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Some higher-order identities for generalized bi-periodic Horadam sequences

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Abstract: In this paper, we derive higher-order identities for the generalized bi-periodic Horadam sequences. Some of our results generalize the identities for the classical Fibonacci sequences and Horadam sequences obtained by Hoggatt, Horadam and Waddill. As an application, we derive some congruence properties satisfied by the generalized bi-periodic Horadam sequences.

Key words: Horadam sequence, generalized Fibonacci sequence, generalized Lucas sequence, congruence

1. Introduction

The generalized bi-periodic Horadam sequence $\{w_n\} := \{w_n(w_0, w_1; a, b, c)\}$ is defined by the following recurrence relations:

$$w_n = \begin{cases} aw_{n-1} + cw_{n-2}, & \text{if } n \text{ is even} \\ bw_{n-1} + cw_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \quad n \geq 2 \tag{1.1}$$

with arbitrary initial conditions w_0 and w_1, and nonzero real numbers a, b, and c. The sequence $\{u_n\}$ defined by $\{u_n\} := \{w_n(0, 1; a, b, c)\}$ is called the generalized bi-periodic Fibonacci sequence. The sequence $\{v_n\}$ defined by $\{v_n\} := \{w_n(2, b; a, b, c)\}$ is called the generalized bi-periodic Lucas sequence.

The generalized bi-periodic Horadam sequence $\{w_n\}$ is a natural generalization of the Horadam sequence $\{H_n\} := \{w_n(H_0, H_1; a, a, c)\}$ [5], the bi-periodic Fibonacci sequence $\{q_n\} := \{w_n(0, 1; a, b, 1)\}$ [3], the classical Fibonacci sequence $\{F_n\}$, the Lucas sequence $\{L_n\}$, etc.

The Binet's formula for the generalized bi-periodic Fibonacci sequence $\{u_n\}$ is as follows [12, Theorem 8]:

$$u_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} \right). \tag{1.2}$$

We recall an identity for the sequence $\{w_n\}$ (see Theorem 3 in Section 3 [8]),

$$w_{n+p} = \left(\frac{b}{a}\right)^{\zeta(n+1)\zeta(p)} u_n w_{p+1} + c \left(\frac{b}{a}\right)^{\zeta(n)\zeta(p+1)} u_{n-1} w_p.$$

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By setting $p = 0$, we get the relation

$$w_n = u_n w_1 + c\left(\frac{b}{a}\right)^{\zeta(n)} u_{n-1} w_0.$$

Based on this identity, we easily obtain the Binet's formula for the sequence $\{w_n\}$:

$$w_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^{n-1} - B\beta^{n-1}\right), \quad n > 0 \tag{1.3}$$

where

$$A = \frac{\alpha w_1 + cbw_0}{\alpha - \beta}, \quad B = \frac{\beta w_1 + cbw_0}{\alpha - \beta}.$$

Here the variables α and β are the roots of the polynomial $x^2 - abx - abc$. That is,

$$\alpha = \frac{ab + \sqrt{a^2b^2 + 4abc}}{2}, \quad \beta = \frac{ab - \sqrt{a^2b^2 + 4abc}}{2}.$$

The function $\zeta(n)$ is the parity function of n, i.e. $\zeta(n) = 0$ if n is even, and $\zeta(n) = 1$ if n is odd. We note the following algebraic properties of α and β:

$$\alpha + \beta = ab, \quad \alpha - \beta = \sqrt{a^2b^2 + 4abc}, \quad \alpha\beta = -abc. \tag{1.4}$$

By setting $w_0 = 2$ and $w_1 = b$ in the equation (1.3), we get the Binet's formula for the sequence $\{v_n\}$:

$$v_n = \frac{b^{\zeta(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left(\alpha^n + \beta^n\right). \tag{1.5}$$

In this paper, the phrase 'an identity of order n' refers to an identity for an integer sequence in which every summand is either a constant (associated to the sequence) or a product of n terms of the sequence.

In Section 2, we derive identities of order 2, 3, or 4 based on a combination of matrix methods and algebraic manipulation on the known identities for the generalized bi-periodic Horadam sequences. Along the way we apply our identities to derive new congruence properties for these sequences. Some of our identities generalize the corresponding identities discovered by Hoggatt, Horadam, Waddill, and others, for the Fibonacci sequence and Horadam sequence. It is worthwhile to note that the research on generating new identities and summation formulas for various generalizations of the classical Fibonacci sequence by using matrix methods has been extensive recently. For examples, the work done by Ekin and Tan [4], Keskin and Siar [7], and Tan [9] serve as good references to the subject.

2. Main results

2.1. Some identities of order 2 for the sequence $\{w_n\}$

The matrix $A = \begin{pmatrix} ab & abc \\ 1 & 0 \end{pmatrix}$ has the following property:

$$A^n = \begin{pmatrix} ab & abc \\ 1 & 0 \end{pmatrix}^n = (ab)^{\lfloor \frac{n}{2} \rfloor} \begin{pmatrix} b^{\zeta(n)} u_{n+1} & cba^{\zeta(n)} u_n \\ a^{-\zeta(n+1)} u_n & cb^{\zeta(n)} u_{n-1} \end{pmatrix}. \tag{2.1}$$

By the mathematical induction, we obtain the following: If n is even, then

$$A^n \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = (ab)^{\frac{n}{2}} \begin{pmatrix} w_{n+1} \\ a^{-1}w_n \end{pmatrix}, \quad A^n \begin{pmatrix} cbw_2 \\ cw_1 \end{pmatrix} = (ab)^{\frac{n}{2}} \begin{pmatrix} cbw_{n+2} \\ cw_{n+1} \end{pmatrix}, \tag{2.2}$$

$$\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} A^n = (ab)^{\frac{n}{2}} \begin{pmatrix} w_{n+1} & bcw_n \end{pmatrix}, \begin{pmatrix} a^{-1}w_2 & cw_1 \end{pmatrix} A^n = (ab)^{\frac{n}{2}} \begin{pmatrix} a^{-1}w_{n+2} & cw_{n+1} \end{pmatrix}. \tag{2.3}$$

Similarly, if n is odd, then

$$A^n \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = (ab)^{\frac{n-1}{2}} \begin{pmatrix} bw_{n+1} \\ w_n \end{pmatrix}, \quad A^n \begin{pmatrix} cbw_2 \\ cw_1 \end{pmatrix} = (ab)^{\frac{n-1}{2}} \begin{pmatrix} abcw_{n+2} \\ bcw_{n+1} \end{pmatrix}, \tag{2.4}$$

$$\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} A^n = (ab)^{\frac{n-1}{2}} \begin{pmatrix} bw_{n+1} & abcw_n \end{pmatrix}, \begin{pmatrix} a^{-1}w_2 & cw_1 \end{pmatrix} A^n = (ab)^{\frac{n-1}{2}} \begin{pmatrix} w_{n+2} & bcw_{n+1} \end{pmatrix}. \tag{2.5}$$

By the Cayley–Hamilton theorem, we get the following matrix identity for A:

$$A^2 - tA - uI = 0, \text{ where } t = ab \text{ and } u = abc. \tag{2.6}$$

By the equation (2.6), we do the following computation:

$$A^4 - tA^3 - u^2I = A^2(A^2 - tA) - u^2I = A^2(uI) - u^2I = u(A^2 - uI) = tuA. \tag{2.7}$$

We define the matrix B as follows:

$$B := A^2 = ab \begin{pmatrix} ab + c & abc \\ 1 & c \end{pmatrix}.$$

By the Cayley–Hamilton theorem, we get the following matrix identity for B:

$$B^2 - rB - sI = 0, \text{ where } r = ab(ab + 2c) \text{ and } s = -(abc)^2. \tag{2.8}$$

By using the equation (2.8), we do the following computation:

$$B^4 - rB^3 - s^2I = B^2(B^2 - rB) - s^2I = B^2(sI) - s^2I = s(B^2 - sI) = rsB. \tag{2.9}$$

On the other hand, by using equations (2.2) and (2.3), we get

$$B^n \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = (ab)^n \begin{pmatrix} w_{2n+1} \\ a^{-1}w_{2n} \end{pmatrix}, \quad B^n \begin{pmatrix} cbw_2 \\ cw_1 \end{pmatrix} = (ab)^n \begin{pmatrix} cbw_{2n+2} \\ cw_{2n+1} \end{pmatrix}, \tag{2.10}$$

$$\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^n = (ab)^n \begin{pmatrix} w_{2n+1} & bcw_{2n} \end{pmatrix}, \quad \begin{pmatrix} a^{-1}w_2 & cw_1 \end{pmatrix} B^n = (ab)^n \begin{pmatrix} a^{-1}w_{2n+2} & cw_{2n+1} \end{pmatrix}. \tag{2.11}$$

We state the following theorem.

Theorem 2.1 *Let n be a fixed positive integer. The following identities for the generalized bi-periodic Horadam sequence $\{w_n\}$ are true:*

$$\left(b^{\zeta(n+1)}w_{n+2}^2 + a^{\zeta(n+1)}\left(\frac{b}{a}\right)^{\zeta(n)} cw_{n+1}^2\right) - a^{\zeta(n+1)}b\left(w_{n+2}w_{n+1} + cw_{n+1}w_n\right) -$$
$$c^2\left(b^{\zeta(n+1)}w_n^2 + a^{\zeta(n+1)}\left(\frac{b}{a}\right)^{\zeta(n)} cw_{n-1}^2\right) = a^{\zeta(n+1)}bc(w_1w_{2n} + cw_0w_{2n-1}), \tag{2.12}$$

$$\left(w_{2n+3}^2 + \frac{b}{a}cw_{2n+2}^2\right) - (ab+2c)w_{2n+3}w_{2n+1} - \frac{bc(ab+2c)}{a}w_{2n+2}w_{2n} - c^4\left(w_{2n-1}^2 + \frac{b}{a}cw_{2n-2}^2\right)$$
$$= -c^2(ab+2c)\left(w_1w_{4n-1} + \frac{b}{a}cw_0w_{4n-2}\right). \tag{2.13}$$

Proof

Let $r = ab(ab+2)$ and $s = -(abc)^2$. By the equations (2.8), (2.9), (2.10), (2.11), we compute the following matrix product:

$$\begin{aligned}
&\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} \left(B^{2n+2} - rB^{2n+1} - s^2 B^{2n-2}\right) \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} \\
&= \begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^{2n-2}(rsB) \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = rs \begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^{2n-1} \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} \\
&= rs(ab)^{2n-1} \begin{pmatrix} w_1 & bcw_0 \end{pmatrix} \begin{pmatrix} w_{4n-1} \\ a^{-1}w_{4n-2} \end{pmatrix} \\
&= -a^3b^3c^2(ab+2c)(ab)^{2n-1} \left(w_1 w_{4n-1} + \frac{b}{a} c w_0 w_{4n-2} \right). \qquad (2.14)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^{2n+2} \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = \begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^{n+1} \cdot B^{n+1} \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} \\
&= (ab)^{2n+2} \begin{pmatrix} w_{2n+3} & bcw_{2n+2} \end{pmatrix} \begin{pmatrix} w_{2n+3} \\ a^{-1}w_{2n+2} \end{pmatrix} = (ab)^{2n+2} \left(w_{2n+3}^2 + \frac{b}{a} c w_{2n+2}^2 \right). \qquad (2.15)
\end{aligned}$$

We compute the following two matrix products in an analogous way,

$$\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^{2n+1} \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = (ab)^{2n+1} \left(w_{2n+3}w_{2n+1} + \frac{b}{a} c w_{2n+2} w_{2n} \right), \qquad (2.16)$$

$$\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} B^{2n-2} \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} = (ab)^{2n-2} \left(a_{2n-1}^2 + \frac{b}{a} c w_{2n-2}^2 \right). \qquad (2.17)$$

By the equations (2.15), (2.16), (2.17),

$$\begin{aligned}
&\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} \left(B^{2n+2} - rB^{2n+1} - s^2 B^{2n-2}\right) \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix} \\
&= (ab)^{2n+2} \left(w_{2n+3}^2 + \frac{b}{a} c w_{2n+2}^2 \right) - (ab)^{2n+2}(ab+2c) \left(w_{2n+3}w_{2n+1} + \frac{b}{a} c w_{2n+2} w_{2n} \right) \\
&- (ab)^{2n+2} c^4 \left(w_{2n-1}^2 + \frac{b}{a} c w_{2n-2}^2 \right). \qquad (2.18)
\end{aligned}$$

We get the equation (2.13) by comparing equations (2.14) and (2.18).

The equation (2.12) can be obtained in a similar way by using equations (2.2), (2.3), (2.4), (2.5), (2.6), (2.7) to compute the following matrix product:

$$\begin{pmatrix} w_1 & bcw_0 \end{pmatrix} \left(A^{2n+2} - tA^{2n+1} - u^2 A^{2n-2}\right) \begin{pmatrix} w_1 \\ a^{-1}w_0 \end{pmatrix}.$$

□

Remark 2.2 *For the Horadam sequence* $\{H_n\} = \{w_n(H_0, H_1; a, a, c)\}$, *by Theorem 2.1, we get the following identity (it was first discovered by Waddill [11]),*

$$H_{n+2}^2 + cH_{n+1}^2 - aH_{n+2}H_{n+1} - acH_{n+1}H_n - c^2H_n^2 - c^3H_{n-1}^2 = ac(H_1H_{2n} + cH_0H_{2n-1}).$$

We obtain the following two corollaries immediately by identities (2.12) and (2.13) in Theorem 2.1.

Corollary 2.3 *Let n be an even positive integer. Let w_0, w_1, a, b, and c be any integers such that $a \neq 0$, $b \neq 0$, and $c \neq 0$. For the generalized bi-periodic Horadam sequence $\{w_n\}$,*

$$bw_{n+2}^2 + acw_{n+1}^2 - bc^2w_n^2 - ac^3w_{n-1}^2 \equiv 0 \mod (ab). \tag{2.19}$$

Corollary 2.4 *Let n be a fixed positive integer. Let H_0, H_1, a, and c be any integers such that $a \neq 0$ and $c \neq 0$. For the Horadam sequence $\{H_n\} := \{w_n(H_0, H_1; a, a, c)\}$,*

$$H_{n+2}^2 + cH_{n+1}^2 - c^2H_n^2 - c^3H_{n-1}^2 \equiv 0 \mod a, \tag{2.20}$$

$$H_{2n+3}^2 + cH_{2n+2}^2 - c^4H_{2n-1}^2 - c^5H_{2n-2}^2 \equiv 0 \mod (a^2 + 2c). \tag{2.21}$$

Remark 2.5 *By the classical Fibonacci identity $F_{2n+1} = F_n^2 + F_{n+1}^2$,*

$$F_{4(n+1)} = F_{4n+4} = F_{4n+5} - F_{4n+3} = (F_{2n+3}^2 + F_{2n+2}^2) - (F_{2n-1}^2 + F_{2n-2}^2).$$

Since $4|(4n+4)$ and $F_4 = 3$, by the well-known divisbility of Fibonacci numbers [10], i.e. if $n|m$, then $F_n|F_m$, it is clear that $F_{2n+3}^2 + F_{2n+2}^2 - F_{2n-1}^2 - F_{2n-2}^2 \equiv 0 \mod 3$. Hence, the equation (2.21) is a generalization of this classical result for $\{F_n\}$ to the corresponding result for the Horadam sequence $\{H_n\}$.

By the definition of the sequence $\{w_n\}$ (see (1.1)), we get the following recurrence relations for $n \geq 2$:

$$w_{2n+3} = (ab + 2c)w_{2n+1} - c^2w_{2n-1}, \tag{2.22}$$

$$w_{2n+2} = (ab + 2c)w_{2n} - c^2w_{2n-2}. \tag{2.23}$$

We recall the following identity (see Corollary 1 in Section 3 [8]):

$$\left(\frac{b}{a}\right)^{\zeta(n)} w_{n+1}^2 + \left(\frac{b}{a}\right)^{\zeta(n+1)} cw_n^2 = w_1w_{2n+1} + \left(\frac{b}{a}\right) cw_0w_{2n}. \tag{2.24}$$

Let n be a fixed positive integer. We define the function $G(n)$ as follows:

$$G(n) := \left(\frac{b}{a}\right)^{\zeta(n)} c^{\zeta(n+1)}w_{n+1}^2 - (ab + 2c)\left(\frac{bc}{a}\right)^{\zeta(n+1)} w_n^2 + \left(\frac{b}{a}\right)^{\zeta(n)} c^{2+\zeta(n+1)}w_{n-1}^2. \tag{2.25}$$

Lemma 2.6 *Let n be a fixed positive integer. The function $G(n)$ satisfies the following identity for all n:*

$$G(n) + c^{2\zeta(n+1)}G(n-1) = 0.$$

Proof We will prove it for odd n. By the equations (2.22), (2.23), (2.24), we do the following computation:

$$\begin{aligned}
&G(n)+G(n-1)\\
&=\left(\frac{b}{a}w_{n+1}^2-(ab+2c)w_n^2+\frac{b}{a}c^2w_{n-1}^2\right)+\left(cw_n^2-(ab+2c)\frac{bc}{a}w_{n-1}^2+c^3w_{n-2}^2\right)\\
&=\left(\frac{b}{a}w_{n+1}^2+cw_n^2\right)-(ab+2c)\left(w_n^2+\frac{bc}{a}w_{n-1}^2\right)+c^2\left(\frac{b}{a}w_{n-1}^2+cw_{n-2}^2\right)\\
&=\left(w_1w_{2n+1}+\frac{bc}{a}w_0w_{2n}\right)-(ab+2c)\left(w_1w_{2n-1}+\frac{bc}{a}w_0w_{2n-2}\right)+c^2\left(w_1w_{2n-3}+\frac{bc}{a}w_0w_{2n-4}\right)\\
&=w_1\left(w_{2n+1}-(ab+2c)w_{2n-1}+c^2w_{2n-3}\right)+\frac{bc}{a}w_0\left(w_{2n}-(ab+2c)w_{2n-2}+c^2w_{2n-4}\right)=0.
\end{aligned}$$

For even n, it can be proved by the same method. □

Lemma 2.7 *Let n be a fixed positive integer such that $n\geq 2$. Then,*

$$\left(\frac{b}{a}\right)^{\zeta(n)}w_{n+1}^2-\left(\frac{b}{a}\right)^{\zeta(n+1)}(ab+c)w_n^2-\left(\frac{b}{a}\right)^{\zeta(n)}c(ab+c)w_{n-1}^2+\left(\frac{b}{a}\right)^{\zeta(n+1)}c^3w_{n-2}^2=0.$$

Proof It is clear by Lemma 2.6. □

The following corollary is clear by Lemma 2.7.

Corollary 2.8 *Let H_0, H_1, a, and c be any integers such that $a\neq 0$ and $c\neq 0$. For the Horadam sequence $\{H_n\}:=\{w_n(H_0,H_1;a,a,c)\}$,*

$$H_{n+3}^2+c^3H_n^2\equiv 0 \mod (a^2+c).$$

Let Δ be the following constant for the sequence $\{w_n\}$:

$$\Delta:=w_1^2-bw_0w_1-\frac{b}{a}cw_0^2.$$

We note that

$$\begin{aligned}
G(1)&=\frac{b}{a}w_2^2-(ab+2c)w_1^2+\frac{bc^2}{a}w_0^2=\frac{b}{a}(aw_1+cw_0)^2-(ab+2c)w_1^2+\frac{bc^2}{a}w_0^2\\
&=-2cw_1^2+\frac{2bc^2}{a}w_0^2+2bcw_0w_1=-2c\Delta.
\end{aligned}\tag{2.26}$$

By Lemma 2.6, the following result can be proved by induction:

$$G(n)=\begin{cases}-c^nG(1), & \text{if } n \text{ is even}\\ c^{n-1}G(1), & \text{if } n \text{ is odd}\end{cases},\ n\geq 1\tag{2.27}$$

Theorem 2.9 *For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have the following identity:*

$$\left(\frac{b}{a}\right)^{\zeta(n)}w_{n+1}^2-(ab+2c)\left(\frac{b}{a}\right)^{\zeta(n+1)}w_n^2+\left(\frac{b}{a}\right)^{\zeta(n)}c^2w_{n-1}^2=(-1)^n2c^n\Delta.$$

Proof It is clear by equations (2.26) and (2.27). □

Remark 2.10 *If $a = b = c = 1$, $w_0 = 0$ and $w_1 = 1$, then we obtain the following two Fibonacci identities (which were first discovered by Hoggatt and Bicknell [1, 2]) by Lemma 2.7 and Theorem 2.9 respectively:*

$$F_{n+3}^2 - 2F_{n+2}^2 - 2F_{n+1}^2 + F_n^2 = 0,$$
$$F_{n+1}^2 - 3F_n^2 + F_{n-1}^2 = (-1)^n 2.$$

2.2. Some identities of order 4 for the sequence $\{w_n\}$

We recall the Cassini's identity for the sequence $\{w_n\}$ (see Theorem 4 in Section 3 [8]):

$$\left(\frac{b}{a}\right)^{\zeta(n)} w_{n-1}w_{n+1} - \left(\frac{b}{a}\right)^{\zeta(n+1)} w_n^2 = (-1)^n c^{n-1}\Delta. \tag{2.28}$$

Lemma 2.11 *Let n be a fixed positive integer.*

$$\left(\frac{b}{a}\right)^{\zeta(n)} cw_{n-1}^2 - \left(\frac{b}{a}\right)^{\zeta(n+1)} w_n^2 = (-1)^n c^{n-1}\Delta - bw_nw_{n-1}.$$

Proof It can be proved by straightforward computation. We will prove it for even n only. By the identity (2.28),

$$w_{n-1}w_{n+1} - \frac{b}{a}w_n^2 = w_{n-1}\left(bw_n + cw_{n-1}\right) - \frac{b}{a}w_n^2 = (-1)^n c^{n-1}\Delta$$
$$\left(cw_{n-1}^2 - \frac{b}{a}w_n^2\right) + bw_nw_{n-1} = (-1)^n c^{n-1}\Delta.$$

We get our result as desired. The corresponding identity for odd n can be proved by the same computation. □

Lemma 2.12 *Let n be a fixed positive integer.*

$$\left(\frac{b}{a}\right)^{2\zeta(n)} c^2w_{n-1}^4 + \left(\frac{b}{a}\right)^{2\zeta(n+1)} w_n^4 = c^{2n-2}\Delta^2 - 2(-1)^n c^{n-1}b\Delta w_nw_{n-1} + \left(b^2 + \frac{2bc}{a}\right) w_{n-1}^2w_n^2.$$

Proof Let n be even. By Lemma 2.11, the result is obvious by expanding the square in the following equation:

$$c^2w_{n-1}^4 + \frac{b^2}{a^2}w_n^4 = \left(cw_{n-1}^2 - \frac{b}{a}w_n^2\right)^2 + \frac{2bc}{a}w_{n-1}^2w_n^2.$$

The proof for the case of odd n is similar. □

Let n be a fixed positive integer. We define the function $S(n)$ as follows:

$$S(n) := 2\left(\frac{b}{a}\right)^{\zeta(n)} bcw_{n-1}^3w_n - 2\left(\frac{b}{a}\right)^{\zeta(n+1)} bw_{n-1}w_n^3 + \left(b^2 - \frac{2bc}{a}\right) w_{n-1}^2w_n^2.$$

Lemma 2.13 *Let n be a fixed positive integer.*

$$S(n) = 2(-1)^n c^{n-1} b\Delta w_{n-1} w_n - \left(b^2 + \frac{2bc}{a}\right) w_{n-1}^2 w_n^2..$$

Proof For even n, we apply Lemma 2.11 to do the following computation,

$$\begin{aligned} S(n) &= 2bw_{n-1}w_n \left(cw_{n-1}^2 - \frac{b}{a} w_n^2\right) + \left(b^2 - \frac{2bc}{a}\right) w_{n-1}^2 w_n^2 \\ &= 2bw_{n-1}w_n \left((-1)^n c^{n-1}\Delta - bw_{n-1}w_n\right) + \left(b^2 - \frac{2bc}{a}\right) w_{n-1}^2 w_n^2 \\ &= 2(-1)^n c^{n-1} b\Delta w_{n-1} w_n - \left(b^2 + \frac{2bc}{a}\right) w_{n-1}^2 w_n^2. \end{aligned}$$

The same result can be proved for odd n by essentially the same computation. □

By combining Lemma 2.12 and Lemma 2.13, we obtain the following theorem immediately.

Theorem 2.14 *Let n be a fixed positive integer. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have the following identity:*

$$\left(\frac{b}{a}\right)^{2\zeta(n)} c^2 w_{n-1}^4 + \left(\frac{b}{a}\right)^{2\zeta(n+1)} w_n^4 + 2\left(\frac{b}{a}\right)^{\zeta(n)} bcw_{n-1}^3 w_n - 2\left(\frac{b}{a}\right)^{\zeta(n+1)} bw_{n-1}w_n^3 + \left(b^2 - \frac{2bc}{a}\right) w_{n-1}^2 w_n^2 = c^{2n-2}\Delta^2.$$

Remark 2.15 *For the generalized Fibonacci sequence $\{h_n\} = \{w_n(h_0, h_1; 1, 1, 1)\}$, by Theorem 2.14, we get the following identity (which was first discovered by Horadam and Walton [6, p.272]):*

$$h_{n-1}^4 + h_n^4 + 2h_{n-1}^3 h_n - 2h_{n-1}h_n^3 - h_{n-1}^2 h_n^2 = \Delta^2.$$

For the Fibonacci sequence $\{F_n\}$, the corresponding result was first stated in Zeitlin's paper [13].

Based on the definition of the sequence $\{w_n\}$ (1.1), we obtain the following recurrence relations:

$$w_{n+1} = a^{\zeta(n)} b^{\zeta(n+1)} w_n + cw_{n-1}, \tag{2.29}$$

$$w_{n+2} = (ab + c)w_n + a^{\zeta(n+1)} b^{\zeta(n)} cw_{n-1}, \tag{2.30}$$

$$w_{n+3} = (a^{\zeta(n)+1} b^{\zeta(n+1)+1} + 2a^{\zeta(n)} b^{\zeta(n+1)} c)w_n + (abc + c^2)w_{n-1}. \tag{2.31}$$

We replace the equations (2.29), (2.30), (2.31) into the following expression:

$$c_5 w_{n+3}^4 + c_4 w_{n+2}^4 + c_3 w_{n+1}^4 + c_2 w_n^4 + c_1 w_{n-1}^4$$

to get an expression of the following form

$$d_5 w_n^4 + d_4 w_n^3 w_{n-1} + d_3 w_n^2 w_{n-1}^2 + d_2 w_n w_{n-1}^3 + d_1 w_{n-1}^4 \tag{2.32}$$

where the variable d_i, for each $1 \leq i \leq 5$, is written as a linear combination of the variables c_j for $1 \leq j \leq 5$ such that the coefficients associated to each variable c_j are expressions in terms of a, b and c. For example, for even n, we have the following system of equations:

$$
\begin{aligned}
d_5 &= c_2 + b^4 c_3 + (ab+c)^4 c_4 + (ab^2+2bc)^4 c_5,\\
d_4 &= (4b^3c)c_3 + \big(4(ab+c)^3(ac)\big)c_4 + \big(4(ab^2+2bc)^3(abc+c^2)\big)c_5,\\
d_3 &= (6b^2c^2)c_3 + \big(6(ab+c)^2(ac)^2\big)c_4 + \big(6(ab^2+2bc)^2(abc+c^2)^2\big)c_5,\\
d_2 &= (4bc^3)c_3 + \big(4(ab+c)(ac)^3\big)c_4 + \big(4(ab^2+2bc)(abc+c^2)^3\big)c_5,\\
d_1 &= c_1 + c^4 c_3 + (ac)^4 c_4 + \big((abc+c^2)^4\big)c_5.
\end{aligned}
$$

For odd n, we get a similar system of equations.

By comparing coefficients of the terms $w_n^k w_{n-1}^{4-k}$, for $0 \leq k \leq 4$, in the expression (2.32) and the LHS of the identity in Theorem 2.14, we can solve the corresponding (5×5)-system of linear equations to have the unknowns c_i, for $1 \leq i \leq 5$, written in terms of the constants a, b, and c. We summarize our result as follows:

Theorem 2.16 *Let n be a fixed positive integer. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have the following identity:*

$$
\begin{aligned}
\Big(\frac{b}{a}\Big)^{2\zeta(n)} w_{n+3}^4 - ab(ab+3c)\Big(\frac{b}{a}\Big)^{2\zeta(n+1)} w_{n+2}^4 - c(6a^2b^2c+10abc^2+a^3b^3+2c^3)\Big(\frac{b}{a}\Big)^{2\zeta(n)} w_{n+1}^4\\
-abc^4(ab+3c)\Big(\frac{b}{a}\Big)^{2\zeta(n+1)} w_n^4 + c^8\Big(\frac{b}{a}\Big)^{2\zeta(n)} w_{n-1}^4 = -6abc^{2n+3}\Delta^2.
\end{aligned}
$$

Remark 2.17 *For the generalized Fibonacci sequence $\{h_n\} = \{w_n(h_0,h_1;1,1,1)\}$, by Theorem 2.16, we get the following identity (which was first discovered by Horadam and Walton [6, p.273]):*

$$
h_{n+4}^4 - 4h_{n+3}^4 - 19h_{n+2}^4 - 4h_{n+1}^4 + h_n^4 = -6\Delta^2.
$$

2.3. Some identities of order 3 for the sequence $\{w_n\}$

It is difficult to come up with identities similar to those found in Theorem 2.9 and Theorem 2.14 for cubes of the terms in the sequence $\{w_n\}$. We state one result related to identities of order 3 for the sequence $\{w_n\}$. First, we state two lemmas.

Lemma 2.18 *Let n be a fixed nonnegative integer.*

$$
w_n w_{n+1} w_{n+2} = (-1)^{n+1}\left(\frac{a}{b}\right)^{\zeta(n+1)} c^n \Delta w_{n+1} + \left(\frac{a}{b}\right)^{1-2\zeta(n)} w_{n+1}^3.
$$

Proof By (2.28), for even n, we have

$$
\begin{aligned}
\frac{b}{a} w_{n+2} w_n - w_{n+1}^2 &= -c^n \Delta\\
\frac{b}{a} w_n w_{n+1} w_{n+2} &= -w_{n+1} c^n \Delta + w_{n+1}^3.
\end{aligned}
$$

The identity for the case of odd n can be proved by essentially the same computation. □

Lemma 2.19 *Let n be a fixed nonnegative integer.*

$$w_{n+2}^3 = (a^{\zeta(n+1)}b^{\zeta(n)})^3 w_{n+1}^3 + c^3 w_n^3 + 3a^{\zeta(n+1)}b^{\zeta(n)} c w_n w_{n+1} w_{n+2}.$$

Proof We will prove it for even n.

$$\begin{aligned} w_{n+2}^3 &= (aw_{n+1} + cw_n)^3 \\ &= a^3 w_{n+1}^3 + c^3 w_n^3 + 3a^2 c w_{n+1}^2 w_n + 3ac^2 w_{n+1} w_n^2 \\ &= a^3 w_{n+1}^3 + c^3 w_n^3 + 3acw_n w_{n+1}(aw_{n+1} + cw_n) \\ &= a^3 w_{n+1}^3 + c^3 w_n^3 + 3acw_n w_{n+1} w_{n+2}. \end{aligned}$$

The identity for the case of odd n can be proved by the same computation. □

We state an identity of order 3 for the sequence $\{w_n\}$.

Theorem 2.20 *Let n be a fixed positive integer. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have the following identity:*

$$w_{n+2}^3 - \left(3a^{\zeta(n+1)}b^{\zeta(n)} c \left(\frac{a}{b}\right)^{1-2\zeta(n)} + (a^{\zeta(n+1)}b^{\zeta(n)})^3\right) w_{n+1}^3 - c^3 w_n^3$$
$$= 3a^{\zeta(n+1)}b^{\zeta(n)}(-c)^{n+1}\left(\frac{a}{b}\right)^{\zeta(n+1)} \Delta w_{n+1}.$$

Proof It is clear by Lemma 2.18 and Lemma 2.19. □

Remark 2.21 *For the Horadam sequence $\{H_n\} = \{w_n(H_0, H_1; a, a, c)\}$, by Theorem 2.20, we get the following identity (which was first discovered by Horadam [5, p.174]):*

$$H_{n+2}^3 - (3ac + a^3)H_{n+1}^3 - c^3 H_n^3 = 3a(-c)^{n+1}\Delta H_{n+1}.$$

2.4. Some power identities for the sequence $\{w_n\}$ by matrix methods

In this subsection, we explore a matrix method to obtain power identities for the sequence $\{w_n\}$. For the Fibonacci numbers $\{F_n\}$, this matrix approach was used by Hoggatt and Bicknell [1, 2] to obtain some power identities for the sequence $\{F_n\}$.

Let $(v_a)_n$ and $(v_b)_n$ be the following column vectors for $\{w_n\}$.

$$\overrightarrow{(v_a)_n} := \begin{pmatrix} c^2 w_n^2 \\ 2acw_n w_{n+1} \\ a^2 w_{n+1}^2 \end{pmatrix}, \quad \overrightarrow{(v_b)_n} := \begin{pmatrix} c^2 w_n^2 \\ 2bcw_n w_{n+1} \\ b^2 w_{n+1}^2 \end{pmatrix}.$$

Let E_a and E_b be the generalized bi-periodic Pascal's Triangles for the sequence $\{w_n\}$, which are defined by the following (3×3) matrices:

$$E_a = \begin{pmatrix} 0 & 0 & \frac{c^2}{a^2} \\ 0 & c & 2c \\ a^2 & a^2 & a^2 \end{pmatrix}, \quad E_b = \begin{pmatrix} 0 & 0 & \frac{c^2}{b^2} \\ 0 & c & 2c \\ b^2 & b^2 & b^2 \end{pmatrix}.$$

If n is even, we obtain the following result by straightforward matrix multiplication:

$$E_a\overrightarrow{(v_a)_n} = \overrightarrow{(v_a)_{n+1}}. \tag{2.33}$$

Similarly, if n is odd, we have

$$E_b\overrightarrow{(v_b)_n} = \overrightarrow{(v_b)_{n+1}}. \tag{2.34}$$

Let T_{ab} and T_{ba} be the following (3×3) matrices:

$$T_{ab} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{b}{a} & 0 \\ 0 & 0 & \frac{b^2}{a^2} \end{pmatrix}, \quad T_{ba} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{a}{b} & 0 \\ 0 & 0 & \frac{a^2}{b^2} \end{pmatrix}.$$

It is clear that $T_{ab}^{-1} = T_{ba}$. The matrices T_{ab} and T_{ba} act as transformations between $\overrightarrow{(v_a)_n}$ and $\overrightarrow{(v_b)_n}$, i.e.

$$T_{ab}\overrightarrow{(v_a)_n} = \overrightarrow{(v_b)_n}, \quad T_{ba}\overrightarrow{(v_b)_n} = \overrightarrow{(v_a)_n}. \tag{2.35}$$

By equations (2.33), (2.34), (2.35), we note the following properties among the matrices defined so far:

$$T_{ba}E_bT_{ab}E_a\overrightarrow{(v_a)_n} = \overrightarrow{(v_a)_{n+2}}, \text{ if } n \text{ is even,} \tag{2.36}$$

$$T_{ab}E_aT_{ba}E_b\overrightarrow{(v_b)_n} = \overrightarrow{(v_b)_{n+2}}, \text{ if } n \text{ is odd.} \tag{2.37}$$

By computing the matrix products, we note that

$$T_{ba}E_bT_{ab}E_a = T_{ab}E_aT_{ba}E_b = \begin{pmatrix} c^2 & c^2 & c^2 \\ 2abc & c^2 + 2abc & 2c^2 + 2abc \\ a^2b^2 & abc + a^2b^2 & c^2 + 2abc + a^2b^2 \end{pmatrix}.$$

To simplify the notation, we denote the matrix $T_{ba}E_bT_{ab}E_a$ (or $T_{ab}E_aT_{ba}E_b$) by M. The characteristic equation of M is

$$M^3 - (3c^2 + 4abc + a^2b^2)M^2 - (3c^4 + 4abc^3 + a^2b^2c^2)M - c^6I = 0. \tag{2.38}$$

Theorem 2.22 *Let n be a fixed nonnegative integer. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have the following identities:*

$$w_{n+6}^2 - (3c^2 + 4abc + a^2b^2)w_{n+4}^2 - (3c^4 + 4abc^3 + a^2b^2c^2)w_{n+2}^2 - c^6w_n^2 = 0,$$

$$w_{n+6}w_{n+7} - (3c^2 + 4abc + a^2b^2)w_{n+4}w_{n+5} - (3c^4 + 4abc^3 + a^2b^2c^2)w_{n+2}w_{n+3} - c^6w_nw_{n+1} = 0.$$

Proof It is clear by an application of the matrix identity (2.38) and by comparing the $(2,1)$-entry and $(3,1)$-entry of the matrix equations in (2.36) and (2.37). □

We obtain the following corollary immediately by Theorem 2.22.

Corollary 2.23 *Let n be a fixed nonnegative integer. Let w_0, w_1, a, b, and c be any integers such that $a \neq 0$, $b \neq 0$, and $c \neq 0$. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have*

$$w_{n+6}^2 - c^6w_n^2 \equiv 0 \mod (3c^2 + 4abc + a^2b^2),$$

$$w_{n+6}w_{n+7} - c^6w_nw_{n+1} \equiv 0 \mod (3c^2 + 4abc + a^2b^2).$$

Remark 2.24 *We note that* $3c^2 + 4abc + a^2b^2 = a^{-1}u_6$. *By Corollary 2.23, we get*

$$aw_{n+6}^2 - ac^6 w_n^2 \equiv 0 \mod u_6,$$
$$aw_{n+6}w_{n+7} - ac^6 w_n w_{n+1} \equiv 0 \mod u_6.$$

Indeed, this matrix approach to obtain power identities for the sequence $\{w_n\}$ can be generalized to obtain identities of higher order.

Let $\overrightarrow{(d_a)_n}$ and $\overrightarrow{(d_b)_n}$ be the following column vectors for the sequence $\{w_n\}$:

$$\overrightarrow{(d_a)_n} := \begin{pmatrix} c^3w_n^3 \\ 3ac^2w_n^2w_{n+1} \\ 3a^2cw_nw_{n+1}^2 \\ a^3w_{n+1}^3 \end{pmatrix}, \quad \overrightarrow{(d_b)_n} := \begin{pmatrix} c^3w_n^3 \\ 3bc^2w_n^2w_{n+1} \\ 3b^2cw_nw_{n+1}^2 \\ b^3w_{n+1}^3 \end{pmatrix}.$$

Let M_a and M_b be the generalized bi-periodic Pascal's Triangles defined by the following (4×4) matrices:

$$M_a := \begin{pmatrix} 0 & 0 & 0 & \frac{c^3}{a^3} \\ 0 & 0 & \frac{c^2}{a} & \frac{3c^2}{a} \\ 0 & ac & 2ac & 3ac \\ a^3 & a^3 & a^3 & a^3 \end{pmatrix}, \quad M_b := \begin{pmatrix} 0 & 0 & 0 & \frac{c^3}{b^3} \\ 0 & 0 & \frac{c^2}{b} & \frac{3c^2}{b} \\ 0 & bc & 2bc & 3bc \\ b^3 & b^3 & b^3 & b^3 \end{pmatrix}.$$

Let N_{ab} and N_{ba} be the following (4×4) matrices:

$$N_{ab} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{b}{a} & 0 & 0 \\ 0 & 0 & \frac{b^2}{a^2} & 0 \\ 0 & 0 & 0 & \frac{b^3}{a^3} \end{pmatrix}, \quad N_{ba} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{a}{b} & 0 & 0 \\ 0 & 0 & \frac{a^2}{b^2} & 0 \\ 0 & 0 & 0 & \frac{a^3}{b^3} \end{pmatrix}.$$

It is clear that $N_{ab} = N_{ba}^{-1}$. The matrices N_{ab} and N_{ba} act as transformations between $\overrightarrow{(d_a)_n}$ and $\overrightarrow{(d_b)_n}$, i.e.

$$N_{ab}\overrightarrow{(d_a)_n} = \overrightarrow{(d_b)_n}, \quad N_{ba}\overrightarrow{(d_b)_n} = \overrightarrow{(d_a)_n}.$$

We have the following properties:

$$N_{ba}M_bN_{ab}M_a\overrightarrow{(d_a)_n} = \overrightarrow{(d_a)_{n+2}}, \text{ if } n \text{ is even,} \tag{2.39}$$

$$N_{ab}M_aN_{ba}M_b\overrightarrow{(d_b)_n} = \overrightarrow{(d_b)_{n+2}}, \text{ if } n \text{ is odd.} \tag{2.40}$$

We note that $N_{ba}M_bN_{ab}M_a = N_{ab}M_aN_{ba}M_b$ and this matrix is equal to

$$\begin{pmatrix} c^3 & c^3 & c^3 & c^3 \\ 3abc^2 & c^3 + 3abc^2 & 2c^3 + 3abc^2 & 3c^3 + 3abc^2 \\ 3a^2b^2c & 2abc^2 + 3a^2b^2c & c^3 + 4abc^2 + 3a^2b^2c & 3c^3 + 6abc^2 + 3a^2b^2c \\ a^3b^3 & a^2b^2c + a^3b^3 & abc^2 + 2a^2b^2c + a^3b^3 & c^3 + 3abc^2 + 3a^2b^2c + a^3b^3 \end{pmatrix}.$$

The characteristic polynomial for this matrix is as follows:

$$x^4 - (4c^3 + a^3b^3 + 6a^2b^2c + 10abc^2)x^3 + (2c^6 + (2c^3 + abc^2)(2c^3 + a^3b^3 + 6a^2b^2c + 9abc^2))x^2$$
$$-c^6(4c^3 + a^3b^3 + 6a^2b^2c + 10abc^2)x + c^{12}.$$

By the matrix equations (2.39), (2.40), the Cayley-Hamilton theorem and comparing the entries in the matrix equations, we obtain the following result.

Theorem 2.25 *Let n be a fixed nonnegative integer. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have the following identities:*

$$w_{n+8}^3-(4c^3+a^3b^3+6a^2b^2c+10abc^2)w_{n+6}^3+(2c^6+(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))\cdot$$
$$w_{n+4}^3-c^6(4c^3+a^3b^3+6a^2b^2c+10abc^2)w_{n+2}^3+c^{12}w_n^3=0,$$
$$w_{n+9}^2w_{n+8}-(4c^3+a^3b^3+6a^2b^2c+10abc^2)w_{n+7}^2w_{n+6}+(2c^6+(2c^3+abc^2)(2c^3+a^3b^3$$
$$+6a^2b^2c+9abc^2))w_{n+5}^2w_{n+4}-c^6(4c^3+a^3b^3+6a^2b^2c+10abc^2)w_{n+3}^2w_{n+2}+c^{12}w_{n+1}^2w_n=0,$$
$$w_{n+9}w_{n+8}^2-(4c^3+a^3b^3+6a^2b^2c+10abc^2)w_{n+7}w_{n+6}^2+(2c^6+(2c^3+abc^2)(2c^3+a^3b^3$$
$$+6a^2b^2c+9abc^2))w_{n+5}w_{n+4}^2-c^6(4c^3+a^3b^3+6a^2b^2c+10abc^2)w_{n+3}w_{n+2}^2+c^{12}w_{n+1}w_n^2=0.$$

We obtain the following corollary immediately by Theorem 2.25.

Corollary 2.26 *Let n be a fixed nonnegative integer. Let w_0, w_1, a, b, and c be any integers such that $a\neq 0$, $b\neq 0$, and $c\neq 0$. For the generalized bi-periodic Horadam sequence $\{w_n\}$, we have*

$$w_{n+8}^3+(2c^6+(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))w_{n+4}^3+c^{12}w_n^3$$
$$\equiv 0 \mod (4c^3+a^3b^3+6a^2b^2c+10abc^2),$$
$$w_{n+9}^2w_{n+8}+(2c^6+(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))w_{n+5}^2w_{n+4}+c^{12}w_{n+1}^2w_n$$
$$\equiv 0 \mod (4c^3+a^3b^3+6a^2b^2c+10abc^2),$$
$$w_{n+9}w_{n+8}^2+(2c^6+(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))w_{n+5}w_{n+4}^2+c^{12}w_{n+1}w_n^2$$
$$\equiv 0 \mod (4c^3+a^3b^3+6a^2b^2c+10abc^2).$$

Remark 2.27 *We note that $4c^3+a^3b^3+6a^2b^2c+10abc^2=a^{-1}u_8$. By Corollary 2.26, we get*

$$aw_{n+8}^3+a(2c^6+(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))w_{n+4}^3+ac^{12}w_n^3$$
$$\equiv 0 \mod u_8,$$
$$aw_{n+9}^2w_{n+8}+a(2c^6+(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))w_{n+5}^2w_{n+4}+ac^{12}w_{n+1}^2w_n$$
$$\equiv 0 \mod u_8,$$
$$aw_{n+9}w_{n+8}^2+a(2c^6+a(2c^3+abc^2)(2c^3+a^3b^3+6a^2b^2c+9abc^2))w_{n+5}w_{n+4}^2+ac^{12}w_{n+1}w_n^2$$
$$\equiv 0 \mod u_8.$$

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