Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1701 – 1707
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doi:10.3906/mat-2001-58

An algorithm to check the equality of total domination number and double of domination number in graphs

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Received: 17.01.2020 • **Accepted/Published Online:** 18.06.2020 • **Final Version:** 21.09.2020

Abstract: In graph theory, domination number and its variants such as total domination number are studied by many authors. Let the domination number and the total domination number of a graph G without isolated vertices be $\gamma(G)$ and $\gamma_t(G)$, respectively. Based on the inequality $\gamma_t(G) \leq 2\gamma(G)$, we investigate the graphs satisfying the upper bound, that is, graphs G with $\gamma_t(G) = 2\gamma(G)$. In this paper, we present some new properties of such graphs and provide an algorithm which can determine whether $\gamma_t(G) = 2\gamma(G)$ or not for a family of graphs not covered by the previous results in the literature.

Key words: Domination number, total domination number

1. Introduction

A *graph* G is a pair of sets $V(G)$ and $E(G)$. $V(G)$ is a nonempty set and an element of $V(G)$ is called a *vertex* of G. $E(G)$ is a set of unordered pairs of vertices of G and an element of $E(G)$ is called an *edge* of G. An edge $\{u,v\}$ is denoted as uv for convenience in the text. Whenever uv is an edge in a graph, we say u and v are *adjacent* or *neighbors* and also say u and v are *endpoints* of uv.

The neighborhood of a vertex $v \in V(G)$, denoted by $N_G(v)$, is the set of all neighbors of v in G. For any subset $S \subseteq V(G)$, the neighborhood of S is the union $\cup_{v\in S} N_G(v)$ and is denoted by $N_G(S)$. The closed neighborhood of a subset $S \subseteq V(G)$, denoted by $N_G[S]$, is $N_G(S) \cup S$. In particular, the closed neighborhood of a vertex v of G is denoted by $N_G[v]$. A vertex v is called *isolated* if v is adjacent to no vertex in the graph G, i.e. $N_G(v) = \emptyset$. Along this paper, we use $N(S)$ and $N[S]$ instead of $N_G(S)$ and $N_G[S]$, respectively, as long as the graph G is implicit in the text.

A set $S \subseteq V(G)$ of vertices is called a *dominating set* of G if every vertex not in S is adjacent to at least one member of S, in short, when closed neighborhood of S is $V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G. A subset S of $V(G)$ is called a γ-set of G whenever S is a minimum dominating set, that is, if S is a dominating set of G satisfying $|S| = \gamma(G)$.

If G has no isolated vertices, a subset $S \subseteq V(G)$ is called a *total dominating set* of G if every member of $V(G)$ is adjacent to a vertex in S, i.e., neighborhood of S is $V(G)$. The *total domination number* of G without isolated vertices, denoted by $\gamma_t(G)$, is the minimum size of a total dominating set of G. Note that

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2010 *AMS Mathematics Subject Classification:* 05C69, 05C85

there is no total dominating set for graphs having an isolated vertex and therefore, total domination number is defined only for graphs with no isolated vertex. A minimum total dominating set of G (a total dominating set of G with cardinality $\gamma_t(G)$) is called a γ_t-set of G. See Figure 1 for γ-sets and γ_t-sets of a graph.

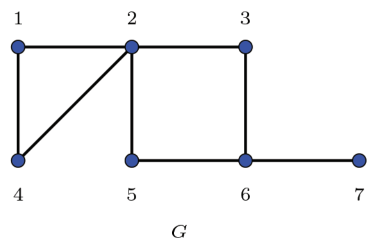

Figure 1. In the given graph G, $\gamma(G) = 2$ and $\gamma_t(G) = 3$. Note that γ-sets of G are $\{1,6\}$, $\{2,6\}$ and $\{4,6\}$, and γ_t-sets of G are $\{2,3,6\}$ and $\{2,5,6\}$.

Since every γ_t-set is also a dominating set, we obtain the inequality $\gamma(G) \leq \gamma_t(G)$. For any γ-set S, one can expand S to a total dominating set of cardinality at most $2\gamma(G)$ by inserting a neighbor of each vertex in S and consequently, we get the inequality $\gamma_t(G) \leq 2\gamma(G)$. Therefore, we have the inequality chain $\gamma(G) \leq \gamma_t(G) \leq 2\gamma(G)$ involving domination and total domination numbers which is first beheld by [2]. We say a graph G is a $(\gamma_t, 2\gamma)$-graph whenever $\gamma_t(G) = 2\gamma(G)$. Characterization of all $(\gamma_t, 2\gamma)$-graphs still appears to be an unsolved problem, however, some results on specific families are provided by several authors. [5] characterized $(\gamma_t, 2\gamma)$-trees in a constructive way, whereas [6] solved the problem for block graphs and gave necessary and sufficient conditions for a graph to be a $(\gamma_t, 2\gamma)$-block graph. [1] extended these two results for a larger family of graphs (including the class of chordal graphs) and showed that the problem of determining whether a given graph from that family is a $(\gamma_t, 2\gamma)$-graph or not is polynomial time solvable. In addition to these three works, [3] characterized all $(\gamma_t, 2\gamma)$-cubic graphs.

In this paper, we study determining whether a given graph is a $(\gamma_t, 2\gamma)$-graph or not. In general, both of finding the domination number and calculating the total domination number are NP-complete problems (see, [4] and [7], respectively.) Therefore, there does not exist an algorithm with polynomial time complexity which computes both γ and γ_t. By using some structural properties of $(\gamma_t, 2\gamma)$-graphs we establish an algorithm which runs in polynomial time to check whether a given graph is a $(\gamma_t, 2\gamma)$-graph or not. Even though the algorithm we provide does not work for some graphs, it works for a family of graphs which belongs to none of the classes above.

The rest of this paper is organized as follows: Preliminaries are provided in Section 2, main results of this paper are presented in Section 3, and concluding remarks together with discussion are given in Section 4.

2. Preliminaries

We first provide some definitions required for the statements of the results in this paper. Two vertices u and v in G are called *true twins* if $N[u] = N[v]$ holds, i.e. u and v are true twins if and only if any vertex other than u and v is adjacent to either both or none of u and v.

We borrow some notation and definition from [1] required for the building blocks of this paper. For every vertex v in G, split $N[v]$ into three sets $T(v), D(v)$ and $M(v)$. The set $T(v)$ consists of v itself and its true twins. Let $D(v) = \{u \in N[v] : N[u] \subset N[v]\}$. That is, $u \in D(v)$ if and only if every neighbor of u other than

v is also a neighbor of v but u is not a true twin of v. All other vertices adjacent to v belong to $M(v)$, i.e. a neighbor of v is in $M(v)$ if and only if it is adjacent to a vertex (other than v) which is not a neighbor of v.

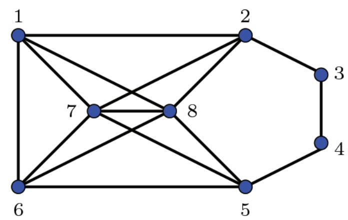

Figure 2. In the given graph, $N[7] = \{1, 2, 5, 6, 7, 8\}$, $T(7) = \{7, 8\}$, $D(7) = \{1, 6\}$ and $M(7) = \{2, 5\}$. Since none of $N(2) = \{1, 3, 7, , 8\}$ and $N(5) = \{4, 6, 7, 8\}$ includes $D(7) = \{1, 6\}$, we see that 7 is a special vertex. Note also that 7 and 8 are true twins and they are the only special vertices in this graph.

In [1], the results are based on a set of vertices (named *special* vertices) satisfying some structural properties. Special vertices are also the keystones of this paper. A vertex v is called special if none of the vertices in $M(v)$ has a neighborhood containing $D(v)$. In other words, a vertex v is special if and only if the set of vertices adjacent to every vertex in $D(v)$ is $T(v)$. Note that if $D(v)$ is an empty set and $M(v)$ is not, then v is not special. Moreover, note that if a vertex is special, then any true twin of it is special as well. See Figure 2 for an example of a special vertex.

Notice that being true twins induces an equivalence relation on $V(G)$ and we consider associated equivalence classes of the set of special vertices. In other words, we partition the set of special vertices of G in such a way that two vertices are in the same part if and only if they are true twins. A set obtained by selecting exactly one vertex from each part is called an $S(G)$*-set*. Hence, for any special vertex v, every $S(G)$-set contains exactly one member in $T(v)$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. For a subset $A \subset V(G)$, $G - A$ is the graph obtained by removing the vertices in A together with the edges incident to them from the graph G. A subgraph H of G is an *induced subgraph* of G whenever $H = G - A$ for some A. The subgraph of G induced by the set $A \subseteq V(G)$ is the graph $G - B$ where $B = V(G) \backslash A$. In other words, the subgraph of G induced by A is the graph whose vertex set is A and whose edge set consists of all the edges in $E(G)$ that have both endpoints in A.

A graph is called $(G_1, \ldots, G_k)$*-free* if none of $G_1, \ldots, G_k$ is an induced subgraph of G. A set $S \subseteq V(G)$ is called a *packing* in G if $N[u]$ and $N[v]$ share no element for every pair of vertices u and v in S. A set is called an *efficient dominating set* of G if it is both a packing and a dominating set in G. Let H_1, H_2 and C_6 be the graphs shown in Figure 3.

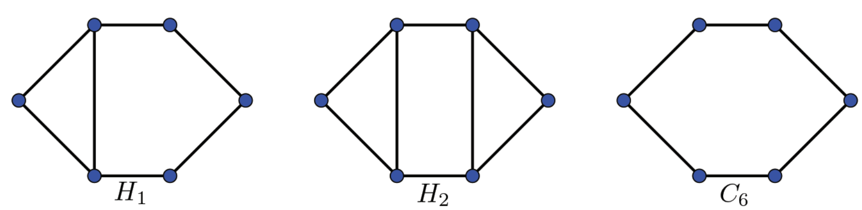

Figure 3. The graphs H_1, H_2 and C_6.

[1] provided the following three results.

Lemma 2.1 *Let G be a $(\gamma_t, 2\gamma)$-graph and A be a subset of $V(G)$. Then, A is a γ-set of G if and only if A is an efficient dominating set of G.*

Lemma 2.2 *Let S be an $S(G)$-set. If S is an efficient dominating set of G, then G is a $(\gamma_t, 2\gamma)$-graph.*

Theorem 2.3 *Let S be an $S(G)$-set of a (H_1, H_2, C_6)-free graph G. Then, G is a $(\gamma_t, 2\gamma)$-graph if and only if S is an efficient dominating set of G.*

Lemma 2.2 gives a sufficient condition for G to be a $(\gamma_t, 2\gamma)$-graph. However, this sufficient condition is not always necessary. For example, C_6 is a $(\gamma_t, 2\gamma)$-graph but it has no special vertices. On the other hand, Theorem 2.3 shows that this sufficient condition is also necessary whenever the graph has no H_1, H_2 or C_6 as an induced subgraph. Therefore, if the given graph G has an induced H_1, H_2 or C_6, then Theorem 2.3 has no conclusion. In this paper, we fill some part of the gap on determining whether a given graph is a $(\gamma_t, 2\gamma)$-graph or not.

3. Main results

In this section, we present an algorithm which might detect $(\gamma_t, 2\gamma)$- graphs and non-$(\gamma_t, 2\gamma)$- graphs. We first provide some results on $(\gamma_t, 2\gamma)$-graphs and special vertices which justify why the algorithm works.

Lemma 3.1 *Let G be a $(\gamma_t, 2\gamma)$-graph and D be a γ-set of G. If v is a special vertex, then $|T(v) \cap D| = 1$, that is, exactly one true twin (or itself) of v is in D.*

Proof Let $A = \{v_1, \ldots, v_k\}$. By Lemma 2.1 we see that A is a packing and a dominating set. Therefore, $N[v_1], \ldots, N[v_k]$ is a partition of $V(G)$ and hence v belongs to exactly one of them, say $N[v_1]$. Note that it suffices to show that $v \in T(v_1)$. We will show that v belongs to neither $D(v_1)$ nor $M(v_1)$.

Suppose that $v \in D(v_1)$. Then, since v and v_1 are not true twins, v_1 has a neighbor which is not adjacent to v. Thus, v_1 must be in $M(v)$. On the other hand, as $v \in D(v_1)$ we have $D(v) \subseteq N[v] \subset N[v_1]$ which contradicts with the fact that v is special. Hence, we get $v \notin D(v_1)$.

Now assume that $v \in M(v_1)$. Clearly v and v_i are not adjacent for every $i \geq 2$. If w is a neighbor of v in $N(v_i)$ for some $i \geq 2$, then $w \in M(v)$, because w is adjacent to v_i which is not a neighbor of v. Therefore, the set $D(v)$ is contained in $N[v_1]$. Then as v is special, v_1 cannot be in $M(v)$. Moreover, v and v_1 are not true twins and hence, v_1 must be in $D(v)$. Then we see that $N[v_1] \subseteq N[v]$, that is, every neighbor of v_1 other than v is also a neighbor of v. However, in this case, the set $\{v, v_2, w_2, \ldots, v_k, w_k\}$ (where $w_i \in N(v_i)$ for every $i \geq 2$ and one of $w_2, \ldots, w_k$ is a neighbor of v which exists since $v \in M(v_1)$) is a total dominating set of size $2k - 1$, a contradiction. $\square$

Notice that in a dominating set, if a vertex is replaced by one of its true twins then the resulting set is a dominating set as well. Consequently, we obtain the following corollaries.

Corollary 3.2 *Let G be a $(\gamma_t, 2\gamma)$-graph and S be an $S(G)$-set. Then, S is a packing and there exists a γ-set of G containing S.*

Corollary 3.3 *If an $S(G)$-set is not a packing, then G is not a $(\gamma_t, 2\gamma)$-graph.*

Considering the empty graph as a $(\gamma_t, 2\gamma)$-graph, we have the following lemma.

Lemma 3.4 *Let G be a $(\gamma_t, 2\gamma)$-graph, X be a subset of a γ-set of G and $G' = G - N[X]$. Then, $\gamma(G') = \gamma(G) - |X|$ and $\gamma_t(G') = 2(\gamma(G) - |X|)$ and hence, G' is a $(\gamma_t, 2\gamma)$-graph as well.*

Proof First note that if G' is empty, then the claim is trivial. Let $X = \{x_1, \ldots, x_k\}$ and $\{x_1, \ldots, x_k, y_1, \ldots, y_m\}$ be a γ-set of G (so, $\gamma(G) = k + m$ and $\gamma_t(G) = 2k + 2m$). Recall that $N[x_1], \ldots, N[x_k], N[y_1], \ldots, N[y_m]$ is a partition of $V(G)$.

Now let D be a γ-set of G'. Note that $D \cup X$ is a dominating set of G and hence, we get $\gamma(G') + k \geq \gamma(G) = k + m$ i.e., $\gamma(G') \geq m$. On the other hand, $\{y_1, \ldots, y_m\}$ is a dominating set of G' and therefore, we obtain $\gamma(G') = m$.

Next, let T be a γ_t-set of G' and w_i be a neighbor of x_i in G for $i = 1, \ldots, k$. Then, $T \cup \{x_1, w_1, \ldots, x_k, w_k\}$ is a γ_t-set of G and hence, we get $\gamma_t(G') + 2k \geq \gamma_t(G) = 2k + 2m$, that is, $\gamma_t(G') \geq 2m$. Moreover, we have $\gamma_t(G') \leq 2\gamma(G') = 2m$ and thus, we get $\gamma_t(G') = 2m = 2\gamma(G')$. $\square$

Combining Corollary 3.2 and Lemma 3.4 yields the following result.

Proposition 3.5 *If G is a $(\gamma_t, 2\gamma)$-graph, then $G - N[S]$ is a $(\gamma_t, 2\gamma)$-graph and $\gamma(G - N[S]) = \gamma(G) - |S|$ for any $S(G)$-set S.*

Proposition 3.5 implies that removing the vertices of an $S(G)$-set together with their neighbors from a $(\gamma_t, 2\gamma)$-graph G reveals another $(\gamma_t, 2\gamma)$-graph which is an induced subgraph of G. However, the converse of this implication is not always true. If $G - N[S]$ is a $(\gamma_t, 2\gamma)$-graph for an $S(G)$-set, then G itself does not have to be a $(\gamma_t, 2\gamma)$-graph. See Figure 4 for an example.

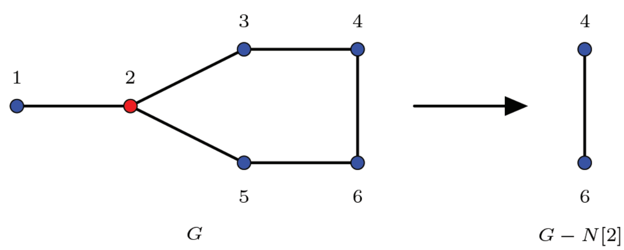

Figure 4. In the given graph G, 2 is the unique special vertex. Note that $\gamma(G - N[2]) = 1$ and $\gamma_t(G - N[2]) = 2$ and hence, $G - N[2]$ is a $(\gamma_t, 2\gamma)$-graph. However, G is not a $(\gamma_t, 2\gamma)$-graph since $\gamma(G) = 2$ and $\gamma_t(G) = 3$.

Proposition 3.6 *Let G be a $(\gamma_t, 2\gamma)$-graph. Let $G_0 = G$ and $G_{i+1} = G_i - N[S_i]$ for $i = 0, \ldots, m$ where each S_i is a nonempty $S(G_i)$-set. If v is a special vertex of G_{m+1}, then $N_G[v] \cap N_G[S_i] = \emptyset$ for $i = 0, \ldots, m$.*

Proof By Proposition 3.5 we see that $G_1, \ldots, G_{m+1}$ are $(\gamma_t, 2\gamma)$-graphs. By Corollary 3.2, there exists a γ-set of G_{m+1} containing v, say D. Proposition 3.5 yields that $\gamma(G_m) = |D| + |S_m|$. On the other hand, $D \cup S_m$ is a dominating set of G_m and therefore, $D \cup S_m$ is a γ-set of G_m. Applying Proposition 3.5 recursively gives that $D \cup (\cup_{i=0}^{m} S_m)$ is a γ-set of $G_0 = G$. As G is a $(\gamma_t, 2\gamma)$-graph, $D \cup (\cup_{i=0}^{m} S_m)$ is a packing in G and the result follows. $\square$

By using Proposition 3.6 we can construct a procedure which concludes $\gamma_t(G) = 2\gamma(G)$, $\gamma_t(G) \neq 2\gamma(G)$ or no comment, see Algorithm 1.

```
Data: A graph G
Result: γt(G) = 2γ(G), γt(G) ≠ 2γ(G) or "test fails"
Let A be an S(G)-set;
if A is an efficient dominating set of G then
|   γt(G) = 2γ(G)
end
else
    initialization;
    H = G;
    T = ∅;
    while H ≠ ∅ do
        Let S_H be the set of all special vertices of H;
        if S_H = ∅ then
        |   Test fails
        end
        if T ∩ N_G(s) ≠ ∅ for some s ∈ S_H then
        |   γt(G) ≠ 2γ(G)
        end
        Let X_H be an S(H)-set;
        if X_H is not a packing in H then
        |   γt(G) ≠ 2γ(G)
        else
            H ← H − N_G[X_H];
            T ← T ∪ N_G[X_H];
        end
    end
    if H = ∅ then
    |   Test fails
    end
end
```

Algorithm 1: A test that might detect $(\gamma_t, 2\gamma)$-graphs and non-$(\gamma_t, 2\gamma)$-graphs.

4. Conclusion and discussion

In this paper, some new properties of $(\gamma_t, 2\gamma)$-graphs and based on these results we provide an algorithm to designate whether a given graph G satisfies $\gamma_t(G) = 2\gamma(G)$ or not. Pseudo code of the procedure is given in Algorithm 1. It is easy to see that in Algorithm 1 the number of operations is bounded above by a polynomial in terms of the number of vertices in the given graph. Therefore, Algorithm 1 has a polynomial time complexity.

Moreover, Algorithm 1 works for some graphs which belong to none of the classifications in previous papers on $(\gamma_t, 2\gamma)$-graphs. As an example, consider the graph G shown in Figure 5. Since G is not a cubic graph, [3] has no conclusion on G. In addition, G has both induced C_6 and H_1 and hence, [1] provides no result for G. However, Algorithm 1 implies that G is not a $(\gamma_t, 2\gamma)$-graph. On the other hand, there are some graphs, such as C_6, whose output is "test fails" in Algorithm 1 although it is a $(\gamma_t, 2\gamma)$-graph. Improving the existing algorithm to make it work for a larger family of graphs and investigating more properties of $(\gamma_t, 2\gamma)$-graphs are topics of ongoing research.

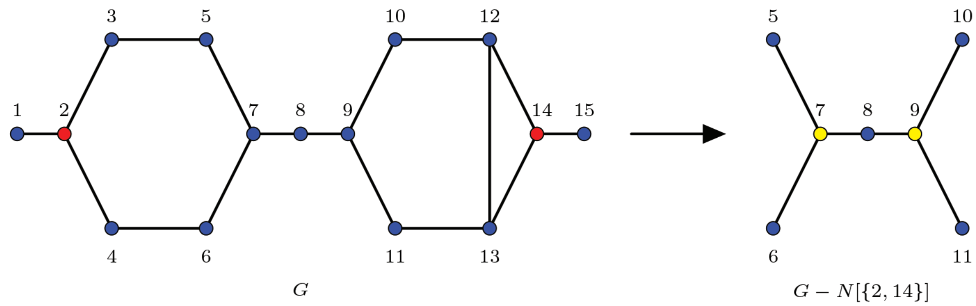

Figure 5. In graph G, special vertices are 2 and 14, and hence $\{2, 14\}$ is the only $S(G)$-set. Special vertices of $G - N[\{2, 14\}]$ are 7 and 9, but they share a common neighbor. Therefore, G is not a $(\gamma_t, 2\gamma)$-graph by Algorithm 1. Indeed, $\gamma(G) = 4$ ($\{2, 7, 9, 14\}$ is a γ-set) and $\gamma_t(G) = 7$ ($\{2, 3, 7, 8, 9, 13, 14\}$ is a γ_t-set).

Acknowledgments

This work is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under grant no. 118E799.

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