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Existence of self-similar solutions to Smoluchowski's coagulation equation with product kernel

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Abstract: We explore, by using formal analysis, the existence of mass conserving self-similar solutions for Smoluchowski's coagulation equation when kernel $K(x,y) = x^{\lambda}y^{\mu} + x^{\mu}y^{\lambda}$ with $0 < \lambda + \mu < 1$.

Key words: Asymptotic behavior of solutions, coagulation equation, self-similar solutions, mass conservation

1. Introduction

In the study of collision processes the following infinite set of nonlinear differential equations, which are also known as the discrete form of the coagulation equation, are considered.

$$\frac{dy_i}{dt} = -y_i \sum_{j=1}^{\infty} a_{i,j} y_j + \frac{1}{2} \sum_{j=1}^{i-1} a_{i-j,j} y_j y_{i-j} \quad (i = 1, 2, 3, \cdots),$$

where $a_{i,j} = a_{j,i} > 0$ and $t \geq 0$. Existence and uniqueness results with initial conditions

$$y_1(0) = 1, \quad y_i(0) = 0 \quad (i = 2, 3, 4, \cdots),$$

for the case $a_{i,j} \leq ij$ were first proved long ago, with under certain reasonable convergence criteria, by McLeod in [1–3] where the explicit solution is valid only for $0 \leq t \leq 1$. Furthermore, existence and uniqueness results related to different cases of the $a_{i,j}$ were also studied. Firstly, existence of solution in the case $a_{i,j} \leq c_i c_j$, where $c_1 = 1$ and $c_i > 0$ for all i, was proved. Secondly, let $a_{i,j} \leq Kij$ be for all i, j and some positive constant K, then it was shown that there is at least one solution of the above equations analytic sufficiently near $t = 0$. Existence of the mass conserving solution was proved in [8] for coagulation kernels with at most linear growth at infinity. The Smoluchowski coagulation equation describing mean-field model [12] is given by

$$\begin{aligned} f_t(x,t) = \frac{1}{2} \int_0^x K(x-y,y) f(x-y,t) f(y,t) dy \\ - \int_0^{\infty} K(x,y) f(x,t) f(y,t) dy, \end{aligned} \tag{1.1}$$

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*In memory of my mentor J. Bryce McLeod

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$$f(x,0) = F(x) \geq 0 \quad a.e.,$$

where the variables $x > 0$ and $t \geq 0$ denote the size of the particles and time, respectively. The number density of particles of size x at time t is denoted by $f(x,t)$. The coagulation kernel $K(x,y)$ represents the rate at which particles of size x coalesce with particles of size y.

Existence and uniqueness of solutions to (1.1) when $K(x,y) = xy$ and fairly general $f(x,0)$ were studied by McLeod in [4]. The following explicit solutions were also established. With the initial conditions

$$F(x) = \frac{\alpha e^{-\alpha x}}{x} \quad \text{and} \quad F(x) = \frac{a e^{-bx} J_1(ax)}{x^2\sqrt{a^2+b^2-b}},$$

where J_1 is the Bessel function of order one, then (1.1) has the explicit solution

$$f(x,t) = \frac{\alpha e^{-(t+\alpha)x} I_1\left(2x\sqrt{t\alpha}\right)}{x^2\sqrt{t}}$$

on $0 \leq t \leq \alpha$, where I_1 is the Modified Bessel function, and

$$f(x,t) = \frac{a e^{-(t+b)x} J_1\left(ax\sqrt{1-2t/B}\right)}{x^2\sqrt{B^2-2tB}}$$

on $0 \leq t \leq \frac{B}{2}$ where $B = \sqrt{a^2+b^2-b}$ respectively. When kernel $K(x,y) = xy$ is bounded then existence of solutions to (1.1) was studied in [10].

The existence of self-similar solutions of (1.1) when kernel $K(x,y) = x^\lambda y^\lambda$ with $0 < \lambda < 1/2$ was studied analytically by employing topological shooting technique and for some other techniques, see [5]. For further information on topological shooting technique, see [6]. In particular, the existence of self-similar solutions with finite mass has been established for a huge class of kernels and some properties of those solutions have been investigated in [14, 15]

Equation (1.1) arises in a number of problems in physics, meteorology, chemistry, engineering, biology, and astrophysics, see [7, 9, 11, 13] and references therein. In the case of the product kernel small particles interact mostly with the ones having a comparable size.

In this paper, we also analyze mass conserving self-similar solutions to the coagulation equation (1.1) when kernel $K(x,y) = x^\lambda y^\mu + x^\mu y^\lambda$ with $0 < \lambda + \mu < 1$ and introduce clasical analysis different from the previously published papers.

2. Main results

We are interested in a solution of the form

$$f(x,t) = \frac{1}{s^2(t)} h\left(\frac{x}{s(t)}\right).$$

Then from (1.1) we see that

$$
\begin{aligned}
&-2s^{-3}s'h\left(\frac{x}{s}\right)-xs^{-4}s'h'\left(\frac{x}{s}\right)\\
&=\frac{1}{2}\frac{1}{s^4}\int_0^x y^{\mu}(x-y)^{\lambda}h\left(\frac{x-y}{s}\right)h\left(\frac{y}{s}\right)dy-\frac{x^{\lambda}}{s^4}h\left(\frac{x}{s}\right)\int_0^{\infty}y^{\mu}h\left(\frac{y}{s}\right)dy\\
&+\frac{1}{2}\frac{1}{s^4}\int_0^x y^{\lambda}(x-y)^{\mu}h\left(\frac{x-y}{s}\right)h\left(\frac{y}{s}\right)dy-\frac{x^{\mu}}{s^4}h\left(\frac{x}{s}\right)\int_0^{\infty}y^{\lambda}h\left(\frac{y}{s}\right)dy.
\end{aligned}
$$

Set $\frac{y}{s}=Y$, $\frac{x}{s}=X$, and we get, after multiplying both sides by s^4, and using symmetry in the first and the third integrals on the right-hand side,

$$
\begin{aligned}
&ss'\{-2h(X)-Xh'(X)\}\\
&=s^{\lambda+\mu+1}\Big\{\int_0^{X/2}Y^{\mu}(X-Y)^{\lambda}h(X-Y)h(Y)dY-X^{\lambda}h(X)\int_0^{\infty}Y^{\mu}h(Y)dY\\
&+\int_0^{X/2}Y^{\lambda}(X-Y)^{\mu}h(X-Y)h(Y)dY-X^{\mu}h(X)\int_0^{\infty}Y^{\lambda}h(Y)dY\Big\}.
\end{aligned}
\tag{2.1}
$$

Since the equation cannot involve t, we see that

$$
s(t)s'(t)=Const\times s^{\lambda+\mu+1}(t),\quad s^{1-\lambda-\mu}(t)=At+B,
$$

for arbitrary constants A and B. However, the (1.1) is translation invariant in t.Therefore, we may without loss of generality take $B=0$. Furthermore, the choice of A may be subsumed into the definition of $h(x)$. Therefore, we may take

$$
s(t)=(1-\lambda-\mu)t^{\frac{1}{1-\lambda-\mu}}.
$$

Equation (2.1) becomes

$$
\begin{aligned}
&-2h(X)-Xh'(X)\\
&=\int_0^{X/2}Y^{\mu}(X-Y)^{\lambda}h(X-Y)h(Y)dY-X^{\lambda}h(X)\int_0^{\infty}Y^{\mu}h(Y)dY\\
&+\int_0^{X/2}Y^{\lambda}(X-Y)^{\mu}h(X-Y)h(Y)dY-X^{\mu}h(X)\int_0^{\infty}Y^{\lambda}h(Y)dY.
\end{aligned}
\tag{2.2}
$$

Our belief (which we have, of course, to verify) is that

$$
h(X)\sim h_0X^{-1-\lambda-\mu}
\tag{2.3}
$$

as $X \to 0$, for some constant h_0. Since this would cause the integrals on the right-hand side of (2.2) to diverge, we must rewrite (2.2) in the form

$$
\begin{aligned}
&-2h(X) - Xh'(X)\\
&= \int_0^{X/2} Y^{\mu} h(Y)\left((X-Y)^{\lambda} h(X-Y) - X^{\lambda} h(X)\right) dY\\
&- X^{\lambda} h(X) \int_{X/2}^{\infty} Y^{\mu} h(Y) dY\\
&+ \int_0^{X/2} Y^{\lambda} h(Y)\left((X-Y)^{\mu} h(X-Y) - X^{\mu} h(X)\right) dY\\
&- X^{\mu} h(X) \int_{X/2}^{\infty} Y^{\lambda} h(Y) dY.
\end{aligned}
$$

We now set

$$
h(X) = X^{-1-\lambda-\mu} H(X),
$$

and the equation for $H(X)$ is

$$
\begin{aligned}
&- X^{-\lambda-\mu} H'(X) - (1-\lambda-\mu) X^{-1-\lambda-\mu} H(X)\\
&= \int_0^{X/2} Y^{-1-\lambda} H(Y)\Big((X-Y)^{-1-\mu}[H(X-Y) - H(X)]\\
&+ H(X)[(X-Y)^{-1-\mu} - X^{-1-\mu}]\Big) dY - X^{-1-\mu} H(X) \int_{X/2}^{\infty} Y^{-1-\lambda} H(Y) dY\\
&+ \int_0^{X/2} Y^{-1-\mu} H(Y)\Big((X-Y)^{-1-\lambda}[H(X-Y) - H(X)]\\
&+ H(X)[(X-Y)^{-1-\lambda} - X^{-1-\lambda}]\Big) dY\\
&- X^{-1-\lambda} H(X) \int_{X/2}^{\infty} Y^{-1-\mu} H(Y) dY\\
&= I_1 + I_2,
\end{aligned}
\tag{2.4}
$$

where I_1 is the first two terms and I_2 is the last two terms.

We can now work out, at least formally, what the value of h_0 must be in (2.3). We first work out I_1 for $H(x) \to h_0$ as $X \to 0$, and so

$$
\begin{aligned}
I_1 \quad &\sim \quad h_0 \int_0^{X/2} Y^{-1-\lambda}\Big((X-Y)^{-1-\mu}[H(X-Y) - H(X)]\\
&+ \quad H(X)[(X-Y)^{-1-\mu} - X^{-1-\mu}]\Big) dY - h_0 X^{-1-\mu} \int_{X/2}^{\infty} Y^{-1-\lambda} H(Y) dY.
\end{aligned}
$$

Now

$$
H(X-Y) - H(X) \asymp YH'(X) \quad \text{and} \quad H'(X) = o(X^{-1}),
$$

as $X \to 0$, since $H(0)$ exists. Thus,

$$\begin{aligned} I_1 \sim h_0 &\int_0^{X/2} Y^{-1-\lambda}(X-Y)^{-1-\mu}[H(X-Y)-H(X)]dY \\ &+ h_0^2 \int_0^{X/2} Y^{-1-\lambda}[(X-Y)^{-1-\mu} - X^{-1-\mu}]dY \\ &- h_0 X^{-1-\mu} \int_{X/2}^{\infty} Y^{-1-\lambda} H(Y)dY, \end{aligned}$$

$$\begin{aligned} I_1 \sim\ & o\left(X^{-1}\int_0^{X/2} Y^{-\lambda}(X-Y)^{-1-\mu}dY\right) \\ &+ h_0^2 \int_0^{X/2} Y^{-1-\lambda}[(X-Y)^{-1-\mu} - X^{-1-\mu}]dY - \frac{2^\lambda h_0^2 X^{-1-\lambda-\mu}}{\lambda}. \end{aligned}$$

By setting $Y = Xu$, we see that the first term in the last line $o(X^{-1-\lambda-\mu})$. Therefore,

$$\begin{aligned} I_1 &\sim h_0^2 X^{-1-\lambda-\mu} \int_0^{1/2} u^{-1-\lambda}[(1-u)^{-1-\mu} - 1]du - \frac{2^\lambda h_0^2 X^{-1-\lambda-\mu}}{\lambda} \\ &= h_0^2 X^{-1-\lambda-\mu} A(\lambda,\mu) - \frac{2^\lambda h_0^2 X^{-1-\lambda-\mu}}{\lambda}, \end{aligned}$$

say. Here

$$A(\lambda,\mu) = \int_0^{1/2} u^{-1-\lambda}[(1-u)^{-1-\mu} - 1]du$$

can be expressed in terms of hypergeometric functions. Similarly, calculating I_2 we see that

$$\begin{aligned} I_2 &\sim h_0^2 X^{-1-\lambda-\mu} \int_0^{1/2} u^{-1-\mu}[(1-u)^{-1-\lambda} - 1]du - \frac{2^\mu h_0^2 X^{-1-\lambda-\mu}}{\mu} \\ &= h_0^2 X^{-1-\lambda-\mu} A(\mu,\lambda) - \frac{2^\mu h_0^2 X^{-1-\lambda-\mu}}{\mu}. \end{aligned}$$

Substituting this into (2.4) and letting $X \to 0$, we see that

$$-(1-\lambda-\mu)h_0 = h_0^2\left(A(\lambda,\mu) - \frac{2^\lambda}{\lambda} + A(\mu,\lambda) - \frac{2^\mu}{\mu}\right).$$

Therefore,

$$h_0(\lambda,\mu) = \frac{\lambda\mu(1-\lambda-\mu)}{\mu(2^\lambda - \lambda A(\lambda,\mu)) + \lambda(2^\mu - \mu A(\mu,\lambda))}. \tag{2.5}$$

Of course, for this to make sense, we need

$$2^\lambda - \lambda A(\lambda,\mu) > 0 \quad \text{and} \quad 2^\mu - \mu A(\mu,\lambda) > 0 \quad \text{for} \quad 0 < \lambda+\mu < 1. \tag{2.6}$$

For $\lambda = \mu = 0$, this is clearly true, since from its definition $A(\lambda,\mu)$ (similarly $A(\mu,\lambda)$) is bounded as $\lambda \to 0$ and $\mu \to 0$. Indeed, by using the binomial series

$$\begin{aligned} A(\lambda,\mu) &= \int_0^{1/2} u^{-1-\lambda}[(1-u)^{-1-\mu}-1]du \\ &= \int_0^{1/2} u^{-1-\lambda} \sum_{n=1}^{\infty} \left(\frac{u^n(\mu+1)(\mu+2)...(\mu+n)}{n!} \right) du \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-\lambda} \frac{(\mu+1)(\mu+2)...(\mu+n)}{(n-\lambda)n!}. \end{aligned}$$

Similarly,

$$A(\mu,\lambda) = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n-\mu} \frac{(\lambda+1)(\lambda+2)...(\lambda+n)}{(n-\mu)n!}.$$

Therefore, the above series converge for any λ and μ. From this, it is clear that

$$\frac{A(\lambda,\mu)}{2^\lambda} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} \frac{(\mu+1)(\mu+2)...(\mu+n)}{(n-\lambda)n!}$$

and

$$\frac{A(\mu,\lambda)}{2^\mu} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{n} \frac{(\lambda+1)(\lambda+2)...(\lambda+n)}{(n-\mu)n!}$$

are increasing function of λ and μ. We also remark below that (2.6) is true (with equality) for $\lambda = \mu = 1/2$.

Remark 2.1 *If $\lambda = \mu$, then one obtains*

$$h_0(\lambda) = \frac{\lambda(1-2\lambda)}{2\big(2^\lambda - \lambda A(\lambda,\lambda)\big)}.$$

With suitable scaling the above remark becomes $h_0(\lambda) = \frac{\lambda(1-2\lambda)}{2^\lambda - \lambda A(\lambda,\lambda)}$.

Remark 2.2 *If $\lambda = \mu$ and $K(x,y) = (xy)^\lambda$ with $0 < 2\lambda < 1$, by applying the same argument as above, then one obtains*

$$h_0(\lambda) = \frac{\lambda(1-2\lambda)}{2^\lambda - \lambda A(\lambda,\lambda)},$$

the same as in [5].

Remark 2.3 *If $\lambda = \mu = 0$, then one obtains $A(0,0) = \ln 2$.*

Remark 2.4 *If* $\lambda = \mu = \frac{1}{2}$, *then one obtains*

$$\begin{aligned} A(\frac{1}{2}, \frac{1}{2}) &= \int_0^{1/2} u^{-3/2} \left[(1-u)^{-3/2} - 1 \right] du \\ &= \frac{2(2\sqrt{1-x}+1)\sqrt{x}}{-x+\sqrt{1-x}+1} |_0^{1/2} = 2^{3/2}. \end{aligned}$$

We now need to investigate the desired behavior as $X \to \infty$. We are looking for exponential behavior. Therefore, on the left-hand side of (2.4), the dominant term should be $-X^{-\lambda-\mu}H'(X)$. The last term on the right-hand side will be a higher-order exponential, and so negligible. Then

$$\begin{aligned} &- X^{-\lambda-\mu}H'(X) \\ &\sim \int_0^{X/2} Y^{-1-\lambda} H(Y) \Big((X-Y)^{-1-\mu} [H(X-Y) - H(X)] \\ &+ H(X)[(X-Y)^{-1-\mu} - X^{-1-\mu}] \Big) dY \\ &+ \int_0^{X/2} Y^{-1-\mu} H(Y) \Big((X-Y)^{-1-\lambda} [H(X-Y) - H(X)] \\ &+ H(X)[(X-Y)^{-1-\lambda} - X^{-1-\lambda}] \Big) dY. \end{aligned}$$

If Y is bounded, say $Y \leq Y_0$, Y_0 large but fixed independent of X, then the contribution to the right-hand side is of order

$$\begin{aligned} &\int_0^{Y_0} Y^{-1-\lambda} H(Y) \Big((X-Y)^{-1-\mu} [H(X-Y) - H(X)] \\ &+ H(X)[(X-Y)^{-1-\mu} - X^{-1-\mu}] \Big) dY \\ &+ \int_0^{Y_0} Y^{-1-\mu} H(Y) \Big((X-Y)^{-1-\lambda} [H(X-Y) - H(X)] \\ &+ H(X)[(X-Y)^{-1-\lambda} - X^{-1-\lambda}] \Big) dY. \\ &= O(X^{-2-\mu} H(X)) + O(X^{-1-\mu} H'(X)) \\ &+ O(X^{-2-\lambda} H(X)) + O(X^{-1-\lambda} H'(X)), \end{aligned}$$

which is of smaller order than the right-hand side (for large X). Hence,

$$\begin{aligned} &- X^{-\lambda-\mu}H'(X) \\ &\sim \int_{Y_0}^{X/2} Y^{-1-\lambda} H(Y)(X-Y)^{1-\mu} H(X-Y) dY \\ &+ \int_{Y_0}^{X/2} Y^{-1-\mu} H(Y)(X-Y)^{1-\lambda} H(X-Y) dY. \end{aligned}$$

If we set

$$H(X) = X\Phi(X),$$

we get

$$\begin{aligned}
&-X^{1-\lambda-\mu}\Phi'(X) \\
\sim\ & \int_{Y_0}^{X/2} Y^{-\lambda}(X-Y)^{-\mu}\Phi(Y)\Phi(X-Y)dY \\
+\ & \int_{Y_0}^{X/2} Y^{-\mu}(X-Y)^{-\lambda}\Phi(Y)\Phi(X-Y)dY \\
\sim\ & \frac{1}{2}\int_{0}^{X} Y^{-\lambda}(X-Y)^{-\mu}\Phi(Y)\Phi(X-Y)dY \\
+\ & \frac{1}{2}\int_{0}^{X} Y^{-\mu}(X-Y)^{-\lambda}\Phi(Y)\Phi(X-Y)dY.
\end{aligned}$$

It is readily verified that an exact solution of this

$$\Phi(X) = ce^{-X},$$

where

$$\begin{aligned}
c(\lambda,\mu) &= \frac{2}{\int_0^1 s^{-\lambda}(1-s)^{-\mu}ds + \int_0^1 s^{-\mu}(1-s)^{-\lambda}ds} \\
&= \frac{\Gamma(2-\lambda-\mu)}{\Gamma(1-\lambda)\Gamma(1-\mu)},
\end{aligned} \tag{2.7}$$

for Gamma and Beta integrals, see [16]. This is, of course, not the only solution. We can also have

$$\Phi(X) = ac(\lambda,\mu)e^{-aX}, \quad \text{for any} \quad a.$$

That is,

$$H(X) = c(\lambda,\mu)(aX)e^{-aX}, \quad \text{for any} \quad a > 0.$$

We can now formally formulate the following existence theorem.

Theorem 2.5 *There exists a solution to* (2.4) *which is* $C^1(0,\infty)$*, and has the properties that*

(i) $H(X) \to h_0(\lambda,\mu)$ *as* $X \to 0$*, where* $h_0(\lambda,\mu)$ *is given by* (2.5)*.*

(ii) $H(X) \sim c(\lambda,\mu)(aX)e^{-aX}$ *as* $X \to \infty$*, for some* $a > 0$*, where* $c(\lambda,\mu)$ *is given by* (2.7)*.*

Remark 2.6 *If* $\lambda = \mu$*, then from* (2.7) *one obtains*

$$c(\lambda) = \frac{\Gamma(2-2\lambda)}{\Gamma^2(1-\lambda)}.$$

Remark 2.7 *If* $\lambda = \mu$*, then Theorem 2.5 case* (*ii*) *is obtained as in [5].*

3. Further investigation of the solution as $X \to 0$

We aim to find a formula to determine r in the following expansion. Set

$$H(X) = h_0 + aX^r + bX^{r+1} + \cdots .$$

Then from (2.4), considering just the first two terms, we get

$$\begin{aligned}
&- arX^{r-1-\lambda-\mu} - (1-\lambda-\mu)X^{-1-\lambda-\mu}(h_0 + aX^r) \\
&+ X^{-1-\mu}(h_0 + aX^r)\int_{\frac{1}{2}X}^{\infty} Y^{-1-\lambda}\{h_0 + (H(Y) - h_0)\}dY \\
&+ X^{-1-\lambda}(h_0 + aX^r)\int_{\frac{1}{2}X}^{\infty} Y^{-1-\mu}\{h_0 + (H(Y) - h_0)\}dY \\
&= \int_0^{\frac{1}{2}X} Y^{-1-\lambda}(h_0 + aY^r)[((X-Y)^{-1-\mu} - X^{-1-\mu})h_0 \\
&+ a((X-Y)^{r-1-\mu} - X^{r-1-\mu})]dY \\
&+ \int_0^{\frac{1}{2}X} Y^{-1-\mu}(h_0 + aY^r)[((X-Y)^{-1-\lambda} - X^{-1-\lambda})h_0 \\
&+ a((X-Y)^{r-1-\lambda} - X^{r-1-\lambda})]dY.
\end{aligned}$$

Therefore, multiplying by $X^{1+\lambda+\mu-r}$, we have

$$\begin{aligned}
&- ar + (\lambda+\mu-1)h_0X^{-r} + a(\lambda+\mu-1) \\
&+ X^{\lambda-r}(h_0 + aX^r)\{\frac{1}{\lambda}(\frac{X}{2})^{-\lambda}h_0 + \int_{\frac{1}{2}X}^{\infty} Y^{-1-\lambda}(H(Y) - h_0)dY\} \\
&+ X^{\mu-r}(h_0 + aX^r)\{\frac{1}{\mu}(\frac{X}{2})^{-\mu}h_0 + \int_{\frac{1}{2}X}^{\infty} Y^{-1-\mu}(H(Y) - h_0)dY\} \\
&= X^{-r}h_0^2\int_0^{\frac{1}{2}} u^{-1-\lambda}((1-u)^{-1-\mu} - 1)du \\
&+ ah_0\int_0^{\frac{1}{2}} u^{r-1-\lambda}((1-u)^{-1-\mu} - 1)du \\
&+ ah_0\int_0^{\frac{1}{2}} u^{-1-\lambda}((1-u)^{r-1-\mu} - 1)du \\
&+ X^{-r}h_0^2\int_0^{\frac{1}{2}} u^{-1-\mu}((1-u)^{-1-\lambda} - 1)du \\
&+ ah_0\int_0^{\frac{1}{2}} u^{r-1-\mu}((1-u)^{-1-\lambda} - 1)du \\
&+ ah_0\int_0^{\frac{1}{2}} u^{-1-\mu}((1-u)^{r-1-\lambda} - 1)du + O(X^r).
\end{aligned} \tag{3.1}$$

We now have six cases in (3.1) to consider depending on whether $r > \lambda$, $r = \lambda$, $r < \lambda$, $r > \mu$, $r = \mu$ and $r < \mu$. If we write

$$K_1 = \int_0^\infty Y^{-1-\lambda}(H(Y) - h_0)dY \quad \text{and} \quad K_2 = \int_0^\infty Y^{-1-\mu}(H(Y) - h_0)dY,$$

then the integral certainly exists if $H(Y)$ is ever bounded as $Y \to \infty$, since $H(Y) - h_0 = O(Y^r)$. Thus, we have, using the value of h_0 from (2.5) to cancel out the highest terms,

$$\begin{aligned}
&- ar + a(\lambda + \mu - 1) + \frac{2^\lambda}{\lambda} ah_0 + \frac{2^\mu}{\mu} ah_0 \\
&h_0 X^{\lambda - r}\left(K_1 - \int_0^{\frac{X}{2}} Y^{-1-\lambda}(H(Y) - h_0)dY\right) \\
&+ aX^\lambda \left(K_1 - \int_0^{\frac{X}{2}} Y^{-1-\lambda}(H(Y) - h_0)dY\right) \\
&+ h_0 X^{\mu - r}\left(K_2 - \int_0^{\frac{X}{2}} Y^{-1-\mu}(H(Y) - h_0)dY\right) \\
&+ aX^\mu \left(K_2 - \int_0^{\frac{X}{2}} Y^{-1-\mu}(H(Y) - h_0)dY\right) \\
&= ah_0 \int_0^{\frac{1}{2}} u^{r-1-\lambda}((1-u)^{-1-\mu} - 1)du \\
&+ ah_0 \int_0^{\frac{1}{2}} u^{-1-\lambda}((1-u)^{r-1-\mu} - 1)du \\
&+ ah_0 \int_0^{\frac{1}{2}} u^{r-1-\mu}((1-u)^{-1-\lambda} - 1)du \\
&+ ah_0 \int_0^{\frac{1}{2}} u^{-1-\mu}((1-u)^{r-1-\lambda} - 1)du + O(X^r).
\end{aligned} \tag{3.2}$$

If $r > \lambda$ and $r > \mu$ in (3.2), then the dominant term are now $X^{\lambda - r} h_0 K_1$ and $X^{\lambda - r} h_0 K_2$, so we must have $K_1 = K_2 = 0$. Then

$$\begin{aligned}
\int_0^{\frac{1}{2}X} Y^{-1-\lambda}(H(Y) - h_0)dy &\sim a\int_0^{\frac{1}{2}X} Y^{-1-\lambda} Y^r dY = a\frac{(\frac{1}{2}X)^{r-\lambda}}{r - \lambda}, \\
\int_0^{\frac{1}{2}X} Y^{-1-\mu}(H(Y) - h_0)dy &\sim a\int_0^{\frac{1}{2}X} Y^{-1-\mu} Y^r dY = a\frac{(\frac{1}{2}X)^{r-\mu}}{r - \mu},
\end{aligned}$$

and so

$$
\begin{aligned}
&- r + (\lambda + \mu - 1) + \frac{2^{\lambda}}{\lambda} h_0 + \frac{2^{\mu}}{\mu} h_0 + h_0 \frac{2^{\lambda - r}}{r - \lambda} + h_0 \frac{2^{\mu - r}}{r - \mu} \\
&= h_0 \int_0^{\frac{1}{2}} u^{r-1-\lambda}((1-u)^{-1-\mu} - 1) du \\
&+ h_0 \int_0^{\frac{1}{2}} u^{-1-\lambda}((1-u)^{r-1-\mu} - 1) du \\
&+ h_0 \int_0^{\frac{1}{2}} u^{r-1-\mu}((1-u)^{-1-\lambda} - 1) du \\
&+ h_0 \int_0^{\frac{1}{2}} u^{-1-\mu}((1-u)^{r-1-\lambda} - 1) du + O(X^r).
\end{aligned} \tag{3.3}
$$

This should determine r.

If $r = \lambda$ and $r = \mu$, then this case is impossible since it produces a term $\log X$ in (3.2) which cannot be cancelled.

If $r < \lambda$ and $r < \mu$, then the dominant term are now from (3.2)

$$
\int_0^{\frac{1}{2}X} Y^{-1-\lambda}(H(Y) - h_0) dy \sim a \int_0^{\frac{1}{2}X} Y^{-1-\lambda} Y^r dY = -a \frac{(\frac{1}{2}X)^{r-\lambda}}{r - \lambda},
$$

$$
\int_{\frac{1}{2}X}^{\infty} Y^{-1-\mu}(H(Y) - h_0) dy \sim a \int_{\frac{1}{2}X}^{\infty} Y^{-1-\mu} Y^r dY = -a \frac{(\frac{1}{2}X)^{r-\mu}}{r - \mu},
$$

so that (3.2) leads to

$$
\begin{aligned}
&- r + (\lambda + \mu - 1) + \frac{2^{\lambda - r}}{\lambda - r} h_0 + \frac{2^{\lambda}}{\lambda} h_0 + \frac{2^{\mu - r}}{\mu - r} h_0 + \frac{2^{\mu}}{\mu} h_0 \\
&= h_0 \int_0^{\frac{1}{2}} u^{r-1-\lambda}((1-u)^{-1-\mu} - 1) du \\
&+ h_0 \int_0^{\frac{1}{2}} u^{-1-\lambda}((1-u)^{r-1-\mu} - 1) du \\
&+ h_0 \int_0^{\frac{1}{2}} u^{r-1-\mu}((1-u)^{-1-\lambda} - 1) du \\
&+ h_0 \int_0^{\frac{1}{2}} u^{-1-\mu}((1-u)^{r-1-\lambda} - 1) du,
\end{aligned} \tag{3.4}
$$

which is once again (3.3). This is the either case r determined from (3.4).

Remark 3.1 *If $\lambda = \mu$ and $K(x, y) = (xy)^{\lambda}$ with $0 < 2\lambda < 1$, then it was established in [5] that*

$$
r(\lambda) \sim \frac{\lambda}{2} \pm \sqrt{\lambda} i \quad as \quad \lambda \to 0.
$$

4. Conclusion

Kernel $K(x, y) = x^{\lambda}y^{\mu} + x^{\mu}y^{\lambda}$ we study is more general than the following kernels. $K(x, y) = 1$, $K(x, y) = x + y$, $K(x, y) = x^{\lambda} + y^{\lambda}$, $K(x, y) = xy$ and $K(x, y) = (xy)^{\lambda}$ with $0 < 2\lambda < 1$. For importance of these kernels and their various applications see [7–9, 11–15] and references therein.

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