2}}f \\ &= C'S_\Lambda Cf, \end{aligned}$$

for every $f\in\mathcal{H}$.

3. Controlled dual K-g-Bessel sequences

As it was mentioned in [23, Lemma 3.3], the frame operator of a controlled g-frame is invertible, but it is not the case for a controlled K-g-frame. Let us consider the following example.

Example 3.1 *Suppose that $\{e_n\}_{n=1}^6$ is the canonical orthonormal basis for $\mathcal{H}=\mathbb{C}^6$. Let*

$$\mathcal{H}_i = \overline{span}\{e_{3(i-1)+k}\ ;\ 1\le k\le 3\}, \quad i=1,2.$$

Now define the linear bounded operators $K:\mathcal{H}\to\mathcal{H}$, $\Lambda_i:\mathcal{H}\to\mathcal{H}_i$, $C:\mathcal{H}\to\mathcal{H}$ as follows:

$$Ke_1=e_1, \quad Ke_2=e_2, \quad Ke_n=0, \quad 3\le n\le 6,$$

$$\Lambda_1(f) = \sum_{k=1}^{3} \langle f, e_k \rangle e_k, \quad \Lambda_2(f) = 0, \quad C = I_{\mathcal{H}}.$$

It is easy to calculate

$$K^* e_1 = e_1, \quad K^* e_2 = e_2, \quad K^* e_n = 0, \quad 3 \leq n \leq 6.$$

Clearly, $\{\Lambda_i\}_{i=1}^2$ *is a* $I_{\mathcal{H}}$*-controlled K-g-frame, since*

$$\|K^* f\|^2 = \left\| \sum_{n=1}^{6} \langle f, e_n \rangle K^* e_n \right\|^2 = |\langle f, e_1 \rangle|^2 + |\langle f, e_2 \rangle|^2.$$

On the other hand,

$$\sum_{i=1}^{2} \langle \Lambda_i I_{\mathcal{H}} f, \Lambda_i I_{\mathcal{H}} f \rangle = \sum_{i=1}^{2} \|\Lambda_i I_{\mathcal{H}} f\|^2 = \left\| \sum_{k=1}^{3} \langle f, e_k \rangle e_k \right\|^2 = \sum_{k=1}^{3} |\langle f, e_k \rangle|^2 \geq \|K^* f\|^2.$$

Thus, for each $f \in \mathcal{H}$ *we have*

$$\|K^* f\|^2 \leq \sum_{i=1}^{2} \|\Lambda_i I_{\mathcal{H}}\|^2 \leq \|f\|^2.$$

One can show that the associated frame operator is obtained as

$$S = \begin{pmatrix} [I]_{3\times 3} & 0_{3\times 3} \\ 0_{3\times 3} & 0_{3\times 3} \end{pmatrix},$$

which is not invertible.

Due to this fact, the classical canonical dual for a controlled K-g-frame is absent. This motivates us in this section, inspired by the idea of [16], to introduce the concept of (C, C')-controlled dual K-g-Bessel sequences of a (C, C')-controlled K-g-frame. Moreover, we give some of their characterizations by some operator theory tools.

Definition 3.2 *Suppose that* $K \in \mathcal{B}(\mathcal{H})$*,* $C, C' \in GL^+(\mathcal{H})$ *and* $\{\Lambda_i\}_{i \in \mathbb{I}}$ *is a* (C, C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$*. A* (C, C')*-controlled g-Bessel sequence* $\{\Gamma_i\}_{i \in \mathbb{I}}$ *for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$ *is said to be a* (C, C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i \in \mathbb{I}}$ *if*

$$Kf = \sum_{i \in \mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Gamma_i (CC')^{\frac{1}{2}} f, \quad (f \in \mathcal{H}). \tag{3.1}$$

From now on, we consider $C, C' \in GL^+(\mathcal{H})$ and C, C' and S_Λ mutually commute. In [7], we know that the duals of a frame are necessarily frames, but it is not the case for a controlled K-g-frame. In the next proposition, we prove that every controlled dual K-g-Bessel sequence is a controlled K^*-g-frame. Before proceeding, we need the following lemma which characterizes (C, C')-controlled K-g-frames in terms of a range inclusion property.

Lemma 3.3 *Suppose that* $K \in \mathcal{B}(\mathcal{H})$*. A sequence* $\{\Lambda_i\}_{i \in \mathbb{I}}$ *is a* (C, C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$ *if and only if* $R(K) \subset R(T_{C\Lambda C'})$*.*

Proof Suppose that $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. Then, for every $f \in \mathcal{H}$,

$$A\langle KK^*f, f\rangle = A\|K^*f\|^2 \leq \sum_{i\in\mathbb{I}}\langle \Lambda_i Cf, \Lambda_i C'f\rangle = \langle S_{C\Lambda C'}f, f\rangle = \langle T_{C\Lambda C'}T^*_{C\Lambda C'}f, f\rangle\,.$$

Using Lemma 2.2, we obtain $R(K) \subset R(T_{C\Lambda C'})$. The proof of the opposite direction is similar, and thus omitted. □

Proposition 3.4 *Suppose that* $K \in \mathcal{B}(\mathcal{H})$. *Then every* (C,C')*-controlled dual K-g-Bessel sequence is a* (C,C')*-controlled* K^**-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$.

Proof Suppose that $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. According to Lemma 3.3, it is enough to show that $R(K^*) \subset R(T_{C\Gamma C'})$. For every $f \in \mathcal{H}$,

$$Kf = \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Gamma_i(CC')^{\frac{1}{2}}f = T_{C\Lambda C'}T^*_{C\Gamma C'}f.$$

Hence, $K = T_{C\Lambda C'}T^*_{C\Gamma C'}$ and so $K^* = T_{C\Gamma C'}T^*_{C\Lambda C'}$. This shows that $R(K^*) \subset R(T_{C\Gamma C'})$, as desired. □

The following proposition shows that a controlled dual K-g-Bessel sequence can naturally generate a new controlled K-g-frame, provided K has closed range.

Proposition 3.5 *Assume that* $K \in \mathcal{B}(\mathcal{H})$ *has closed range such that* $CK = KC$ *and* $C'K = KC'$. *Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a* (C,C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$ *and* $\{\Gamma_i\}_{i\in\mathbb{I}}$ *is a* (C,C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$. *Then* $\{\Gamma_i K^*\}_{i\in\mathbb{I}}$ *is a* (C,C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$.

Proof For each $f \in \mathcal{H}$,

$$\begin{aligned}\|Kf\|^2 = \sup_{\|h\|=1}|\langle Kf, h\rangle|^2 &= \sup_{\|h\|=1}\left|\left\langle \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Gamma_i(CC')^{\frac{1}{2}}f, h\right\rangle\right|^2\\ &= \sup_{\|h\|=1}\left|\sum_{i\in\mathbb{I}}\left\langle \Gamma_i(CC')^{\frac{1}{2}}f, \Lambda_i(CC')^{\frac{1}{2}}h\right\rangle\right|^2\\ &\leq \sup_{\|h\|=1}\sum_{i\in\mathbb{I}}\left\|\Gamma_i(CC')^{\frac{1}{2}}f\right\|^2\sum_{i\in\mathbb{I}}\left\|\Lambda_i(CC')^{\frac{1}{2}}h\right\|^2\\ &\leq B_\Lambda\sum_{i\in\mathbb{I}}\left\|\Gamma_i(CC')^{\frac{1}{2}}f\right\|^2.\end{aligned}$$

It follows that for every $f \in \mathcal{H}$,

$$B_\Lambda^{-1}\|Kf\|^2 \leq \sum_{i\in\mathbb{I}}\left\|\Gamma_i(CC')^{\frac{1}{2}}f\right\|^2 \leq B_\Gamma\|f\|^2,$$

and consequently

$$B_\Lambda^{-1}\|KK^*f\|^2 \le \sum_{i\in\mathbb{I}} \left\|\Gamma_i(CC')^{\frac{1}{2}}K^*f\right\|^2 \le B_\Gamma\|K^*f\|^2 \le B_\Gamma\|K\|^2\|f\|^2.$$

By assumption K^* has closed range. Hence, for each $g \in R(K^*)$, we get $g = K^*(K^*)^\dagger g = K^\dagger K g$. Therefore,

$$\|g\|^2 = \|K^\dagger K g\|^2 \le \|K^\dagger\|^2\|Kg\|^2.$$

It is concluded that for each $f \in \mathcal{H}$,

$$B_\Lambda^{-1}\|K^\dagger\|^{-2}\|K^*f\|^2 \le \sum_{i\in\mathbb{I}} \left\|\Gamma_i(CC')^{\frac{1}{2}}K^*f\right\|^2 = \sum_{i\in\mathbb{I}} \left\|\Gamma_i K^*(CC')^{\frac{1}{2}}f\right\|^2 \le B_\Gamma\|K\|^2\|f\|^2,$$

and the result follows. □

The following theorem shows that for any controlled K-g-frame there always exists a controlled dual K-g-Bessel sequence such that they provide a reconstruction formula for any element in the range of K.

Theorem 3.6 *Suppose that $K \in \mathcal{B}(\mathcal{H})$. Then every (C, C')-controlled K-g-frame admits a (C, C')-controlled dual K-g-Bessel sequence.*

Proof Let $\{\Lambda_i\}_{i\in\mathbb{I}}$ be a (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. Then for every $f \in \mathcal{H}$,

$$A\langle KK^*f, f\rangle = A\langle K^*f, K^*f\rangle \le \sum_{i\in\mathbb{I}}\langle \Lambda_i Cf, \Lambda_i C'f\rangle = \langle S_{C\Lambda C'}f, f\rangle = \langle T_{C\Lambda C'}T^*_{C\Lambda C'}f, f\rangle.$$

Hence, $KK^* \le A^{-1}T_{C\Lambda C'}T^*_{C\Lambda C'}$. Thus, by Lemma 2.2, there exists $U \in \mathcal{B}(\mathcal{H}, (\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2})$ such that $K = T_{C\Lambda C'}U$. Suppose that P_n is the projection on $(\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2}$ that maps each element to its n-th component, i.e. $P_n\{f_i\}_{i\in\mathbb{I}} = \{0, \dots, 0, f_n, 0, \dots\}_{i\in\mathbb{I}}$, for each $\{f_i\}_{i\in\mathbb{I}} \in (\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2}$. If we set $\Gamma_i = P_i U(CC')^{\frac{-1}{2}}$, for each $i \in \mathbb{I}$, then for every $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i\in\mathbb{I}}\langle \Gamma_i Cf, \Gamma_i C'f\rangle = \sum_{i\in\mathbb{I}}\left\|\Gamma_i(CC')^{\frac{1}{2}}f\right\|^2 &= \sum_{i\in\mathbb{I}}\|P_iUf\|^2 \\ &= \sum_{i\in\mathbb{I}}\|(Uf)_i\|^2 \\ &= \|Uf\|^2 \le \|U\|^2\|f\|^2.\end{aligned}$$

It follows that $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. Moreover,

$$\begin{aligned}Kf = T_{C\Lambda C'}Uf &= \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\,\Lambda_i^*(Uf)_i \\ &= \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*P_iUf \\ &= \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Gamma_i(CC')^{\frac{1}{2}}f.\end{aligned}$$

□

The following proposition shows that for any controlled K-g-frame, there is a unique controlled dual K-g-Bessel sequence whose synthesis operator obtains the minimal norm of the set of the norms of synthesis operators of all controlled dual K-g-Bessel sequences of the controlled K-g-frame.

Proposition 3.7 *Suppose that* $K \in \mathcal{B}(\mathcal{H})$*. Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a* (C,C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$*. Then, there exists a unique* (C,C')*-controlled dual K-g-Bessel sequence* $\{\Gamma_i\}_{i\in\mathbb{I}}$ *of* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *such that* $\|T_{C\Gamma C'}\| \le \|T_{C\Theta C'}\|$ *for any* (C,C')*-controlled dual K-g-Bessel sequence* $\{\Theta_i\}_{i\in\mathbb{I}}$ *of* $\{\Lambda_i\}_{i\in\mathbb{I}}$*.*

Proof Since $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled K-g-frame, so $KK^* \le (A_\Lambda^{-1})T_{C\Lambda C'}T^*_{C\Lambda C'}$. Therefore, by Lemma 2.2, there exists $V \in \mathcal{B}(\mathcal{H}, (\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i)_{\ell^2})$ such that $K = T_{C\Lambda C'}V$ and $\|V\|^2 = \inf\{\mu : \|K^*f\|^2 \le \mu \|T^*_{C\Lambda C'}\|^2\}$. Define $\Gamma_i = P_i V (CC')^{\frac{-1}{2}}$, for every $i \in \mathbb{I}$. Then by Theorem 3.6, $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. Since $P_i T^*_{C\Gamma C'} f = \Gamma_i (CC')^{\frac{1}{2}} f = P_i V f$, for any $i \in \mathbb{I}$, so $T^*_{C\Gamma C'} = V$.

Now, let $\{\Theta_i\}_{i\in\mathbb{I}}$ be any (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. Therefore, $K = T_{C\Lambda C'}T^*_{C\Theta C'}$ and so $K^* = T_{C\Theta C'}T^*_{C\Lambda C'}$. It follows that $\|K^*\|^2 \le \|T_{C\Theta C'}\|^2 \|T_{C\Lambda C'}\|^2$. Consequently,

$$\|T_{C\Theta C'}\|^2 \ge \|V\|^2 = \|T^*_{C\Gamma C'}\|^2 = \|T_{C\Gamma C'}\|^2.$$

□

The (C,C')-controlled dual K-g-Bessel sequence satisfying in Proposition 3.7 is called the canonical (C,C')-controlled dual K-g-Bessel sequence.

4. Canonical controlled Dual K-g-Bessel sequences for Parseval frames

In this section, we give the exact form of the canonical (C,C')-controlled dual K-g-Bessel sequences for Parseval (C,C')-controlled K-g-frames under the condition that K has closed range. Therefore, throughout this section, we will assume that $R(K)$ is closed, since this can assure that the pseudo-inverse $K^\dagger$ of K exists. Moreover, it is worth mentioning that if $K \in \mathcal{B}(\mathcal{H})$ has closed range and $C \in \mathcal{B}(\mathcal{H})$ is an arbitrary operator which commutes with K, then C commutes with $K^\dagger$ [20].

Proposition 4.1 *Assume that* $K \in \mathcal{B}(\mathcal{H})$*,* $CK = KC$ *and* $C'K = KC'$*. Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a Parseval* (C,C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i \in \mathbb{I}\}$*. Then* $\{\Lambda_i (K^\dagger)^*\}_{i\in\mathbb{I}}$ *is a* (C,C')*-controlled dual g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$*.*

Proof Obviously, $\{\Lambda_i (K^\dagger)^*\}_{i\in\mathbb{I}}$ is a (C,C')-controlled g-Bessel sequence for $\mathcal{H}$, since for every $f \in \mathcal{H}$,

$$\begin{aligned}\sum_{i\in\mathbb{I}} \left\langle \Lambda_i (K^\dagger)^* C f, \Lambda_i (K^\dagger)^* C' f \right\rangle &= \sum_{i\in\mathbb{I}} \left\langle \Lambda_i C (K^\dagger)^* f, \Lambda_i C' (K^\dagger)^* f \right\rangle \\ &= \left\| K^* (K^\dagger)^* f \right\|^2 \\ &\le \|K\|^2 \|K^\dagger\|^2 \|f\|^2.\end{aligned}$$

By Lemma 2.1, for every $g \in R(K^*)$, $g = K^*(K^*)^\dagger g = K^*\left(K^\dagger\right)^* g$. Thus,

$$\begin{aligned} Kg = KK^*(K^\dagger)^* g &= \sum_{i\in\mathbb{I}} C'\Lambda_i^*\Lambda_i C\left(K^\dagger\right)^* g \\ &= \sum_{i\in\mathbb{I}} C'\Lambda_i^*\Lambda_i\left(K^\dagger\right)^* Cg \\ &= C'S_\Lambda\left(K^\dagger\right)^* Cg \\ &= (CC')^{\frac{1}{2}}S_\Lambda(CC')^{\frac{1}{2}}\left(K^\dagger\right)^* g \\ &= \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i\left(K^\dagger\right)^*(CC')^{\frac{1}{2}}g. \end{aligned}$$

If $h \in (R(K^*))^\perp = N(K)$, so by Lemma 2.1, $h \in N((K^*)^\dagger) = N((K^\dagger)^*)$. Therefore,

$$\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(K^\dagger)^*(CC')^{\frac{1}{2}}h = \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(CC')^{\frac{1}{2}}(K^\dagger)^* h = 0 = Kh.$$

Altogether we have

$$Kf = \sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(K^\dagger)^*(CC')^{\frac{1}{2}}f, \qquad (f \in \mathcal{H}),$$

and the proof is complete. □

Proposition 4.2 *Suppose that $K \in \mathcal{B}(\mathcal{H})$, $CK = KC$ and $C'K = KC'$. Moreover, let $\{\Lambda_i\}_{i\in\mathbb{I}}$ be a Parseval (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. Then the sequence $\{\Gamma_i \in \mathcal{B}(\mathcal{H}, \mathcal{H}_i) : i \in \mathbb{I}\}$ is a (C, C')-controlled dual g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$ if and only if there exists $U \in \mathcal{B}(\mathcal{H}, (\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2})$ such that $T_{C\Lambda C'}U = 0$ and $\Gamma_i = \Lambda_i(K^\dagger)^* + P_iU(CC')^{\frac{-1}{2}}$, for every $i \in \mathbb{I}$.*

Proof First, assume that $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a a (C, C')-controlled dual g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. Define $U : \mathcal{H} \to (\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2}$ as

$$(Uf)_i := \Gamma_i(CC')^{\frac{1}{2}}f - \Lambda_i(K^\dagger)^*(CC')^{\frac{1}{2}}f.$$

Then $U \in \mathcal{B}\left(\mathcal{H}, \left(\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i\right)_{\ell^2}\right)$ since for each $f \in \mathcal{H}$,

$$\begin{aligned} \|Uf\| &= \left\|\left\{\Gamma_i(CC')^{\frac{1}{2}}f - \Lambda_i(K^\dagger)^*(CC')^{\frac{1}{2}}f\right\}_{i\in\mathbb{I}}\right\| \\ &\le \left\|\left\{\Gamma_i(CC')^{\frac{1}{2}}f\right\}_{i\in\mathbb{I}}\right\| + \left\|\left\{\Lambda_i(K^\dagger)^*(CC')^{\frac{1}{2}}f\right\}_{i\in\mathbb{I}}\right\| \\ &= \left(\sum_{i\in\mathbb{I}}\langle\Gamma_i Cf, \Gamma_i C'f\rangle\right)^{\frac{1}{2}} + \left(\sum_{i\in\mathbb{I}}\left\langle\Lambda_i(K^\dagger)^* Cf, \Lambda_i(K^\dagger)^* C'f\right\rangle\right)^{\frac{1}{2}} \\ &= \left(\sum_{i\in\mathbb{I}}\langle\Gamma_i Cf, \Gamma_i C'f\rangle\right)^{\frac{1}{2}} + \left(\sum_{i\in\mathbb{I}}\left\langle\Lambda_i C(K^\dagger)^* f, \Lambda_i C'(K^\dagger)^* f\right\rangle\right)^{\frac{1}{2}} \\ &\le \sqrt{B_\Gamma}\|f\| + \left\|K^*(K^\dagger)^* f\right\| \\ &\le \left(\sqrt{B_\Gamma} + \|K\|\|K^\dagger\|\right)\|f\|. \end{aligned}$$

Clearly $\Gamma_i = \Lambda_i(K^\dagger)^* + P_i U (CC')^{\frac{-1}{2}}$, for every $i \in \mathbb{I}$, and

$$\begin{aligned} T_{C\Lambda C'}Uf &= \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* (Uf)_i \\ &= \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* (\Gamma_i (CC')^{\frac{1}{2}} f - \Lambda_i (K^\dagger)^* (CC')^{\frac{1}{2}} f) \\ &= \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Gamma_i (CC')^{\frac{1}{2}} f - \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Lambda_i (K^\dagger)^* (CC')^{\frac{1}{2}} f \\ &= Kf - Kf = 0. \end{aligned}$$

Conversely, $\{\Gamma_i\}_{i\in\mathbb{I}} = \left\{\Lambda_i(K^\dagger)^* + P_i U (CC')^{\frac{-1}{2}}\right\}_{i\in\mathbb{I}}$ is a (C, C')-controlled g-Bessel sequence for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in I\}$, since for every $f \in \mathcal{H}$,

$$\begin{aligned} \sum_{i\in\mathbb{I}} \langle \Gamma_i C f, \Gamma_i C' f \rangle &= \sum_{i\in\mathbb{I}} \left\| \Gamma_i (CC')^{\frac{1}{2}} f \right\|^2 \\ &= \sum_{i\in\mathbb{I}} \left\| \Lambda_i (K^\dagger)^* (CC')^{\frac{1}{2}} f + P_i U f \right\|^2 \\ &\leq \sum_{i\in\mathbb{I}} \left\| \Lambda_i (K^\dagger)^* (CC')^{\frac{1}{2}} f \right\|^2 + \sum_{i\in\mathbb{I}} \|P_i U f\|^2 \\ &\leq \left(\|K\|^2 \|K^\dagger\|^2 + \|U\|^2 \right) \|f\|^2. \end{aligned}$$

Moreover, for every $f \in \mathcal{H}$,

$$\begin{aligned} \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Gamma_i (CC')^{\frac{1}{2}} f &= \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \left(\Lambda_i (K^\dagger)^* + P_i U (CC')^{\frac{-1}{2}} \right) (CC')^{\frac{1}{2}} f \\ &= \sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Lambda_i (K^\dagger)^* (CC')^{\frac{1}{2}} f + T_{C\Lambda C'} U f \\ &= Kf, \end{aligned}$$

and so the result follows. □

Corollary 4.3 *Suppose that $K \in \mathcal{B}(\mathcal{H})$, $CK = KC$ and $C'K = KC'$. Moreover, let $\{\Lambda_i\}_{i\in\mathbb{I}}$ be a Parseval (C, C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$. Then $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a (C, C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$ if and only if there exists a (C, C')-controlled g-Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i : i \in \mathbb{I}\}$ such that $\Gamma_i = \Lambda_i(K^\dagger)^* + \Theta_i$ and $\sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* \Theta_i (CC')^{\frac{1}{2}} f = 0$, for each $f \in \mathcal{H}$.*

Proof Suppose that $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a a (C, C')-controlled dual g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. Then by Proposition 4.2, there exists $U \in \mathcal{B}(\mathcal{H}, (\bigoplus_{i\in\mathbb{I}} \mathcal{H}_i)_{\ell^2})$ such that $\Gamma_i = \Lambda_i(K^\dagger)^* + P_i U (CC')^{\frac{-1}{2}}$, for each $i \in \mathbb{I}$, and $\sum_{i\in\mathbb{I}} (CC')^{\frac{1}{2}} \Lambda_i^* (Uf)_i = 0$, for every $f \in \mathcal{H}$. Put $\Theta_i = P_i U (CC')^{\frac{-1}{2}}$, for each $i \in \mathbb{I}$. Then, $\{\Theta_i\}_{i\in\mathbb{I}}$ is a

(C,C')-controlled g-Bessel sequence, since

$$\sum_{i\in\mathbb{I}}\langle\Theta_iCf,\Theta_iC'f\rangle=\sum_{i\in\mathbb{I}}\left\|\Theta_i(CC')^{\frac{1}{2}}f\right\|^2=\sum_{i\in\mathbb{I}}\|P_iUf\|^2$$
$$=\sum_{i\in\mathbb{I}}\|(Uf)_i\|^2$$
$$=\|Uf\|^2\le\|U\|^2\|f\|^2.$$

Furthermore,

$$\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Theta_i(CC')^{\frac{1}{2}}f=\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\left(\Gamma_i-\Lambda_i(K^\dagger)^*\right)(CC')^{\frac{1}{2}}f$$
$$=\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Gamma_i(CC')^{\frac{1}{2}}f-\sum_{i\in I}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(K^\dagger)^*(CC')^{\frac{1}{2}}f$$
$$=Kf-Kf=0$$

Conversely, let $\{\Gamma_i\}_{i\in\mathbb{I}}=\{\Lambda_i(K^\dagger)^*+\Theta_i\}_{i\in\mathbb{I}}$, where $\{\Theta_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled g-Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ for $\mathcal{H}$ with respect to $\{\mathcal{H}_i:i\in I\}$ such that

$$\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Theta_i(CC')^{\frac{1}{2}}f=0,$$

for every $f\in\mathcal{H}$. Then

$$\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Gamma_i(CC')^{\frac{1}{2}}f=\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\left(\Lambda_i(K^\dagger)^*+\Theta_i\right)(CC')^{\frac{1}{2}}f$$
$$=\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Lambda_i(K^\dagger)^*f+\sum_{i\in\mathbb{I}}(CC')^{\frac{1}{2}}\Lambda_i^*\Theta_i(CC')^{\frac{1}{2}}f$$
$$=Kf,$$

from which we conclude that $\{\Gamma_i\}_{i\in\mathbb{I}}$ is a (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. □

The next result gives the exact form of the canonical (C,C')-controlled dual K-g-Bessel sequence of a Parseval (C,C')-controlled K-g-frame.

Proposition 4.4 *Suppose that $K\in\mathcal{B}(\mathcal{H})$, $CK=KC$ and $C'K=KC'$. Moreover, let $\{\Lambda_i\}_{i\in\mathbb{I}}$ be a Parseval (C,C')-controlled K-g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_i:i\in\mathbb{I}\}$. Then $\left\{\Lambda_i(K^\dagger)^*\right\}_{i\in\mathbb{I}}$ is the canonical (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$.*

Proof By Proposition 4.1, $\tilde{\Lambda}=\left\{\Lambda_i(K^\dagger)^*\right\}_{i\in\mathbb{I}}$ is a (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. Thus, it is enough to show that $\|T_{C\tilde{\Lambda}C'}\|\le\|T_{C\Gamma C'}\|$, for every (C,C')-controlled dual K-g-Bessel sequence $\{\Gamma_i\}_{i\in\mathbb{I}}$. According to Proposition 4.2, there exists $U\in\mathcal{B}(\mathcal{H},(\bigoplus_{i\in\mathbb{I}}\mathcal{H}_i)_{\ell^2})$ such that $T_{C\Lambda C'}U=0$ and $\Gamma_i=\Lambda_i(K^\dagger)^*+P_iU(CC')^{\frac{-1}{2}}$, for every $i\in\mathbb{I}$. It is easy to check that $T^*_{C\Gamma C'}=T^*_{C\tilde{\Lambda}C'}+U$. Now, by the fact that $T_{C\tilde{\Lambda}C'} = K^{\dagger}T_{C\Lambda C'}$, we have

$$\begin{aligned}\|T^*_{C\Gamma C'}f\|^2 &= \langle T^*_{C\Gamma C'}f, T^*_{C\Gamma C'}f\rangle \\ &= \left\langle T^*_{C\tilde{\Lambda}C'}f + Uf, T^*_{C\tilde{\Lambda}C'}f + Uf\right\rangle \\ &= \left\|T^*_{C\tilde{\Lambda}C'}f\right\|^2 + \left\langle T^*_{C\tilde{\Lambda}C'}f, Uf\right\rangle + \left\langle Uf, T^*_{C\tilde{\Lambda}C'}f\right\rangle + \|Uf\|^2 \\ &= \|T^*_{C\tilde{\Lambda}C'}f\|^2 + \|Uf\|^2 \geq \|T^*_{C\tilde{\Lambda}C'}f\|^2.\end{aligned}$$

This completes the proof. □

It is concluded from Corollary 4.3 and Proposition 4.4 that the difference between a (C, C')-controlled dual K-g-Bessel $\{\Gamma_i\}_{i\in\mathbb{I}}$ and the canonical (C, C')-controlled dual g-Bessel $\{\Lambda_i(K^{\dagger})^*\}_{i\in\mathbb{I}}$ of the Parseval (C, C')-controlled K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$ can be considered as a (C, C')-controlled g-Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ which $T_{C\Lambda C'}T^*_{C\Theta C'} = 0$.

Remark 4.5 *(1) The canonical (C, C')-controlled dual K-g-Bessel sequence of the Parseval (C, C')-controlled K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$, which will be denoted by $\{\tilde{\Lambda}_i\}_{i\in\mathbb{I}}$ later, is actually a Parseval (C, C')-controlled g-frame on $(N(K))^{\perp}$, since*

$$\begin{aligned}\sum_{i\in\mathbb{I}}\left\langle \Lambda_i(K^{\dagger})^*Cf, \Lambda_i(K^{\dagger})^*C'f\right\rangle &= \sum_{i\in\mathbb{I}}\left\langle \Lambda_iC(K^{\dagger})^*f, \Lambda_iC'(K^{\dagger})^*f\right\rangle \\ &= \left\langle K^*\left(K^{\dagger}\right)^*f, K^*\left(K^{\dagger}\right)^*f\right\rangle \\ &= \left\langle \left(K^{\dagger}K\right)^*f, \left(K^{\dagger}K\right)^*f\right\rangle \\ &= \left\|\left(K^{\dagger}K\right)^*f\right\|^2 \\ &= \left\|K^{\dagger}Kf\right\|^2 \\ &= \|f\|^2,\end{aligned}$$

for every $f \in (N(K))^{\perp} = R(K^{\dagger})$.

(2) The canonical (C, C')-controlled dual K-g-Bessel sequence of the Parseval (C, C')-controlled K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$ is a Parseval (C, C')-controlled $K^{\dagger}K$-g-frame, since

$$\begin{aligned}\sum_{i\in\mathbb{I}}\left\langle \Lambda_i(K^{\dagger})^*Cf, \Lambda_i(K^{\dagger})^*C'f\right\rangle &= \sum_{i\in\mathbb{I}}\left\langle \Lambda_iC(K^{\dagger})^*f, \Lambda_iC'(K^{\dagger})^*f\right\rangle \\ &= \left\langle K^*(K^{\dagger})^*f, K^*(K^{\dagger})^*f\right\rangle \\ &= \left\langle (K^{\dagger}K)^*f, (K^{\dagger}K)^*f\right\rangle \\ &= \left\|(K^{\dagger}K)^*f\right\|^2.\end{aligned}$$

(3) Although the canonical (C, C')-controlled dual K-g-Bessel sequence of the Parseval (C, C')-controlled K-g-frame $\{\Lambda_i\}_{i\in\mathbb{I}}$ is not a Parseval (C, C')-controlled K-g-frame in general, it can generate a new one in the

form $\{\tilde{\Lambda}_i K^*\}_{i\in\mathbb{I}}$, *because*

$$\begin{aligned}\sum_{i\in\mathbb{I}}\left\langle\tilde{\Lambda}_i K^* C f, \tilde{\Lambda}_i K^* C' f\right\rangle &= \sum_{i\in\mathbb{I}}\left\langle\Lambda_i\left(K^\dagger\right)^* K^* C f, \Lambda_i\left(K^\dagger\right)^* K^* C' f\right\rangle\\ &= \sum_{i\in\mathbb{I}}\left\langle\Lambda_i C\left(KK^\dagger\right)^* f, \Lambda_i C'\left(KK^\dagger\right)^* f\right\rangle\\ &= \left\|K^*\left(KK^\dagger\right)^* f\right\|^2\\ &= \left\|(KK^\dagger K)^* f\right\|^2\\ &= \|K^* f\|^2.\end{aligned}$$

Finally, we give a necessary and sufficient condition for a (C,C')-controlled dual K-g-Bessel sequence of a Parseval (C,C')-controlled K-g-frame to be the canonical (C,C')-controlled dual K-g-Bessel sequence.

Proposition 4.6 *Suppose that* $K\in\mathcal{B}(\mathcal{H})$, $CK=KC$ *and* $C'K=KC'$. *Moreover, let* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *be a Parseval* (C,C')*-controlled K-g-frame for* $\mathcal{H}$ *with respect to* $\{\mathcal{H}_i : i\in\mathbb{I}\}$ *and* $\{\Gamma_i\}_{i\in\mathbb{I}}$ *be a* (C,C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$. *Then* $\{\Gamma_i\}_{i\in\mathbb{I}}$ *is the canonical* (C,C')*-controlled dual K-g-Bessel sequence of* $\{\Lambda_i\}_{i\in\mathbb{I}}$ *if and only if* $T_{C\Gamma C'}T^*_{C\Gamma C'} = T_{C\Gamma C'}T^*_{C\Theta C'}$, *for any* (C,C')*-controlled dual K-g-Bessel sequence* $\{\Theta_i\}_{i\in\mathbb{I}}$ *of* $\{\Lambda_i\}_{i\in\mathbb{I}}$.

Proof First, assume that $\{\Gamma_i\}_{i\in\mathbb{I}} = \{\tilde{\Lambda}_i\}_{i\in\mathbb{I}}$ is the canonical (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. A direct calculation can show that $T^*_{C\Gamma C'} = T^*_{C\Lambda C'}(K^\dagger)^*$. On the other hand, for every $f\in\mathcal{H}$,

$$T_{C\Lambda C'}(T^*_{C\Lambda C'} - T^*_{C\Theta C'})f = T_{C\Lambda C'}T^*_{C\Lambda C'}f - T_{C\Lambda C'}T^*_{C\Theta C'}f = Kf - Kf = 0,$$

where $\{\Theta_i\}_{i\in\mathbb{I}}$ is any (C,C')-controlled dual K-g-Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ of $\{\Lambda_i\}_{i\in\mathbb{I}}$. From these facts, we obtain, for any $f,g\in\mathcal{H}$,

$$\begin{aligned}\langle T_{C\Gamma C'}\left(T^*_{C\Gamma C'} - T^*_{C\Theta C'}\right)f, g\rangle &= \langle\left(T^*_{C\Gamma C'} - T^*_{C\Theta C'}\right)f, T^*_{C\Gamma C'}g\rangle\\ &= \left\langle\left(T^*_{C\Gamma C'} - T^*_{C\Theta C'}\right)f, T^*_{C\Lambda C'}\left(K^\dagger\right)^* g\right\rangle\\ &= \left\langle K^\dagger T_{C\Lambda C'}\left(T^*_{C\Gamma C'} - T^*_{C\Theta C'}\right)f, g\right\rangle = 0.\end{aligned}$$

It follows that $T_{C\Gamma C'}\left(T^*_{C\Gamma C'} - T^*_{C\Theta C'}\right) = 0$, or equivalently $T_{C\Gamma C'}T^*_{C\Gamma C'} = T_{C\Gamma C'}T^*_{C\Theta C'}$.

Conversely, let $T_{C\Gamma C'}T^*_{C\Gamma C'} = T_{C\Gamma C'}T^*_{C\Theta C'}$, for any (C,C')-controlled dual K-g-Bessel sequence $\{\Theta_i\}_{i\in\mathbb{I}}$ of $\{\Lambda_i\}_{i\in\mathbb{I}}$. Then

$$\|T^*_{C\Gamma C'}\|^2 = \|T_{C\Gamma C'}T^*_{C\Gamma C'}\| = \|T_{C\Gamma C'}T^*_{C\Theta C'}\| \leq \|T^*_{C\Gamma C'}\|\,\|T^*_{C\Theta C'}\|,$$

which shows that $\|T^*_{C\Gamma C'}\| \leq \|T^*_{C\Theta C'}\|$. This implies that $\{\Gamma_i\}_{i\in\mathbb{I}}$ is the canonical (C,C')-controlled dual K-g-Bessel sequence of $\{\Lambda_i\}_{i\in\mathbb{I}}$. □

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