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The Meyer function on the handlebody group

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Abstract: We give an explicit formula for the signature of handlebody bundles over the circle in terms of the homological monodromy. This gives a cobounding function of Meyer's signature cocycle on the mapping class group of a 3-dimensional handlebody, i.e. the handlebody group. As an application, we give a topological interpretation for the generator of the first cohomology group of the hyperelliptic handlebody group.

Key words: Signature cocycle, handlebody group, mapping class groups

1. Introduction

Let Σ_g be a closed connected oriented surface of genus g and $\mathrm{Mod}(\Sigma_g)$ the mapping class group of Σ_g, namely the group of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g. Unless otherwise stated, we assume that (co)homology groups have coefficients in $\mathbb{Z}$. The second cohomology of $\mathrm{Mod}(\Sigma_g)$ has been determined for all $g \geq 1$ by works of many people, in particular by the seminal work of Harer [6, 7] for $g \geq 3$. We have $H^2(\mathrm{Mod}(\Sigma_1)) \cong \mathbb{Z}/12\mathbb{Z}$, $H^2(\mathrm{Mod}(\Sigma_2)) \cong \mathbb{Z}/10\mathbb{Z}$, and

$$H^2(\mathrm{Mod}(\Sigma_g)) \cong \mathbb{Z} \quad \text{for } g \geq 3.$$

There are various interesting constructions of nontrivial second cohomology class of $\mathrm{Mod}(\Sigma_g)$; the reader is referred to the survey article [13]. Among others, the remarkable approach of Meyer [16, 17] was to consider the signature of Σ_g-bundles over surfaces. The central object that Meyer used was a normalized 2-cocycle

$$\tau_g\colon \mathrm{Sp}(2g;\mathbb{Z}) \times \mathrm{Sp}(2g;\mathbb{Z}) \to \mathbb{Z}$$

on the integral symplectic group of degree $2g$.

Meyer showed that for $g \geq 3$ the pullback of the cohomology class of τ_g by the homology representation $\rho\colon \mathrm{Mod}(\Sigma_g) \to \mathrm{Sp}(2g;\mathbb{Z})$ is of infinite order in $H^2(\mathrm{Mod}(\Sigma_g))$. On the other hand, if $g = 1, 2$ then $[\rho^*\tau_g]$ is torsion and there exists a (unique) rational valued cobounding function $\phi_g\colon \mathrm{Mod}(\Sigma_g) \to \mathbb{Q}$ of $\rho^*\tau_g$. This means that

$$\tau_g(\rho(\varphi_1), \rho(\varphi_2)) = \phi_g(\varphi_1) + \phi_g(\varphi_2) - \phi_g(\varphi_1\varphi_2) \quad \text{for any } \varphi_1, \varphi_2 \in \mathrm{Mod}(\Sigma_g).$$

Since the case $g = 1$ was extensively studied by Meyer, such a cobounding function is called a Meyer function. Some number-theoretic and differential geometric aspects of the function ϕ_1 were studied by Atiyah [2]. The

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case $g = 2$ was studied by Matsumoto [15], Morifuji [18], and Iida [11]. For $g \geq 3$, there is no cobounding function of $\rho^*\tau_g$ on the whole mapping class group $\mathrm{Mod}(\Sigma_g)$. However, if we restrict $\rho^*\tau_g$ to a subgroup called the hyperelliptic mapping class group $\mathcal{H}(\Sigma_g)$, then it is known that there is a (unique) cobounding function $\phi_g^{\mathcal{H}} \colon \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ of $\rho^*\tau_g$. Note that $\mathcal{H}(\Sigma_g) = \mathrm{Mod}(\Sigma_g)$ for $g = 1, 2$. This function $\phi_g^{\mathcal{H}}$ was studied by Endo [4] and Morifuji [18]. One motivation for studying Meyer functions comes from the localization phenomenon of the signature of fibered 4-manifolds. See, e.g., [1, 14].

In this paper, we study a new example of Meyer functions: the Meyer function on the handlebody group. The handlebody group of genus g, which we denote by $\mathrm{Mod}(V_g)$, is defined as the group of isotopy classes of orientation-preserving self-diffeomorphisms of the 3-dimensional handlebody V_g of genus g. It is well known that the natural homomorphism $\mathrm{Mod}(V_g) \to \mathrm{Mod}(\Sigma_g), \varphi \mapsto \varphi|_{\Sigma_g}$ is injective since V_g is an irreducible 3-manifold. Therefore, we can think of $\mathrm{Mod}(V_g)$ as a subgroup of $\mathrm{Mod}(\Sigma_g)$. For a mapping class $\varphi \in \mathrm{Mod}(V_g)$, we denote by M_φ the mapping torus of φ. It is a compact oriented 4-manifold. We define

$$\phi_g^V(\varphi) := \mathrm{Sign}\, M_\varphi \in \mathbb{Z}.$$

We show in Lemma 4.2 that ϕ_g^V is a cobounding function of the cocycle $\rho^*\tau_g$ on the handlebody group $\mathrm{Mod}(V_g)$. If $g \geq 3$, this is the unique cobounding function since $H_1(\mathrm{Mod}(V_g))$ is torsion (see [21, Theorem 20] and [12, Remark 3.5]).

The value $\phi_g^V(\varphi)$ can be computed from the action of φ on the first homology $H_1(\Sigma_g)$, and our first result gives its explicit description. To state it, we take a suitable basis of $H_1(\Sigma_g)$ so that the homology representation ρ restricted to $\mathrm{Mod}(V_g)$ takes values in a subgroup $\mathrm{urSp}(2g;\mathbb{Z}) \subset \mathrm{Sp}(2g;\mathbb{Z})$. (See Section 2.3 for details.) Then, $\rho(\varphi)$ is of the form $\rho(\varphi) = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix}$, where P, Q, and S are $g \times g$ matrices. We consider a $\mathbb{Q}$-linear space $U_\varphi := \mathrm{Ker}(S - I_g) \subset \mathbb{Q}^g$, and define a bilinear form $\langle \ , \ \rangle_\varphi$ on it by

$$\langle x, y \rangle_\varphi := {}^t x\, {}^t Q\, y, \quad \text{for } x, y \in U_\varphi.$$

It turns out that $\langle \ , \ \rangle_\varphi$ is symmetric, and we have the following:

Theorem 1.1 *The value $\phi_g^V(\varphi)$ coincides with the signature of the symmetric bilinear form $\langle \ , \ \rangle_\varphi$ on U_φ.*

In fact, we will show in Section 3.5 that the intersection form on $H_2(M_\varphi)$ is equivalent to the bilinear form $\langle \ , \ \rangle_\varphi$.

As a corollary, we see that the function ϕ_g^V is bounded by $g = \mathrm{rank}\, H_1(V_g)$. We also give sample calculations of ϕ_g^V in Lemmas 4.4 and 4.5. Walker also constructed a function $j \colon \mathrm{Mod}(\Sigma_g) \to \mathbb{Q}$ whose restriction to $\mathrm{Mod}(V_g)$ coincides with ϕ_g^V. Our description of ϕ_g^V in Theorem 1.1 is similar to but different from a description of j given by Gilmer and Masbaum [5, Proposition 6.9]. See, for details, Remark 3.6.

As an application of the function ϕ_g^V, we obtain a nontrivial first cohomology class in the intersection $\mathcal{H}(\Sigma_g) \cap \mathrm{Mod}(V_g)$ called the hyperelliptic handlebody group, denoted by $\mathcal{H}(V_g)$. The group $\mathcal{H}(V_g)$ is an extension by $\mathbb{Z}/2\mathbb{Z}$ of a subgroup of the mapping class group of a 2-sphere with $(2g+2)$-punctures, called the Hilden group. The Hilden group was introduced in [8], and it is related to the study of links in 3-manifolds. In

[10], Hirose and Kin studied the minimal dilatation of pseudo-Anosov elements in $\mathcal{H}(V_g)$, and gave a presentation of $\mathcal{H}(V_g)$.

We consider the difference

$$\phi_g^{\mathcal{H}} - \phi_g^V \in \mathrm{Hom}(\mathcal{H}(V_g), \mathbb{Q}) = H^1(\mathcal{H}(V_g); \mathbb{Q})$$

of the Meyer functions on $\mathcal{H}(\Sigma_g)$ and on $\mathrm{Mod}(V_g)$. From the abelianization of $\mathcal{H}(V_g)$ obtained in [10, Corollary A.9], we see that the rank of $H^1(\mathcal{H}(V_g))$ is one. Let us denote a generator of $H^1(\mathcal{H}(V_g))$ by μ. Our second result is:

Theorem 1.2 *Let $g \geq 1$. We have*

$$\phi_g^{\mathcal{H}} - \phi_g^V = \begin{cases} \dfrac{2}{2g+1}\mu & \text{if } g \text{ is even,} \\ \dfrac{1}{2g+1}\mu & \text{if } g \text{ is odd.} \end{cases}$$

When $g = 1, 2$, we have $\mathcal{H}(V_g) = \mathrm{Mod}(V_g)$, and $\phi_g^{\mathcal{H}} - \phi_g^V$ gives an abelian quotient of $\mathrm{Mod}(V_g)$.

There is an interpretation of the cohomology class $\phi_g^{\mathcal{H}} - \phi_g^V$ in terms of a kind of connecting homomorphism. We assume that $g \geq 3$. From the diagram

$$\begin{array}{ccc} \mathcal{H}(V_g) & \xrightarrow{i_2} & \mathrm{Mod}(V_g) \\ {\scriptstyle i_1}\downarrow & & \downarrow{\scriptstyle j_2} \\ \mathcal{H}(\Sigma_g) & \xrightarrow[j_1]{} & \mathrm{Mod}(\Sigma_g). \end{array}$$

of groups and their inclusions, we have a natural homomorphism

$$\Upsilon\colon H^2(\mathrm{Mod}(\Sigma_g); \mathbb{Q}) \to H^1(\mathcal{H}(V_g); \mathbb{Q})$$

defined as follows. For $[c] \in H^2(\mathrm{Mod}(\Sigma_g); \mathbb{Q})$, there are cobounding functions $f^{\mathcal{H}}\colon \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ of $j_1^* c$ and $f^V\colon \mathrm{Mod}(V_g) \to \mathbb{Q}$ of $j_2^* c$, respectively. The cochain $i_1^* f^{\mathcal{H}} - i_2^* f^V$ is actually a homomorphism on $\mathcal{H}(V_g)$. It does not depend on the choices of the representatives c, $f^{\mathcal{H}}$, and f^V since $H^1(\mathrm{Mod}(V_g); \mathbb{Q}) = H^1(\mathcal{H}(\Sigma_g); \mathbb{Q}) = 0$ when $g \geq 3$. Then $\Upsilon([c])$ is defined to be $i_1^* f^{\mathcal{H}} - i_2^* f^V$. In this setting, our cohomology class is written as $\Upsilon([\tau_g]) = \phi_g^{\mathcal{H}} - \phi_g^V \in H^1(\mathcal{H}(V_g); \mathbb{Q})$.

The outline of this paper is as follows. In Section 2, we review the definition of Meyer's signature cocycle and the handlebody group $\mathrm{Mod}(V_g)$. We also review the abelianization of the hyperelliptic handlebody group obtained in [10], and describe a generator of the cohomology group $H^1(\mathcal{H}(V_g))$ in Corollary 2.6. In Section 3, we investigate the intersection form of the mapping torus of $\varphi \in \mathrm{Mod}(V_g)$, and prove Theorem 1.1. As it turns out, we can explicitly describe ϕ_g^V as a function on a subgroup $\mathrm{urSp}(2g; \mathbb{Z})$ of the integral symplectic group. In Section 4, we prove Theorem 1.2 by using explicit calculations of the Meyer function $\phi_g^V\colon \mathrm{Mod}(V_g) \to \mathbb{Z}$ in Lemmas 4.4 and 4.5.

2. Preliminaries on mapping class groups

Fix a nonnegative integer g.

2.1. Mapping class group of a surface

Let Σ_g be a closed connected oriented surface of genus g. The *mapping class group* of Σ_g, denoted by $\mathrm{Mod}(\Sigma_g)$, is the group of isotopy classes of orientation-preserving self-diffeomorphisms of Σ_g. To simplify notation, we will use the same letter for a self-diffeomorphism of Σ_g and its isotopy class.

The first homology group $H_1(\Sigma_g)$ is equipped with a nondegenerate skew-symmetric pairing $\langle\cdot,\cdot\rangle$, namely the intersection form. Thus, we can take a symplectic basis $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ for $H_1(\Sigma_g)$. This means that $\langle\alpha_i,\beta_j\rangle=\delta_{ij}$ and $\langle\alpha_i,\alpha_j\rangle=\langle\beta_i,\beta_j\rangle=0$ for any $i,j\in\{1,\dots,g\}$, where δ_{ij} is the Kronecker symbol.

Once a symplectic basis for $H_1(\Sigma_g)$ is fixed, we obtain the homology representation

$$\rho\colon\ \mathrm{Mod}(\Sigma_g)\to\mathrm{Sp}(2g;\mathbb{Z}),\quad \varphi\mapsto\varphi_*.$$

Here, the target is the integral *symplectic group*

$$\mathrm{Sp}(2g;\mathbb{Z})=\{A\in\mathrm{GL}(2g;\mathbb{Z})\mid {}^tAJA=J\},$$

where $J=\begin{pmatrix}O_g & I_g\\ -I_g & O_g\end{pmatrix}$, and $\rho(\varphi)=\varphi_*$ is the matrix presentation of the action of φ on $H_1(\Sigma_g)$ with respect to the fixed symplectic basis. We use block matrices to denote elements in $\mathrm{Sp}(2g;\mathbb{Z})$, e.g., $A=\begin{pmatrix}P & Q\\ R & S\end{pmatrix}$ with $g\times g$ integral matrices P, Q, R, and S.

2.2. Meyer's signature cocycle

Let $A,B\in\mathrm{Sp}(2g;\mathbb{Z})$. We consider an $\mathbb{R}$-linear space

$$V_{A,B}:=\{(x,y)\in\mathbb{R}^{2g}\oplus\mathbb{R}^{2g}\mid (A^{-1}-I_{2g})x+(B-I_{2g})y=0\}$$

and a bilinear form on $V_{A,B}$ given by

$$\langle(x,y),(x',y')\rangle_{A,B}:={}^t(x+y)J(I_{2g}-B)y'.$$

The form $\langle\cdot,\cdot\rangle_{A,B}$ turns out to be symmetric, and thus its signature is defined; we set

$$\tau_g(A,B):=\mathrm{Sign}(V_{A,B},\langle\cdot,\cdot\rangle_{A,B}).$$

The map $\tau_g\colon\ \mathrm{Sp}(2g;\mathbb{Z})\times\mathrm{Sp}(2g;\mathbb{Z})\to\mathbb{Z}$ is called *Meyer's signature cocycle* [16, 17]. It is a normalized 2-cocycle of the group $\mathrm{Sp}(2g;\mathbb{Z})$.

Let P be a compact oriented surface of genus 0 with three boundary components, i.e. a pair of pants. We denote by C_1, C_2, and C_3 the boundary components of P. Choose a base point in P, and let ℓ_1, ℓ_2, and ℓ_3 be based loops in P such that ℓ_i is parallel to the negatively oriented boundary component C_i for any $i\in\{1,2,3\}$ and $\ell_1\ell_2\ell_3=1$ holds in the fundamental group $\pi_1(P)$.

For given two mapping classes $\varphi_1,\varphi_2\in\mathrm{Mod}(\Sigma_g)$, there is an oriented Σ_g-bundle $E(\varphi_1,\varphi_2)\to P$ such that the monodromy along ℓ_i is φ_i for $i=1,2$. It is unique up to bundle isomorphisms. The total space $E(\varphi_1,\varphi_2)$ is a compact 4-manifold equipped with a natural orientation; hence, its signature is defined.

Proposition 2.1 (Meyer [16, 17]) $\mathrm{Sign}(E(\varphi_1, \varphi_2)) = \tau_g(\rho(\varphi_1), \rho(\varphi_2))$.

Remark 2.2 *Turaev [20] independently found the signature cocycle. He also studied its relation to the Maslov index.*

2.3. Handlebody group

Let V_g be a handlebody of genus g. That is, V_g is obtained by attaching g one-handles to the 3-ball D^3. We identify Σ_g and the boundary of V_g by choosing an orientation-preserving diffeomorphism between them. We have the following short exact sequence

$$0 \longrightarrow H_2(V_g, \Sigma_g) \xrightarrow{\partial_*} H_1(\Sigma_g) \xrightarrow{i_*} H_1(V_g) \longrightarrow 0 \tag{2.1}$$

which is a part of the homology exact sequence of the pair (V_g, Σ_g). There are properly embedded, oriented and pairwise disjoint disks $D_1, \dots, D_g$ in V_g whose homology classes (denoted by the same letters) constitute a basis for $H_2(V_g, \Sigma_g)$. We set $\alpha_i := \partial_*(D_i) \in H_1(\Sigma_g)$ for $i \in \{1, \dots, g\}$. Then α_i's extend to a symplectic basis $\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g$ for $H_1(\Sigma_g)$. In what follows, we fix a symplectic basis obtained in this way. The image of the homology classes $\beta_1, \dots, \beta_g$ by the map i_* constitute a basis for $H_1(V_g)$. For simplicity, we denote them by the same letters $\beta_1, \dots, \beta_g$.

We denote by $\mathrm{Mod}(V_g)$ the *handlebody group* of genus g. It can be considered a subgroup of $\mathrm{Mod}(\Sigma_g)$. For any $\varphi \in \mathrm{Mod}(V_g)$, the matrix $\rho(\varphi)$ lies in the subgroup of $\mathrm{Sp}(2g;\mathbb{Z})$ defined by

$$\mathrm{urSp}(2g;\mathbb{Z}) := \left\{ A \in \mathrm{Sp}(2g;\mathbb{Z}) \mid A = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \right\},$$

cf. [3, 9] for details. The matrices P, Q, and S satisfy the following relations:

$$^tPS = I_g, \quad ^tQS = {}^tSQ. \tag{2.2}$$

Remark 2.3 *The group* $\mathrm{Mod}(V_g)$ *acts naturally on the groups in* (2.1)*, and the maps* ∂_* *and* i_* *are* $\mathrm{Mod}(V_g)$*-module homomorphisms. The matrix presentation of the action* φ_* *on* $H_1(V_g)$ *is* S*.*

2.4. Hyperelliptic handlebody group

An involution of Σ_g is called *hyperelliptic* if it acts on $H_1(\Sigma_g)$ as $-\mathrm{id}$. We fix an hyperelliptic involution ι which extends to an involution of V_g, as in Figure 1.

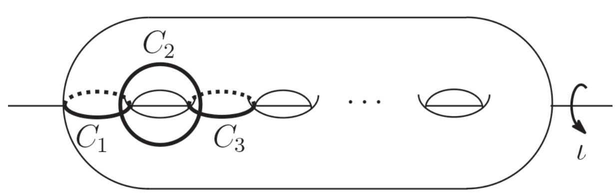

Figure 1. The involution ι of V_g and the curves C_1, C_2, C_3.

The *hyperelliptic mapping class group* $\mathcal{H}(\Sigma_g)$ is the centralizer of ι in $\mathrm{Mod}(\Sigma_g)$:

$$\mathcal{H}(\Sigma_g) := \{\varphi \in \mathrm{Mod}(\Sigma_g) \mid \varphi\iota = \iota\varphi\}.$$

Definition 2.4 ([10]) *The* hyperelliptic handlebody group $\mathcal{H}(V_g)$ *is defined by*

$$\mathcal{H}(V_g) := \mathcal{H}(\Sigma_g) \cap \mathrm{Mod}(V_g).$$

Hirose and Kin [10, Appendix A] gave a finite presentation of the group $\mathcal{H}(V_g)$. Moreover, they determined the abelianization of $\mathcal{H}(V_g)$ as

$$\mathcal{H}(V_g)^{\mathrm{abel}} \cong \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \quad \text{for } g \geq 2.$$

In fact, using their presentation, it is easy to make this result more explicit. Let C_1, C_2, and C_3 be simple closed curves on Σ_g as in Figure 1. For each $i \in \{1,2,3\}$ denote by t_i the *right* handed Dehn twist along C_i. Following [10], set $r_1 = t_2^{-1} t_3^{-1} t_1 t_2$ and $s_1 = t_2 t_3 t_1 t_2$. (Note that in [10], t_C denotes the *left* handed Dehn twist along C.)

Lemma 2.5 *When* $g = 1$, *one has* $\mathcal{H}(V_1) \cong \mathbb{Z}\,[t_1 s_1] \oplus \mathbb{Z}_2\,[t_1^2 s_1]$. *If* $g \geq 2$, *then*

$$\mathcal{H}(V_g)^{\mathrm{abel}} \cong \begin{cases} \mathbb{Z}\,[s_1] \oplus \mathbb{Z}_2\,[t_1 s_1^{\frac{g}{2}}] \oplus \mathbb{Z}_2\,[r_1] & \text{if } g \text{ is even,} \\ \mathbb{Z}\,[t_1 s_1^{\frac{g+1}{2}}] \oplus \mathbb{Z}_2\,[t_1^2 s_1^g] \oplus \mathbb{Z}_2\,[r_1] & \text{if } g \text{ is odd.} \end{cases}$$

Here, $[s_1]$ *is the class of* s_1 *in* $\mathcal{H}(V_g)^{\mathrm{abel}}$, *and* $\mathbb{Z}\,[s_1]$ *is the infinite cyclic group generated by* $[s_1]$, *etc.*

Proof The case $g = 1$ follows from the fact that $\mathcal{H}(V_1) \cong \mathrm{Mod}(V_1)$ and a result of Wajnryb [21, Theorem 14].

Assume that $g \geq 2$. Using [10, Theorem A.8], one sees that $\mathcal{H}(V_g)^{\mathrm{abel}}$ is generated by $[r_1]$, $[s_1]$, and $[t_1]$ with the relations

$$2[r_1] = 0, \quad 4[t_1] + 2g[s_1] = 0, \quad 2(g+1)[t_1] + g(g+1)[s_1] = 0.$$

The assertion follows from these relations by a direct computation. □

The following corollary to Lemma 2.5 will be used in Section 4.4 to prove Theorem 1.2.

Corollary 2.6 *Let* $g \geq 1$. *There is a unique homomorphism* $\mu\colon \mathcal{H}(V_g) \to \mathbb{Z}$ *satisfying the following property:*

(1) *If* g *is even,* $\mu(s_1) = 1$ *and* $\mu(t_1) = -g/2$;

(2) *If* g *is odd,* $\mu(t_1) = -g$, $\mu(s_1) = 2$, *and thus* $\mu(t_1 s_1^{\frac{g+1}{2}}) = 1$.

Moreover, the first cohomology group $H^1(\mathcal{H}(V_g)) = \mathrm{Hom}(\mathcal{H}(V_g), \mathbb{Z})$ *is an infinite cyclic group generated by* μ.

3. Handlebody bundles over S^1

3.1. Mapping torus

Let $I = [0,1]$ be the unit interval. By identifying the endpoints of I, we obtain the circle $S^1 = [0,1]/0 \sim 1$. Let $\ell\colon I \to S^1$ be the natural projection. For $t \in I$, we set $[t] := \ell(t)$. Choose $[0]$ as a base point of S^1. Then the fundamental group $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of ℓ.

In what follows, we use the following cell decomposition of S^1: the 0-cell is $e^0 = [0]$ and the 1-cell is $e^1 = S^1 \setminus e^0$. The map ℓ induces an orientation of e^1.

Let $\varphi \in \mathrm{Mod}(V_g)$. The *mapping torus* of φ is the quotient space

$$M_\varphi := (I \times V_g)/(0,x) \sim (1, \varphi(x)).$$

For $(t,x) \in I \times V_g$, its class in M_φ is denoted by $[t,x]$. The natural projection $\pi\colon M_\varphi \to S^1, [t,x] \mapsto [t]$ is an oriented V_g-bundle, and the total space M_φ is a compact 4-manifold with boundary equipped with a natural orientation. The pullback of $M_\varphi \to S^1$ by ℓ is a trivial V_g-bundle over I, and its trivialization is given by the map

$$\Phi\colon I \times V_g \to M_\varphi, \quad (t,x) \mapsto [t,x]. \tag{3.1}$$

The following composition of maps coincides with φ:

$$V_g \overset{0\times \mathrm{id}}{\cong} \{0\} \times V_g \xrightarrow{\Phi(0,\cdot)} \pi^{-1}([0]) = \pi^{-1}([1]) \xrightarrow{\Phi(1,\cdot)^{-1}} \{1\} \times V_g \overset{1\times \mathrm{id}}{\cong} V_g.$$

Therefore, the monodromy of $M_\varphi \to S^1$ along ℓ is equal to the mapping class φ. As was mentioned in Remark 2.3, the groups $H_2(V_g, \Sigma_g)$, $H_1(\Sigma_g)$, and $H_1(V_g)$ are $\mathrm{Mod}(V_g)$-modules. Thus, these groups become $\pi_1(S^1)$-modules; the homotopy class of ℓ, which is a generator of $\pi_1(S^1)$, acts as the monodromy $\varphi \in \mathrm{Mod}(V_g)$.

3.2. Second homology of the mapping torus

For a nonnegative integer $q \geq 0$, let $\mathscr{H}_q(V_g)$ be the local system on S^1 which comes from the V_g-bundle $\pi\colon M_\varphi \to S^1$, and whose fiber at $x \in S^1$ is the q-th homology group $H_q(\pi^{-1}(x))$. Similarly, we consider the local system $\mathscr{H}_q(V_g, \Sigma_g)$ whose fiber at $x \in S^1$ is the q-th relative homology group $H_q(\pi^{-1}(x), \partial\pi^{-1}(x))$.

Consider the Serre homology spectral sequence of the V_g-bundle $M_\varphi \to S^1$. It degenerates at the E^2 page, which is given by $E^2_{p,q} = H_p(S^1; \mathscr{H}_q(V_g))$. Since $H_2(V_g) = 0$ and the base space S^1 is 1-dimensional, we obtain

$$H_2(M_\varphi) \cong E^\infty_{1,1} \cong E^2_{1,1} = H_1(S^1; \mathscr{H}_1(V_g)).$$

Moreover, using the cellular homology of S^1 with coefficients in $\mathscr{H}_1(V_g)$, we have

$$\begin{aligned} H_1(S^1; \mathscr{H}_1(V_g)) &\cong \mathrm{Ker}(\partial\colon C_1(S^1; \mathscr{H}_1(V_g)) \to C_0(S^1; \mathscr{H}_1(V_g))) \\ &= \mathrm{Ker}(\partial\colon \mathbb{Z}e^1 \otimes H_1(V_g) \to \mathbb{Z}e^0 \otimes H_1(V_g) = H_1(V_g)), \end{aligned}$$

where the boundary map is given by

$$\partial(e^1 \otimes \alpha) = \ell_*(\alpha) - \alpha = (\Phi(0,\cdot)^{-1} \circ \Phi(1,\cdot))_*(\alpha) - \alpha = \varphi_*^{-1}(\alpha) - \alpha.$$

In summary, we have proved the following lemma. In the statement, $H_1(V_g)^{\pi_1(S^1)}$ is the space of invariants under the action of $\pi_1(S^1)$, i.e., $H_1(V_g)^{\pi_1(S^1)} = \{\alpha \in H_1(V_g) \mid \varphi_*(\alpha) = \alpha\}$.

Lemma 3.1 *We have* $H_2(M_\varphi) \cong H_1(S^1; \mathscr{H}_1(V_g)) \cong H_1(V_g)^{\pi_1(S^1)}$.

Similarly, for the relative homology of the pair $(M_\varphi, \partial M_\varphi)$, there is a spectral sequence converging to $H_*(M_\varphi, \partial M_\varphi)$ such that $E^2_{p,q} = H_p(S^1; \mathscr{H}_q(V_g, \Sigma_g))$. This degenerates at the E^2 page, too. Since $H_1(V_g, \Sigma_g) = 0$, we obtain

$$H_2(M_\varphi, \partial M_\varphi) \cong E^\infty_{0,2} \cong E^2_{0,2} = H_0(S^1; \mathscr{H}_2(V_g, \Sigma_g)).$$

By the same argument as above, we obtain the following lemma. In the statement, $H_2(V_g, \Sigma_g)_{\pi_1(S^1)}$ is the space of coinvariants under the action of $\pi_1(S^1)$, i.e. the quotient of $H_2(V_g, \Sigma_g)$ by the subgroup generated by the set $\{\varphi_*(\delta) - \delta \mid \delta \in H_2(V_g, \Sigma_g)\}$.

Lemma 3.2 *We have* $H_2(M_\varphi, \partial M_\varphi) \cong H_0(S^1; \mathscr{H}_2(V_g, \Sigma_g)) \cong H_2(V_g, \Sigma_g)_{\pi_1(S^1)}$.

3.3. Description of the inclusion homomorphism

Recall that the short exact sequence (2.1) is $\mathrm{Mod}(V_g)$-equivariant. Let $\alpha \in H_1(V_g)^{\pi_1(S^1)}$ be a φ_*-invariant homology class. Pick an element $\tilde{\alpha} \in H_1(\Sigma_g)$ such that $i_*(\tilde{\alpha}) = \alpha$. Then $\varphi_*(\tilde{\alpha}) - \tilde{\alpha} \in \mathrm{Ker}(i_*) = \mathrm{Im}(\partial_*)$.

Definition 3.3 $d(\alpha) := [\partial_*^{-1}(\varphi_*(\tilde{\alpha}) - \tilde{\alpha})] \in H_2(V_g, \Sigma_g)_{\pi_1(S^1)}$.

It is easy to see that $d(\alpha)$ is independent of the choice of $\tilde{\alpha}$. Thus, we obtain a well-defined map $d\colon H_1(V_g)^{\pi_1(S^1)} \to H_2(V_g, \Sigma_g)_{\pi_1(S^1)}$.

Proposition 3.4 *The following diagram is commutative:*

$$\begin{array}{ccc} H_1(V_g)^{\pi_1(S^1)} & \xrightarrow{d} & H_2(V_g, \Sigma_g)_{\pi_1(S^1)} \\ \downarrow \cong & & \downarrow \cong \\ H_2(M_\varphi) & \xrightarrow{i_*} & H_2(M_\varphi, \partial M_\varphi), \end{array}$$

where the bottom horizontal arrow is the inclusion homomorphism, and the vertical arrows are the isomorphisms in Lemmas 3.1 and 3.2.

3.4. Proof of Proposition 3.4

In this section, for a topological space X, we denote by $S_n(X)$ and $Z_n(X)$ the groups of singular n-chains and singular n-cycles, respectively.

Let $\alpha \in H_1(V_g)^{\pi_1(S^1)}$. Pick its lift $\tilde{\alpha} \in H_1(\Sigma_g)$ such that $i_*(\tilde{\alpha}) = \alpha$. Take a singular 1-cycle $\tilde{a} \in Z_1(\Sigma_g)$ representing the homology class $\tilde{\alpha}$. Then, $\varphi_\sharp^{-1}(\tilde{a}) - \tilde{a}$ is a singular 1-boundary in V_g since $\varphi_*^{-1}(\tilde{\alpha}) - \tilde{\alpha} \in \mathrm{Ker}(i_*)$. Therefore, there exists $\sigma_{\varphi,\alpha} \in S_2(V_g)$ such that $\partial \sigma_{\varphi,\alpha} = \varphi_\sharp^{-1}(\tilde{a}) - \tilde{a}$.

First we compute the composition of d and the right vertical map. We claim that $d(\alpha)$ is represented by the relative 2-cycle $-\sigma_{\varphi,\alpha} \in Z_2(V_g, \Sigma_g)$. This follows from the equality $\varphi_*(\tilde{\alpha}) - \tilde{\alpha} = -(\varphi_*^{-1}(\tilde{\alpha}) - \tilde{\alpha})$ in $H_1(\Sigma_g)_{\pi_1(S^1)}$ and the relation $\partial \sigma_{\varphi,\alpha} = \varphi_\sharp^{-1}(\tilde{a}) - \tilde{a}$. Hence, the right vertical map sends $d(\alpha)$ to the homology class represented by the relative 2-cycle $-e^0 \times \sigma_{\varphi,\alpha} \in Z_2(M_\varphi, \partial M_\varphi)$, where the symbol $\times$ means the cross product.

Next we compute the composition of the left vertical map and i_*. For this purpose, we set

$$\mathcal{Z}_\alpha := \Phi_\sharp(I \times \tilde{a}) - e^0 \times \sigma_{\varphi,\alpha} \in S_2(M_\varphi).$$

Here, Φ is the map defined in (3.1), and the unit interval is regarded as a singular 1-chain in the obvious way. Actually, $\mathcal{Z}_\alpha$ is a 2-cycle in M_φ.

Lemma 3.5 *The isomorphism in Lemma 3.1 sends* α *to the homology class of* $\mathcal{Z}_\alpha$.

Proof We need to inspect the spectral sequence involved in Lemma 3.1. For simplicity we denote $M = M_\varphi$, and for every nonnegative integer $q \geq 0$ let $M^{(q)}$ be the inverse image of the q-skeleton of S^1 by the projection map π. Thus, we have $\emptyset \subset M^{(0)} = \pi^{-1}([0]) \subset M^{(1)} = M$. Accordingly, the singular chain complex $S_*(M)$ has an increasing filtration: $\{0\} \subset S_*(M^{(0)}) \subset S_*(M^{(1)}) = S_*(M)$. The associated spectral sequence is the one that we consider.

Now let $\alpha \in H_1(V_g)^{\pi_1(S^1)}$. There is an isomorphism

$$E^2_{1,1} = H_1(S^1; \mathscr{H}_1(V_g)) \cong \mathrm{Ker}(\partial_* \colon H_2(M, M^{(0)}) \to H_1(M^{(0)})),$$

under which the homology class $[e^1 \otimes \alpha]$ is mapped to the homology class of the relative 2-cycle $\Phi_\sharp(I \times \tilde{a})$. However, since $e^0 \times \sigma_{\varphi,\alpha} \in S_2(M^{(0)})$, it holds that

$$[\Phi_\sharp(I \times \tilde{a})] = [\Phi_\sharp(I \times \tilde{a}) - e^0 \times \sigma_{\varphi,\alpha}] = [\mathcal{Z}_\alpha] \in H_2(M, M^{(0)}).$$

Thus, the homology class under consideration is now represented by a *genuine* 2-cycle in M. Finally, we observe that the natural map

$$H_2(M) \cong E^\infty_{1,1} \xrightarrow{\cong} E^2_{1,1} \subset H_2(M, M^{(0)})$$

coincides with the inclusion homomorphism. This completes the proof. □

By Lemma 3.5, it is enough to compute $i_*([\mathcal{Z}_\alpha])$. Since $\tilde{a}$ is a 1-cycle in $\Sigma_g = \partial V_g$, the 2-chain $\Phi_\sharp(I \times \tilde{a})$ lies in ∂M_φ. Hence,

$$\mathcal{Z}_\alpha = -e^0 \times \sigma_{\varphi,\alpha} \in Z_2(M_\varphi, \partial M_\varphi).$$

This shows that $i_*([\mathcal{Z}_\alpha])$ is represented by the relative 2-cycle $-e^0 \times \sigma_{\varphi,\alpha}$. This completes the proof of Proposition 3.4.

3.5. Proof of Theorem 1.1

We describe the intersection form of M_φ and prove Theorem 1.1.

First we claim that the second homology group $H_2(M_\varphi)$ is naturally isomorphic to $U^{\mathbb{Z}}_\varphi := \mathrm{Ker}(S - I_g) \subset \mathbb{Z}^g$. In fact, by Lemma 3.1 we have $H_2(M_\varphi) \cong H_1(V_g)^{\pi_1(S^1)}$, and the action of φ on $H_1(V_g) \cong \mathbb{Z}^g$ is given by the matrix S. Thus, the claim follows.

We next claim that under the isomorphism $H_2(M_\varphi) \cong U^{\mathbb{Z}}_\varphi$, the intersection form on $H_2(M_\varphi)$ is transferred to the bilinear form $\langle\ ,\ \rangle_\varphi$. Since $\phi^V_g(\varphi) = \mathrm{Sign}\, M_\varphi$, this will complete the proof of Theorem 1.1. The proof of this claim consists of two steps.

Step 1. We give a description of the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ that is obtained by transferring the intersection form on $H_2(M_\varphi)$. Let $\langle \cdot, \cdot \rangle_V \colon H_2(V_g, \Sigma_g) \times H_1(V_g) \to \mathbb{Z}$ be the intersection product of the compact oriented 3-manifold V_g. We have

$$\langle D_i, \beta_j \rangle_V = \delta_{ij} \quad \text{for any } i, j \in \{1, \ldots, g\}. \tag{3.2}$$

Let

$$H_0(S^1; \mathscr{H}_2(V_g, \Sigma_g)) \times H_1(S^1; \mathscr{H}_1(V_g)) \longrightarrow \mathbb{Z} \tag{3.3}$$

be the intersection product of $H_0(S^1; \mathscr{H}_2(V_g, \Sigma_g))$ and $H_1(S^1; \mathscr{H}_1(V_g))$ followed by the contraction of the coefficients by the form $\langle \cdot, \cdot \rangle_V$. Under the isomorphisms in Lemmas 3.1 and 3.2, this is equivalent to the intersection product $H_2(M_\varphi) \times H_2(M_\varphi, \partial M_\varphi) \to \mathbb{Z}$. By composing (3.3) and the homomorphism

$$\begin{aligned} H_1(V_g)^{\pi_1(S^1)} \times H_1(V_g)^{\pi_1(S^1)} &\overset{d\otimes \mathrm{id}}{\longrightarrow} H_2(V_g, \Sigma_g)_{\pi_1(S^1)} \times H_1(V_g)^{\pi_1(S^1)} \\ &\cong \; H_0(S^1; \mathscr{H}_2(V_g, \Sigma_g)) \times H_1(S^1; \mathscr{H}_1(V_g)), \end{aligned}$$

we obtain a bilinear form on $H_1(V_g)^{\pi_1(S^1)}$. Proposition 3.4 implies that this is equivalent to the intersection form on $H_2(M_\varphi)$.

Step 2. We prove that the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ described in the previous paragraph is equivalent to $\langle \ , \ \rangle_\varphi$ under the identification $H_1(V_g)^{\pi_1(S^1)} \cong U_\varphi^{\mathbb{Z}}$. Let $x = (x_1, \ldots, x_g)$, $y = (y_1, \ldots, y_g) \in U_\varphi^{\mathbb{Z}} \subset \mathbb{Z}^g$. We regard x as an element of $H_1(V_g)^{\pi_1(S^1)}$. Then, we can take $\tilde{x} = \sum_{i=1}^g x_i \beta_i \in H_1(\Sigma_g)$ as a lift of x which we need to compute $d(x)$. Thus, we have

$$\varphi_*(\tilde{x}) - \tilde{x} = (\alpha_1, \ldots, \alpha_g)\, Q\, {}^t(x_1, \ldots, x_g) = (x_1, \ldots, x_g)\, {}^tQ\, {}^t(\alpha_1, \ldots, \alpha_g),$$

and hence $d(x) = (x_1, \ldots, x_g)\, {}^tQ\, {}^t(D_1, \ldots, D_g)$. Therefore, the pairing of x and y by the bilinear form on $H_1(V_g)^{\pi_1(S^1)}$ described above is equal to

$$\big\langle (x_1, \ldots, x_g)\, {}^tQ\, {}^t(D_1, \ldots, D_g), (\beta_1, \ldots, \beta_g)\, {}^t(y_1, \ldots, y_g) \big\rangle_V = {}^tx\, {}^tQ\, y = \langle x, y \rangle_\varphi.$$

Here we used the equality (3.2). This completes the proof of Theorem 1.1.

Remark 3.6 *There is a 2-cocycle m_λ on $\mathrm{Sp}(2g;\mathbb{Z})$ constructed by Turaev [20] which satisfies $[m_\lambda] = -[\tau_g] \in H^2(\mathrm{Sp}(2g;\mathbb{Z}))$, and Walker, in page 124 of his note*, constructed a (unique) cobounding function $j\colon \mathrm{Mod}(\Sigma_g) \to \mathbb{Q}$ of the sum $\rho^*\tau_g + \rho^* m_\lambda$ of 2-cocycles. The 2-cocycle m_λ and the function j depend on the choice of a lagrangian $\lambda \subset H_1(\Sigma_g;\mathbb{Q})$. If we choose a suitable lagrangian λ, the restriction of j to $\mathrm{Mod}(V_g)$ is known to be a cobounding function of $\rho^*\tau_g$, and coincides with our function ϕ_g^V. Gilmer and Masbaum [5, Proposition 6.9] described j explicitly in a way which is similar to but different from ours.*

Remark 3.7 *Since $Sy = y$ for any $y \in U_\varphi$, we have $\langle x, y \rangle_\varphi = {}^tx\, {}^tQS\, y$ for any $x, y \in U_\varphi$. Since ${}^tQS$ is symmetric by (2.2), this gives a purely algebraic explanation for the symmetric property of the form $\langle \ , \ \rangle_\varphi$ on U_φ.*

*K. Walker (1991). *On Witten's 3-manifold invariants*, Preliminary Version [online]. Website https://canyon23.net/math/1991TQFTNotes.pdf [accessed 1 May 2020].

Remark 3.8 *By Theorem 1.1, one can regard* ϕ_g^V *as a* 1*-cochain on* $\mathrm{urSp}(2g;\mathbb{Z})$*. For* $g \geq 3$*, it is the unique* 1*-cochain which cobounds* τ_g *on* $\mathrm{urSp}(2g;\mathbb{Z})$ *since* $H^1(\mathrm{urSp}(2g;\mathbb{Z})) = 0$*; see [19, Corollary 4.4].*

4. Evaluation of Meyer functions

4.1. The Meyer function on the hyperelliptic mapping class group

There is a unique 1-cochain $\phi_g^{\mathcal{H}}\colon \mathcal{H}(\Sigma_g) \to \mathbb{Q}$ such that for any $\varphi_1, \varphi_2 \in \mathcal{H}(\Sigma_g)$,

$$\phi_g^{\mathcal{H}}(\varphi_1) + \phi_g^{\mathcal{H}}(\varphi_2) - \phi_g^{\mathcal{H}}(\varphi_1\varphi_2) = \tau_g(\rho(\varphi_1), \rho(\varphi_2)). \tag{4.1}$$

The 1-cochain $\phi_g^{\mathcal{H}}$ is called the *Meyer function on the hyperelliptic mapping class group of genus* g; see [4, 18].

Recall the element $s_1 = t_2t_3t_1t_2 \in \mathcal{H}(V_g) \subset \mathcal{H}(\Sigma_g)$ which was defined in Section 2.4.

Lemma 4.1 $\phi_g^{\mathcal{H}}(s_1) = (2g+3)/(2g+1)$.

Proof Set $T_i = \rho(t_i)$ for every $i \in \{1,2,3\}$. Using (4.1), we have

$$\begin{aligned}\phi_g^{\mathcal{H}}(s_1) = \phi_g^{\mathcal{H}}(t_2) + \phi_g^{\mathcal{H}}(t_3) + \phi_g^{\mathcal{H}}(t_1) + \phi_g^{\mathcal{H}}(t_2)\\ - \tau_g(T_1,T_2) - \tau_g(T_3, T_1T_2) - \tau_g(T_2, T_3T_1T_2).\end{aligned}$$

As was shown in [4, Lemma 3.3] and [18, Proposition 1.4], we have $\phi_g^{\mathcal{H}}(t_i) = (g+1)/(2g+1)$ for all $i \in \{1,2,3\}$. Also, by a direct computation we obtain $\tau_g(T_1,T_2) = 0$, $\tau_g(T_3,T_1T_2) = 0$, and $\tau_g(T_2,T_3T_1T_2) = 1$. The result follows from these equalities. □

4.2. The Meyer function on the handlebody group

Recall from the introduction that we defined $\phi_g^V\colon \mathrm{Mod}(V_g) \to \mathbb{Z}$ by $\varphi \mapsto \mathrm{Sign}\, M_\varphi$, where M_φ is the mapping torus of φ.

Lemma 4.2 *The function* $\phi_g^V\colon \mathrm{Mod}(V_g) \to \mathbb{Z}$ *cobounds the cocycle* $\rho^*\tau_g$ *in the handlebody group* $\mathrm{Mod}(V_g)$*. If* $g \geq 3$*,* ϕ_g^V *is the unique cobounding function of* $\rho^*\tau_g$*.*

Proof The uniqueness follows from the fact that $H_1(\mathrm{Mod}(V_g))$ is torsion when $g \geq 3$.

For given two mapping classes $\varphi, \psi \in \mathrm{Mod}(V_g)$, there is an oriented V_g-bundle $W(\varphi,\psi) \to P$ such that the monodromy along ℓ_1, ℓ_2, and ℓ_3 are φ, ψ, and $(\varphi\psi)^{-1}$, respectively. The boundary of $W(\varphi,\psi)$ is written as

$$\partial W(\varphi,\psi) = E(\varphi,\psi) \cup (M_{\varphi^{-1}} \sqcup M_{\psi^{-1}} \sqcup M_{\varphi\psi}).$$

Note that $M_{\varphi^{-1}}$ is diffeomorphic to $-M_\varphi$ under an orientation-preserving diffeomorphism, where $-M_\varphi$ denotes the mapping torus M_φ with orientation reversed. Since the signature of $\partial W(\varphi,\psi)$ is zero, Novikov additivity implies that

$$\mathrm{Sign}\, E(\varphi,\psi) - \mathrm{Sign}\, M_\varphi - \mathrm{Sign}\, M_\psi + \mathrm{Sign}\, M_{\varphi\psi} = 0.$$

This shows that ϕ_g^V is a cobounding function of $\rho^*\tau_g$ restricted to $\mathrm{Mod}(V_g)$. □

Since $\dim V_{A,B} \leq 4g$ for any $A, B \in \mathrm{Sp}(2g;\mathbb{Z})$, the signature cocycle τ_g is a bounded 2-cocycle. Therefore, it represents a class in the second bounded cohomology group $H_b^2(\mathrm{Mod}(\Sigma_g))$. The image of $[\tau_g]$ under the natural homomorphism $H_b^2(\mathrm{Mod}(\Sigma_g);\mathbb{Q}) \to H_b^2(\mathcal{H}(\Sigma_g);\mathbb{Q})$ is nontrivial since the Meyer function $\phi_g^{\mathcal{H}}$ is unbounded. In contrast, we have:

Proposition 4.3 *Under the natural homomorphism $H_b^2(\mathrm{Mod}(\Sigma_g);\mathbb{Q}) \to H_b^2(\mathrm{Mod}(V_g);\mathbb{Q})$, the image of the cohomology class $[\tau_g]$ vanishes.*

Proof The restriction of the signature cocycle τ_g to $\mathrm{Mod}(V_g)$ is cobounded by the function ϕ_g^V, and ϕ_g^V is a bounded function since the rank of $H_2(M_\varphi)$ is at most g. □

4.3. Computation of the Meyer function on the handlebody group

Theorem 1.1 shows that the bilinear form $\langle\ ,\ \rangle_\varphi$ on U_φ, whose signature coincides with $\phi_g^V(\varphi)$, can be computed from the homological monodromy $\rho(\varphi) \in \mathrm{urSp}(2g;\mathbb{Z})$. In more detail, if $\rho(\varphi) = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix}$, then $U_\varphi = \mathrm{Ker}(S - I_g) \subset \mathbb{Q}^g$ and $\langle x, y\rangle_\varphi = {}^t x\, {}^t Q\, y$ for $x, y \in U_\varphi$.

The 1-cochain ϕ_g^V, regarded as the one defined on $\mathrm{urSp}(2g;\mathbb{Z})$, is *stable* with respect to g in the following sense. For every nonnegative integer $g \geq 0$, there is a natural embedding $\iota\colon \mathrm{urSp}(2g;\mathbb{Z}) \hookrightarrow \mathrm{urSp}(2(g+1);\mathbb{Z})$;

$$A = \begin{pmatrix} P & Q \\ O_g & S \end{pmatrix} \mapsto \iota(A) = \begin{pmatrix} \tilde{P} & \tilde{Q} \\ O_{g+1} & \tilde{S} \end{pmatrix},$$

where

$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & 0 \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} S & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\phi_{g+1}^V(\iota(A)) = \phi_g^V(A)$ for any $A \in \mathrm{urSp}(2g;\mathbb{Z})$.

Lemma 4.4 *For any positive integer m, we have $\phi_g^V(t_1^m) = 1$.*

Proof Since the action of $\rho(t_1^m)$ on $H_1(\Sigma_g)$ is given by

$$\rho(t_1)\colon \alpha_i \mapsto \alpha_i \quad (i = 1, \dots, g), \qquad \beta_1 \mapsto m\alpha_1 + \beta_1, \qquad \beta_i \mapsto \beta_i \quad (i = 2, \dots, g),$$

we may assume that $g = 1$. Then $\rho(t_1^m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}$, and $\mathrm{Ker}(S - I_1) = \mathbb{Z}$ on which the pairing is given by the 1×1 matrix $\begin{pmatrix} m \end{pmatrix}$. Hence, $\phi_g^V(t_1^m) = 1$, as required. □

Lemma 4.5 $\phi_g^V(s_1) = 1$.

Proof The proof proceeds as in the same way as the previous lemma. In this case we may assume that $g = 2$. Then

$$\rho(s_1) = \begin{pmatrix} P & Q \\ O_2 & S \end{pmatrix} \quad \text{with} \quad P = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix}.$$

The rest of computation is straightforward, so we omit it. □

4.4. Proof of Theorem 1.2

Since both the 1-cochains $\phi_g^{\mathcal{H}}$ and ϕ_g^V cobound the signature cocycle, their difference becomes a $\mathbb{Q}$-valued homomorphism on $\mathcal{H}(V_g) = \mathcal{H}(\Sigma_g) \cap \mathrm{Mod}(V_g)$.

We compare the homomorphism $\phi_g^{\mathcal{H}} - \phi_g^V$ with the generator $\mu \in H^1(\mathcal{H}(V_g))$ in Corollary 2.6. It is sufficient to evaluate $\phi_g^{\mathcal{H}} - \phi_g^V$ on s_1 if g is even, and on $t_1 s_1^{\frac{g+1}{2}}$ if g is odd. By Lemmas 4.1 and 4.5 we immediately obtain

$$(\phi_g^{\mathcal{H}} - \phi_g^V)(s_1) = \frac{2}{2g+1}. \tag{4.2}$$

This settles the case where g is even. When g is odd, we compute

$$\begin{aligned}(\phi_g^{\mathcal{H}} - \phi_g^V)(t_1 s_1^{\frac{g+1}{2}}) &= (\phi_g^{\mathcal{H}} - \phi_g^V)(t_1) + \frac{g+1}{2}(\phi_g^{\mathcal{H}} - \phi_g^V)(s_1) \\ &= \left(\frac{g+1}{2g+1} - 1\right) + \frac{g+1}{2} \cdot \frac{2}{2g+1} \\ &= \frac{1}{2g+1}.\end{aligned}$$

Here, we used the fact that $\phi_g^{\mathcal{H}} - \phi_g^V$ is a homomorphism on $\mathcal{H}(V_g)$ in the first line; we used the fact that $\phi_g^{\mathcal{H}}(t_1) = (g+1)/(2g+1)$ (see the proof of Lemma 4.1), Lemma 4.4 and (4.2) in the second line. This completes the proof of Theorem 1.2.

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