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Benedicks and Donoho-Stark type theorems

Lotfi KAMOUN[1,2,*], **Raoudha LAFFI**[2]
[1]Department of Mathematics, Faculty of Science, University of Monastir, Monastir, Tunisia
[2]Laboratory LR11ES11, Department of Mathematics, Faculty of Sciences of Tunis,
University of Tunis El Manar, Tunis, Tunisia

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Abstract: In this paper, we prove a Benedicks type theorem and a Donoho-Stark type theorem, for the generalized Fourier transform $\mathcal{F}_\alpha$ associated to some differential operators that we call Flensted-Jensen operators, in various spaces such $\mathrm{L}^1_\alpha(\mathbb{K})$, $\mathrm{L}^2_\alpha(\mathbb{K})$ and $\mathrm{L}^1_\alpha(\mathbb{K}) \cap \mathrm{L}^2_\alpha(\mathbb{K})$, where $\mathbb{K} = \mathbb{R}_+ \times \mathbb{R}$.

Key words: Generalized Fourier transform, Benedicks theorem, Donoho-Stark theorem, uncertainty principle

1. Introduction

The uncertainty principle is a characterization of a quantum mechanical system. This principle says that one cannot measure, simultaneously and as accurately as one wants, the position and momentum of a quantum particle. In harmonic analysis, the uncertainty principle can be summarized by the following sentence:

a nonzero function and its Fourier transform cannot be localized as precisely as one wishes.

We can distinguish two formulations of this principle, quantitative and qualitative. In 1927, W. Heisenberg [13] gave a physical interpretation of the quantitative uncertainty principle that he wrote in the form of the following formula called Heisenberg inequality:

$$\forall f \in \mathrm{L}^2(\mathbb{R}), \quad \int_{\mathbb{R}} x^2 |f(x)|^2 dx \ . \ \int_{\mathbb{R}} y^2 |\widehat{f}(y)|^2 dy \geq \frac{1}{4} \left(\int_{\mathbb{R}} |f(x)|^2 dx \right)^2,$$

where $\widehat{f}$ is the Fourier transform of f, defined for all $f \in \mathrm{L}^2(\mathbb{R}) \cap \mathrm{L}^1(\mathbb{R})$, by

$$\widehat{f}(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(x) e^{-ixy} dx.$$

Equality cases are realized only by Gaussians of the form

$$f(x) = Ce^{-ax^2}, \qquad x \in \mathbb{R},$$

where C and a are constants with $a > 0$.

*Correspondence: kamoun.lotfi@planet.tn
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By a qualitative uncertainty principle one means a result that, without giving quantitative estimates for a function f and its Fourier transform $\widehat{f}$, says that f and $\widehat{f}$ cannot both be sharply localized unless $f = 0$. Several authors have published works in the context of the qualitative uncertainty principle. We can cite for example, [1–7, 12, 14–16]. For further references about uncertainty principle, we refer the reader to the book [11] and the survey [10].

In [7], Donoho and Stark studied a new version of qualitative uncertainty principle. This uncertainty principle relies on the notion of $\varepsilon-$concentrated, where a function f belongs to $\mathrm{L}^2(\mathbb{R})$ called ε-concentrated on a measurable set E if

$$\|f - f_{|E}\|_2 \leq \varepsilon \|f\|_2.$$

Both others in [7] established that if f is ε_1-concetrated on E and $\widehat{f}$ is ε_2-concentrated on F, then

$$m(E)m(F) \geq (1 - \varepsilon_1 - \varepsilon_2)^2$$

The aim of this paper is to establish Benedicks and Donoho-Stark type theorems associated to the following operators, that we call Flensted-Jensen operators,

$$\begin{cases} D &= \dfrac{\partial}{\partial \theta}, \\ D_\alpha &= \dfrac{\partial^2}{\partial y^2} + [(2\alpha+1)\coth y + \tanh y]\dfrac{\partial}{\partial y} - \dfrac{1}{\cosh^2 y}\dfrac{\partial^2}{\partial \theta^2} + (\alpha+1)^2, \end{cases}$$

where $\alpha > 0$ and $(y, \theta) \in \mathbb{K} = [0, +\infty[\times\mathbb{R}$.
This system was first considered by Flensted-Jensen in [9] for $\alpha = n-1$, where n is a positive integer, in the frame work of simply connected semisimple Lie group. The operators D and $[D_{n-1} - n^2]$ with the identity generate the algebra $\mathbf{D}(\widetilde{G}/K)$ of left invariant differential operators on $\widetilde{G}/K$, where $\widetilde{G}$ is the universal covering group of $G = \mathbf{U}(n,1)$ and K is the subgroup $\mathbf{U}(n)$. A several works on the theory of uncertainty principle, related to the operators D and D_α were studiedin [15–17].

The outline of this paper is given as follows:
Section 2 is devoted to recall some results concerning the harmonic analysis associated to the operators D and D_α. In section 3, we prove a Benedicks type theorem. In the last section we obtain a various versions of Donoho-Stark theorem.

2. Preliminaries

For $(y, \theta) \in \mathbb{K}$, the following system

$$\begin{cases} Du(y,\theta) &= i\lambda u(y,\theta), \\ D_\alpha u(y,\theta) &= -\mu^2 u(y,\theta), \qquad \lambda, \mu \in \mathbb{R} \\ u(0,0) &= 1, \quad \dfrac{\partial u}{\partial y}(0,\theta) = 0, \quad \theta \in \mathbb{R} \end{cases}$$

has a unique solution given by

$$\varphi_{\lambda,\mu}(y,\theta) = e^{i\lambda\theta}(\cosh y)^\lambda\, \varphi_\mu^{\alpha,\lambda}(y),$$

where $\varphi_\mu^{\alpha,\lambda}$ is the Jacobi function defined by

$$\varphi_\mu^{\alpha,\lambda}(y) = {}_2F_1\left(\frac{\alpha+\lambda+1+i\mu}{2}, \frac{\alpha+\lambda+1-i\mu}{2}; \alpha+1; -\sinh^2 y\right).$$

Recall that ${}_2F_1$ is the Gaussian hypergeometric function (see [8]).
From [18], we have

$$\sup_{(y,\theta)\in\mathbb{K}} |\varphi_{\lambda,\mu}(y,\theta)| = 1, \quad (\lambda,\mu)\in\widehat{\mathbb{K}} \tag{2.1}$$

Let $\mathbb{L} = \mathbb{R}\times[0,+\infty[$ and $\widehat{\mathbb{K}} = \mathbb{L}\cup\Omega$, where

$$\Omega = \bigcup_{m\in\mathbb{N}} D_m^+ \cup D_m^-.$$

with

$$D_m^+ = \{(\alpha+2m+1+\eta, i\eta)\,|\,\eta>0\} \qquad \text{and} \qquad D_m^- = \{(-\alpha-2m-1-\eta, i\eta)\,|\,\eta>0\}.$$

Let $1\le p<+\infty$. Consider $\mathrm{L}_\alpha^p(\mathbb{K})$, the space of measurable functions f on $\mathbb{K}$ verifying

$$\|f\|_{p,m_\alpha} = \left(\int_{\mathbb{K}} |f(y,\theta)|^p \, dm_\alpha(y,\theta)\right)^{\frac{1}{p}} < +\infty,$$

where

$$dm_\alpha(y,\theta) = 2^{2(\alpha+1)}(\sinh y)^{2\alpha+1}\cosh y \, dy \, d\theta.$$

For $p=\infty$, we put

$$\|f\|_{\infty,m_\alpha} = \mathbf{ess} \sup_{(y,\theta)\in\mathbb{K}} |f(y,\theta)|.$$

The generalized Fourier transform of f associated to Flensted-Jensen operators is given by

$$\forall\,(\lambda,\mu)\in\widehat{\mathbb{K}}, \qquad \mathcal{F}_\alpha f(\lambda,\mu) = \int_{\mathbb{K}} f(y,\theta)\varphi_{-\lambda,\mu}(y,\theta), dm_\alpha(y,\theta).$$

where $f\in\mathrm{L}_\alpha^1(\mathbb{K})$.

Denote γ_α the positive measure, defined on $\widehat{\mathbb{K}}$ by

$$\begin{aligned}\int_{\widehat{\mathbb{K}}} g\,d\gamma_\alpha(\lambda,\mu) &= \frac{1}{(2\pi)^2}\int_{\mathbb{R}\times[0,+\infty[} g(\lambda,\mu)\frac{d\lambda d\mu}{|C_1(\lambda,\mu)|^2} \\ &+\frac{1}{(2\pi)^2}\sum_{m=0}^{+\infty}\left\{\int_0^{+\infty} g(\alpha+2m+1+\eta,i\eta)C_2(\alpha+2m+1+\eta,i\eta)d\eta\right. \\ &\left.\qquad+\int_0^{+\infty} g(-\alpha-2m-1-\eta,i\eta)C_2(-\alpha-2m-1-\eta,i\eta)d\eta\right\},\end{aligned}$$

where

$$C_1(\lambda,\mu) = \frac{2^{\alpha+1-i\mu}\Gamma(\alpha+1)\Gamma(i\mu)}{\Gamma\left(\frac{\alpha+\lambda+1+i\mu}{2}\right)\Gamma\left(\frac{\alpha-\lambda+1+i\mu}{2}\right)}, \quad (\lambda,\mu)\in\mathbb{R}\times]0,+\infty[$$

and

$$C_2(\lambda,\mu) = -i \operatorname*{Res}_{z=\mu}\Big[C_1(\lambda,z)C_1(\lambda,-z)\Big]^{-1}, \quad (\lambda,\mu)\in\Omega.$$

We have from [18] the following inversion formula

$$\mathcal{F}_\alpha^{-1} g(y,\theta) = \int_{\widehat{\mathbb{K}}} g(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta)\, d\gamma_\alpha(\lambda,\mu), \tag{2.2}$$

For $1 \le p < +\infty$, denote $\mathrm{L}^p_\alpha(\widehat{\mathbb{K}})$ the space of measurable functions $g:\widehat{\mathbb{K}} \longmapsto \mathbb{C}$ verifying

$$\|g\|_{p,\gamma_\alpha} = \left(\int_{\widehat{\mathbb{K}}} |g(\lambda,\mu)|^p\, d\gamma_\alpha(\lambda,\mu)\right)^{\frac{1}{p}} < +\infty.$$

For $p=\infty$, we denote

$$\|g\|_{\infty,\gamma_\alpha} = \mathbf{ess}\sup_{(\lambda,\mu)\in\widehat{\mathbb{K}}} |g(\lambda,\mu)|$$

The generalized Fourier transform $\mathcal{F}_\alpha$ extended to an isometry between $\mathrm{L}^2_\alpha(\mathbb{K})$ and $\mathrm{L}^2_\alpha(\widehat{\mathbb{K}})$. In particular, for $f\in \mathrm{L}^2_\alpha(\widehat{\mathbb{K}})$, we have the Plancherel formula

$$\|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha} = \|f\|_{2,m_\alpha}. \tag{2.3}$$

For $f\in \mathrm{L}^1_\alpha(\mathbb{K})$, we have

$$\|\mathcal{F}_\alpha f\|_{\infty,\gamma_\alpha} \le \|f\|_{1,m_\alpha}. \tag{2.4}$$

In the following sections, we consider $E\subset\mathbb{K}$ and $F\subset\widehat{\mathbb{K}}$ tow measurable subsets. For a function $f\in \mathrm{L}^2_\alpha(\mathbb{K})$, we denote by

T_E the time-limiting operator

$$T_E f = \chi_E\, f,$$

P_F the frequency-limiting operator

$$\mathcal{F}_\alpha(P_F\, f) = \chi_F\, \mathcal{F}_\alpha(f),$$

where χ_A is the characteristic function of the set A.
If $0<\gamma_\alpha(F)<\infty$, then for $f\in \mathrm{L}^2_\alpha(\mathbb{K})$ we have

$$P_F f(y,\theta) = \int_F \mathcal{F}_\alpha f(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta) d\gamma_\alpha(\lambda,\mu). \tag{2.5}$$

The operators P_F is bounded from $\mathrm{L}^2_\alpha(\mathbb{K})$ into itself and

$$\|P_F f\|_{2,m_\alpha} \le \|f\|_{2,m_\alpha}. \tag{2.6}$$

3. Benedicks type theorem

In order to prove the main theorem of this section, we start by proving the following lemmas.

Lemma 3.1 *If* $0 < m_\alpha(E) < \infty$ *and* $\gamma_\alpha(F) < \infty$, *then the Hilbert-Schmidt norm of* $P_F T_E$ *is finite and we have*

$$\|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}.$$

Proof Let $f \in \mathrm{L}^2_\alpha(\mathbb{K})$, from relation (2.5)

$$\begin{aligned} P_F T_E f(y,\theta) &= \int_F \mathcal{F}_\alpha T_E f(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu) \\ &= \int_{\widehat{\mathbb{K}}} \chi_F(\lambda,\mu)\left\{\int_{\mathbb{K}} \chi_E(s,t)f(s,t)\varphi_{-\lambda,\mu}(s,t)dm_\alpha(s,t)\right\}\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu)\,. \end{aligned}$$

Denote

$$g_{s,t}(\lambda,\mu) = \chi_F(\lambda,\mu)\varphi_{-\lambda,\mu}(s,t)$$

and

$$\mathcal{N}(s,t,y,\theta) = \chi_E(s,t)\mathcal{F}_\alpha^{-1}(g_{s,t})(y,\theta).$$

Using Fubini's theorem, we obtain

$$P_F T_E f(y,\theta) = \int_{\mathbb{K}} f(s,t)\mathcal{N}(s,t,y,\theta)dm_\alpha(s,t).$$

$\mathcal{N}$ is called the kernel of integral operator $P_F T_E$ and the Hilbert-Schmidt norm of this operator is given by

$$\|P_F T_E\|_{HS} = \|\mathcal{N}\|_{\mathrm{L}^2_\alpha(\mathbb{K})\otimes\mathrm{L}^2_\alpha(\mathbb{K})}.$$

Therefore,

$$\|\mathcal{N}\|_{\mathrm{L}^2_\alpha(\mathbb{K})\otimes\mathrm{L}^2_\alpha(\mathbb{K})} = \left(\int_{\mathbb{K}} |\chi_E(s,t)|^2 \left(\int_{\mathbb{K}} |\mathcal{F}_\alpha^{-1}(g_{s,t})(y,\theta)|^2 dm_\alpha(y,\theta)\right) dm_\alpha(s,t))\right)^{\frac{1}{2}}.$$

By applying Plancherel formula (2.3), we get

$$\|\mathcal{N}\|_{\mathrm{L}^2_\alpha(\mathbb{K})\otimes\mathrm{L}^2_\alpha(\mathbb{K})} = \left(\int_{\mathbb{K}} \chi_E(s,t)\left(\int_{\mathbb{K}} \chi_F(\lambda,\mu)|\varphi_{-\lambda,\mu}(s,t)|^2 d\gamma_\alpha(\lambda,\mu)\right) dm_\alpha(s,t)\right)^{\frac{1}{2}}.$$

We deduce from relation (2.1) that

$$\|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}.$$

□

Lemma 3.2 *Let* $f \in \mathrm{L}^2_\alpha(\mathbb{K})$. *Then*

$$(1 - \|P_F T_E\|)\|f\|_{2,m_\alpha} \leq \left(\|T_{E^c} f\|^2_{2,m_\alpha} + \|P_{F^c} f\|^2_{2,m_\alpha}\right)^{\frac{1}{2}}$$

Proof Let I be the identity operator, we have

$$I = P_F T_E + P_F T_{E^c} + P_{F^c}.$$

For $f \in \mathrm{L}^2_\alpha(\mathbb{K})$, we get

$$\begin{aligned} \|f - P_F T_E f\|^2_{2,m_\alpha} &= \|P_F T_{E^c} f + P_{F^c} f\|^2_{2,m_\alpha} \\ &= \|P_F T_{E^c} f\|^2_{2,m_\alpha} + \|P_{F^c} f\|^2_{2,m_\alpha} \end{aligned}$$

It follows by using (2.6) that

$$\|f - P_F T_E f\|^2_{2,m_\alpha} \leq \|T_{E^c} f\|^2_{2,m_\alpha} + \|P_{F^c} f\|^2_{2,m_\alpha}. \tag{3.1}$$

On the other hand, we have

$$\|f - P_F T_E f\|_{2,m_\alpha} \geq \|f\|_{2,m_\alpha} - \|P_F T_E f\|_{2,m_\alpha}.$$

Since

$$\|P_F T_E f\| \leq \|P_F T_E\| \ \|f\|_{2,m_\alpha},$$

therefore

$$\|f - P_F T_E f\|_{2,m_\alpha} \geq (1 - \|P_F T_E\|)\|f\|_{2,m_\alpha}. \tag{3.2}$$

Combining relations (3.1) and (3.2) we obtain the wanted result.

□

Theorem 3.3 *Let* $f \in \mathrm{L}^2_\alpha(\mathbb{K})$. *If* $supp(f) \subset E$, $supp(\mathcal{F}_\alpha f) \subset F$ *and* $0 < m_\alpha(E)\gamma_\alpha(F) < 1$ *then* $f = 0$.

Proof Let $f \in \mathrm{L}^2_\alpha(\mathbb{K})$. from lemma 3.1, we obtain

$$\|P_F T_E\| \leq \|P_F T_E\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)} < 1.$$

Applying lemma 3.2, we get

$$\begin{aligned} \|f\|^2_{2,m_\alpha} &\leq (1 - \|P_F T_E\|)^{-2} \left(\|T_{E^c} f\|^2_{2,m_\alpha} + \|P_{F^c} f\|^2_{2,m_\alpha}\right) \\ &\leq \left(1 - \sqrt{m_\alpha(E)\gamma_\alpha(F)}\right)^{-2} \left(\|T_{E^c} f\|^2_{2,m_\alpha} + \|P_{F^c} f\|^2_{2,m_\alpha}\right). \end{aligned}$$

Hence $suppf \subset E$ and $supp\mathcal{F}_\alpha f \subset F$ then

$$T_{E^c} f = 0 \quad \text{and} \quad P_{F^c} f = 0.$$

Therefore $f = 0$.

□

4. Donoho-Stark uncertainty principle

4.1. L^2 version of Donoho-Stark theorem

We start by giving the definition of ε-concentrated functions.

Definition 4.1 *Let $f \in \mathrm{L}^2_\alpha(\mathbb{K})$, E and F be measurable subsets, respectively, of $\mathbb{K}$ and $\widehat{\mathbb{K}}$. We call*

1. *f is an ε_E-concentrated on E if there exists a vanishing function g on $\mathbb{K} \setminus E$, such that*

$$\|f - g\|_{2,m_\alpha} \leq \varepsilon_E \, \|f\|_{2,m_\alpha}.$$

2. *$\mathcal{F}_\alpha(f)$ is an ε_F-concentrated on F if there exists a vanishing function h on $\widehat{\mathbb{K}} \setminus F$, such that*

$$\|\mathcal{F}_\alpha(f) - h\|_{2,\gamma_\alpha} \leq \varepsilon_F \, \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}.$$

Lemma 4.2 *Let $f \in \mathrm{L}^2_\alpha(\mathbb{K})$, E and F be measurable subsets, respectively, of $\mathbb{K}$ and $\widehat{\mathbb{K}}$. We have*

1. *f is ε_E-concentrated on E if and only if*

$$\|f - T_E f\|_{2,m_\alpha} \leq \varepsilon_E \|f\|_{2,m_\alpha}. \tag{4.1}$$

2. *$\mathcal{F}_\alpha f$ is ε_F-concentrated on F if and only if*

$$\|f - P_F f\|_{2,m_\alpha} \leq \varepsilon_F \|f\|_{2,m_\alpha}. \tag{4.2}$$

Proof

1. Let f be a ε_E-concentrated on E. There exits a vanishing function g on E^c, such that

$$\|f - g\|_{2,m_\alpha} \leq \varepsilon_E \, \|f\|_{2,m_\alpha}. \tag{4.3}$$

On the other hand, we have

$$f(y,\theta) - T_E f = \chi_{E^c} f.$$

Then

$$\begin{aligned}
\|f - T_E f\|^2_{2,m_\alpha} &= \int_{\mathbb{K}} |f(y,\theta) - T_E f(y,\theta)|^2 dm_\alpha(y,\theta) \\
&= \int_{E^c} |f(y,\theta) - g(y,\theta)|^2 dm_\alpha(y,\theta) \\
&\leq \|f - g\|^2_{2,m_\alpha}.
\end{aligned}$$

Then from relation (4.3), we get

$$\|f - T_E f\|_{2,m_\alpha} \leq \varepsilon_E \|f\|_{2,m_\alpha}.$$

2. Let $\mathcal{F}_\alpha f$ be a $\varepsilon_{_F}$-concentrated to F, then there exists a vanishing function h on F^c, such that
$$\|\mathcal{F}_\alpha(f) - h\|_{2,\gamma_\alpha} \leq \varepsilon_{_F}\, \|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}. \tag{4.4}$$
Moreover
$$\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_{_F} f) = \mathcal{F}_\alpha f - \chi_{_F}\mathcal{F}_\alpha f = \chi_{_{F^c}}\mathcal{F}_\alpha f.$$
Then
$$\begin{aligned}\|\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_{_F} f)\|^2_{2,\gamma_\alpha} &= \int_{\widehat{\mathbb{K}}} |\mathcal{F}_\alpha f(\lambda,\mu) - \mathcal{F}_\alpha(P_F f)(\lambda,\mu)|^2 d\gamma_\alpha(\lambda,\mu)\\ &= \int_{F^c} |\mathcal{F}_\alpha f(\lambda,\mu) - h(\lambda,\mu)|^2 d\gamma_\alpha(\lambda,\mu)\\ &\leq \|\mathcal{F}_\alpha f - h\|^2_{2,\gamma_\alpha}.\end{aligned}$$
By relation (4.4), we obtain the following result
$$\|\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_{_F} f)\|_{2,\gamma_\alpha} \leq \varepsilon_{_F}\|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha}.$$
Applying Plancherel's formula (2.3) on both terms of the above inequality we get
$$\|f - P_{_F} f\|_{2,m_\alpha} \leq \varepsilon_{_F}\|f\|_{2,m_\alpha}.$$

□

Lemma 4.3 *For $f \in \mathrm{L}^2_\alpha(\mathbb{K})$ we have*
$$\|P_{_F} T_{_E} f\|_{2,m_\alpha} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}\ \|f\|_{2,m_\alpha}.$$

Proof Assume that $m_\alpha(E)$ and $\gamma_\alpha(F)$ are finite. Applying Lemma 3.1 we get
$$\|P_{_F} T_{_E}\|_{HS} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}$$
considering
$$\|P_{_F} T_{_E}\| = \sup_{f\in \mathrm{L}^2_\alpha(\mathbb{K})\setminus\{0\}} \frac{\|P_{_F} T_E f\|_{2,m_\alpha}}{\|f\|_{2,m_\alpha}} \leq \|P_{_F} T_{_E}\|_{HS}$$
then for $f \in \mathrm{L}^2_\alpha(\mathbb{K}) \setminus \{0\}$ we have
$$\frac{\|P_{_F} T_{_E} f\|_{2,m_\alpha}}{\|f\|_{2,m_\alpha}} \leq \sqrt{m_\alpha(E)\gamma_\alpha(F)}$$
which allows us to deduce the wanted result.

□

Theorem 4.4 *Consider a nonzero function $f \in \mathrm{L}^2_\alpha(\mathbb{K})$. If f is an $\varepsilon_{_E}-$concentrated on E, $\mathcal{F}_\alpha f$ is an $\varepsilon_{_F}-$concentrated on F and $\varepsilon_{_E} + \varepsilon_{_F} < 1$, then*
$$\sqrt{m_\alpha(E)\gamma_\alpha(F)} \geq 1 - \varepsilon_{_E} - \varepsilon_{_F}.$$

Proof Let $f \in \mathrm{L}^2_\alpha(\mathbb{K}) \setminus \{0\}$, we have

$$\|f - P_{_F} T_{_E} f\|_{2,m_\alpha} \leq \|f - P_{_F} f\|_{2,m_\alpha} + \|P_{_F} f - P_{_F} T_{_E} f\|_{2,m_\alpha}.$$

From relations(4.2), (2.6) and (4.1), we obtain

$$\begin{aligned}\|f - P_{_F} T_{_E} f\|_{2,m_\alpha} &\leq \varepsilon_{_F} \|f\|_{2,m_\alpha} + \|f - T_{_E} f\|_{2,m_\alpha} \\ &\leq (\varepsilon_{_E} + \varepsilon_{_F})\|f\|_{2,m_\alpha}.\end{aligned}$$

which allows us to get the following inequality

$$\begin{aligned}\|P_{_F} T_{_E} f\|_{2,m_\alpha} &\geq \|f\|_{2,m_\alpha} - \|f - P_{_F} T_{_E} f\|_{2,m_\alpha} \\ &\geq (1 - \varepsilon_{_E} - \varepsilon_F)\|f\|_{2,m_\alpha}.\end{aligned}$$

Applying lemma 4.3 we conclude that

$$\sqrt{m_\alpha(E)\gamma_\alpha(F)} \geq (1 - \varepsilon_E - \varepsilon_F).$$

□

4.2. L^1 version of Donoho-Stark theorem

In this section, we study the case of a function $f \in \mathrm{L}^1_\alpha(\mathbb{K})$.

The operator $T_{_E}$ verifies the following inequality on $\mathrm{L}^1_\alpha(\mathbb{K})$.

$$\|T_{_E} f\|_{1,m_\alpha} \leq \|f\|_{1,m_\alpha} \tag{4.5}$$

We say that f is an $\varepsilon_{_E}-$concentrated on E in $\mathrm{L}^1_\alpha(\mathbb{K})$ if

$$\|f - T_{_E} f\|_{1,m_\alpha} \leq \varepsilon_{_E} \|f\|_{1,m_\alpha}.$$

We denote by $\mathrm{B}^1_\alpha(F)$ the following subset

$$\mathrm{B}^\alpha_1(F) = \left\{g \in \mathrm{L}^1_\alpha(\mathbb{K}) | \ P_{_F} g = g\right\}.$$

We say that f is an $\varepsilon_{_F}-$bandlimited on F if there is a function $g \in \mathrm{B}^\alpha_1(F)$ such that

$$\|f - g\|_{1,m_\alpha} \leq \varepsilon_{_F} \|f\|_{1,m_\alpha}.$$

We begin with the following lemma in order to prove the Donoho-Stark type theorem on $\mathrm{L}^1_\alpha(\mathbb{K})$.

Lemma 4.5 *Consider a nonzero function $f \in \mathrm{B}^\alpha_1(F)$, we have*

$$\frac{\|T_{_E} f\|_{1,m_\alpha}}{\|f\|_{1,m_\alpha}} \leq m_\alpha(E)\gamma_\alpha(F).$$

Proof Let $f \in \mathrm{B}_1^{\alpha}(F) \setminus \{0\}$, according to relation (2.5) we get

$$f(y,\theta) = \int_{\widehat{\mathbb{K}}} \chi_F(\lambda,\mu)\mathcal{F}_\alpha f(\lambda,\mu)\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu).$$

Therefore by Fubini's theorem, we obtain

$$f(y,\theta) = \int_{\mathbb{K}} f(s,t)\left(\int_F \varphi_{-\lambda,\mu}(s,t)\varphi_{\lambda,\mu}(y,\theta)d\gamma_\alpha(\lambda,\mu)\right)dm_\alpha(s,t).$$

From relation(2.1), we get

$$\|f\|_{\infty,m_\alpha} \leq \gamma_\alpha(F)\|f\|_{1,m_\alpha}. \tag{4.6}$$

Furthermore,

$$\|T_E f\|_{1,m_\alpha} = \int_{\mathbb{K}} \chi_E(y,\theta)|f(y,\theta)|dm_\alpha(y,\theta) \leq m_\alpha(E)\|f\|_{\infty,m_\alpha}$$

by using the relation(4.6), we get

$$\|T_E f\|_{1,m_\alpha} \leq m_\alpha(E)\gamma_\alpha(F)\|f\|_{1,m_\alpha}.$$

Then, we gain the wanted result.

□

Theorem 4.6 *Consider a nonzero function* $f \in \mathrm{L}^1_\alpha(\mathbb{K})$ *and* ε_E, ε_F *two real numbers such that* $\varepsilon_E + \varepsilon_F < 1$. *If* f *is* ε_E-*concentrated on* E *and* ε_F-*bandlimited on* F *in* $\mathrm{L}^1_\alpha(\mathbb{K})$ *then*

$$m_\alpha(E)\gamma_\alpha(F) \geq \frac{1-\varepsilon_E-\varepsilon_F}{1+\varepsilon_F}.$$

Proof We consider $f \in \mathrm{L}^1_\alpha(\mathbb{K}) \setminus \{0\}$, we have

$$\|T_E f\|_{1,m_\alpha} = \|f + T_E f - f\|_{1,m_\alpha}.$$

By applying the triangular inequality, we obtain

$$\|T_E f\|_{1,m_\alpha} \geq \|f\|_{1,m_\alpha} - \|f - T_E f\|_{1,m_\alpha}.$$

Since f is ε_E-concentrated on E, then

$$\|T_E f\|_{1,m_\alpha} \geq (1-\varepsilon_E)\|f\|_{1,m_\alpha}. \tag{4.7}$$

On the other hand, f is ε_F-bandlimited so there exists a function $g \in \mathrm{B}^1_\alpha(F)$ such that

$$\|f-g\|_{1,m_\alpha} \leq \varepsilon_F\|f\|_{1,m_\alpha}. \tag{4.8}$$

Furthermore, from relation (4.5) we get

$$\|T_E g\|_{1,m_\alpha} \geq \|T_E f\|_{1,m_\alpha} - \|T_E f - T_E g\|_{1,m_\alpha} \geq \|T_E f\|_{1,m_\alpha} - \|f-g\|_{1,m_\alpha}.$$

Using both relations (4.7) and (4.8), we get

$$\|T_{E}g\|_{1,m_\alpha} \geq (1-\varepsilon_{E}-\varepsilon_{F})\|f\|_{1,m_\alpha}.$$

On the other hand, we have

$$\|g\|_{1,m_\alpha} \leq (1+\varepsilon_{F})\|f\|_{1,m_\alpha}.$$

Therefore,

$$\frac{\|T_{F}g\|_{1,m_\alpha}}{\|g\|_{1,m_\alpha}} \geq \frac{1-\varepsilon_{E}-\varepsilon_{F}}{1+\varepsilon_{F}}.$$

Then, by lemma 4.5 we obtain the wanted result.

□

In the sequel, we give an $L^1_\alpha \cap L^2_\alpha$ version of Donoho-Stark theorem for the generalized Fourier transform $\mathcal{F}_\alpha$.

Theorem 4.7 *Consider a nonzero function* $f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})$. *If* f *is* $\varepsilon_{E}-$*concentrated on* E *in* $L^1_\alpha(\mathbb{K})$ *and* $\mathcal{F}_\alpha f$ *is* $\varepsilon_{F}-$*cocentrated on* F *in* $L^2_\alpha(\mathbb{K})$ *then*

$$m_\alpha(E)\gamma_\alpha(F) \geq (1-\varepsilon_{E})^2(1-\varepsilon_{F})^2.$$

Proof Assume that $m_\alpha(E)$ and $\gamma_\alpha(F)$ are finite. For a nonzero function $f \in L^1_\alpha(\mathbb{K}) \cap L^2_\alpha(\mathbb{K})$, we have

$$\|f\|_{2,m_\alpha} \leq \|f-P_{F}f\|_{2,m_\alpha} + \|P_{F}f\|_{2,m_\alpha}.$$

Plancherel's formula (2.3) gives us the following inequality

$$\|f\|_{2,m_\alpha} \leq \|\mathcal{F}_\alpha f - \mathcal{F}_\alpha(P_F f)\|_{2,\gamma_\alpha} + \|\chi_{F}\mathcal{F}_\alpha(f)\|_{2,\gamma_\alpha}.$$

Since $\mathcal{F}_\alpha f$ is $\varepsilon_{F}-$cocentrated on F in $L^2_\alpha(\mathbb{K})$, we obtain by using relation(4.2)

$$\begin{aligned}
\|f\|_{2,m_\alpha} &\leq \varepsilon_{F}\|\mathcal{F}_\alpha f\|_{2,\gamma_\alpha} + \left(\int_F |\mathcal{F}_\alpha f(\lambda,\mu)|^2 d\gamma_\alpha(\lambda,\mu)\right)^{\frac{1}{2}} \\
&\leq \varepsilon_{F}\|f\|_{2,m_\alpha} + \sqrt{\gamma_\alpha(F)}\|\mathcal{F}_\alpha f\|_{\infty,\gamma_\alpha}.
\end{aligned}$$

Furthermore from relation (2.4), we obtain

$$(1-\varepsilon_F)\|f\|_{2,m_\alpha} \leq \sqrt{\gamma_\alpha(F)}\|f\|_{1,m_\alpha}. \tag{4.9}$$

On the other hand, we have

$$\|f\|_{1,m_\alpha} \leq \|f-T_{E}f\|_{1,m_\alpha} + \|T_{E}f\|_{1,m_\alpha}.$$

Seeing that f is $\varepsilon_{E}-$concentrated on E in $L^1_\alpha(\mathbb{K})$, we conclude from relation (4.1) that

$$\begin{aligned}
\|f\|_{1,m_\alpha} &\leq \varepsilon_{E}\|f\|_{1,m\alpha} + \int_E |f(y,\theta)| dm_\alpha(y,\theta) \\
&\leq \varepsilon_{E}\|f\|_{1,m_\alpha} + \sqrt{m_\alpha(E)}\|f\|_{2,m_\alpha}.
\end{aligned}$$

Therefore,

$$(1-\varepsilon_{E})\|f\|_{1,m_\alpha} \leq \sqrt{m_\alpha(E)}\|f\|_{2,m_\alpha}. \tag{4.10}$$

Combining (4.9) and (4.10) we reach the needed result.

□

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