Some characterizations of KB-operators on Banach lattices and ordered Banach spaces

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Abstract: We determine that two recent classes of KB-operators and weak KB-operators and the well-known class of b-weakly compact operators, from a Banach lattice into a Banach space, are the same. We extend our study to the ordered Banach space setting by showing that a weak chain-preserving operator between two ordered Banach spaces is a KB-operator if and only if it is a weak KB-operator.

Key words: KB-operator, b-weakly compact operator, Banach lattice, ordered Banach space, normal cone

1. Introduction

In this paper, an operator version of the property of being a KB-space for Banach lattices (i.e. positive increasing and norm bounded sequences are convergent) is studied. The operators that we will deal with are those from a Banach lattice E into a Banach space X that map positive increasing bounded sequences to ones that have convergent subsequences. These operators were recently introduced in [5] under the name of KB-operators. Moreover, the authors in the same paper also introduced the class of weak KB-operators by replacing the convergence of subsequences appearing in the definition of KB-operators with weak convergence. They showed that the three classes of KB, weak KB and b-weakly compact operators coincide with positive operators between Banach lattices; see [5, Prop 2.1]. Recall that an operator $T : E \to X$ is said to be b-weakly compact (bw-compact) if it maps any b-order bounded set in E (i.e. order bounded as a subset of the bidual E'') to a weakly compact set in X, equivalently, if it maps any b-order bounded positive disjoint sequence to a norm null one. This class of operators was studied in the Banach lattice literature (see for instance [3, 4]), and it turns out that for a Banach lattice E, being a KB-space is nothing but, the identity operator of E is a bw-compact (resp. a KB-) operator; see [3, Prop 2.10] and [5, Prop 2.11]. At the end of their paper [5], the authors asked two questions: Is there an operator from a Banach lattice into a Banach space which is KB but fails to be bw-compact? Is there an operator from a Banach lattice into a Banach space which is weak KB but fails to be KB? The authors of [11] have recently answered the first question negatively by establishing that the two classes of KB and bw-compact operators from E into X are in fact the same. Here, we continue to investigate all these classes of operators by showing, in Section 2, that the answer of the second question is also negative, and in fact KB, weak KB and bw-compact operators from E into X are the same (Theorem 3.2).

Moreover, since one does not need any lattice structure in the definition of (weak) KB-operators, we extend in the last section of the paper our study by showing in Theorem 4.10 that the two classes of KB and weak KB-operators between ordered Banach spaces (with some reasonable compatibility conditions for the order structure) still coincide with operators preserving chains in some weak sense (in particular, on positive operators). Our extension to the ordered Banach space setting is motivated by the fact that many subspaces of function spaces are of this type and do not inherit the lattice structure (see for instance Examples 2.4 hereafter). However, due to the lack of many powerful tools guaranteed by the lattice structure of Banach lattices, we could not answer the question whether the two classes of KB and weak KB-operators between ordered Banach spaces are the same.

2. Preliminaries

Recall that a subset K of a real vector space E is said to be a wedge if $K + K \subset K$, $\alpha K \subset K$ for all $\alpha \geq 0$. If furthermore $K \cap (-K) = \{0\}$, then K is called a cone, and E endowed with the (partial) order relation

$$x \leq y \Leftrightarrow y - x \in K$$

is called an ordered vector space (OVS). We will denote the cone $K = \{x \in E : x \geq 0\}$ by E^+.

We will use the following fundamental extension lemma for additive mappings between cones.

Lemma 2.1 (Lem 1.26, [2]) *Suppose that E and F are two OVSs with F Archimedean. Then, any additive mapping $T : E^+ \to F^+$ extends uniquely to a positive operator (denoted again by T) from $E^+ - E^+$ to F defined by*

$$Tx = Tx_1 - Tx_2 \ , \ x = x_1 - x_2, \ \ x_1, x_2 \in E^+.$$

Note that the operator T is well defined since the preceding expression does not depend on the representation $x = x_1 - x_2$, $x_1, x_2 \in E^+$.

We say that E^+ is generating in E, if $E = E^+ - E^+$, i.e. for every $x \in E$,

$$x = x_1 - x_2, \text{ for some } x_1, x_2 \in E^+.$$

A vector $e \in E^+$ is said to be an order unit of an OVS E, if for every $x \in E$ there exists some $\lambda > 0$ such that $x \leq \lambda e$. It is clear that for an OVS E, E^+ is generating iff it majorizes E, that is, for every $x \in E$ there exists $y \in E^+$ with $x \leq y$. It follows that the cone of an OVS with order unit is automatically generating.

If the OVS $(E, \leq)$ is a normed space, then E is said to be an ordered normed space (ONS) if the cone E^+ is closed. An ordered Banach space (OBS) is an ONS which is furthermore a Banach space (BS). Let us recall that the cone E^+ of an ONS $(E, \leq, \|.\|)$ is said to be normal, if

$$\exists N > 0, \ \forall x, y \in E, \ 0 \leq x \leq y \Rightarrow \|x\| \leq N \|y\|.$$

The following two theorems give us the characterizations of generating cones and normal cones in OBSs.

Theorem 2.2 (Thm 2.37, [2]) *For an OBS E, E^+ is generating iff there exists some constant $M > 0$ such that for each $x \in E$ there exist $x_1, x_2 \in E^+$ with*

$$x = x_1 - x_2 \ \ and \ \ \|x_i\| \leq M \|x\|, \ \ i = 1, 2.$$

Theorem 2.3 (Thms 2.38 and 2.40, [2]) *1. For an ONS E, E^+ is normal iff E admits an equivalent monotone norm $|||.|||$, i.e. a norm $|||.|||$ satisfying for every $x, y \in E$*

$$0 \le x \le y \Rightarrow |||x||| \le |||y|||.$$

2. For an OBS E, E^+ is normal iff each order interval $[x, y]$ of E, $x \le y$, is norm bounded.

Recall that a vector lattice (or Riesz space) is an OVS E such that the set E is a lattice with respect to the order structure. A normed vector lattice is an ONS $(E, \le, \|.\|)$ such that E^+ is a lattice cone in the sense that $(E, \le)$ is a vector lattice, and the norm $\|.\|$ is a Riesz norm, that is, it is monotone and absolute. An absolute norm $\|.\|$ on a vector lattice E, is a norm such that for every $x \in E$, $\||x|\| = \|x\|$. A Banach lattice (BL) is a normed vector lattice which is complete with respect to its norm.

The following example gives an ONS with a normal and generating cone which fails to be a vector lattice.

Example 2.4 *Let $E = D[0, 1]$ be the real vector space of differentiable functions $f : [0, 1] \to \mathbb{R}$, endowed with the pointwise ordering and the standard supremum norm. It is clear that the cone E^+ is closed, and hence E is an ONS, the supremum norm is monotone (and hence E^+ is normal), and that the cone E^+ is generating for E has an order unit. However, E fails to be a vector lattice; see [6, Exple 5.9 (5), p 44].*

By the same considerations of the above example, the real vector spaces of polynomials on $[0, 1]$, rational functions on $[0, 1]$ (i.e. functions $f : [0, 1] \to \mathbb{R}$ such that $f = \frac{p}{q}$ for some two polynomials p, q on $[0, 1]$ with $q(x) \ne 0$ for all $x \in [0, 1]$) and continuously differentiable functions on $[0, 1]$ are ONSs with a normal and generating cone. All these spaces fail to be a vector lattice; see [2, Exercise 4 p 19 and Exercise 9 p 52].

It is worth noting that for an ONS E, the set of positive linear functionals on E is only a wedge in E' called the dual wedge of E^+. It is easily seen that in case E^+ is generating, the dual wedge of E^+ is a norm closed cone in E' that makes E' an OBS.

The following theorem will also be useful in later.

Theorem 2.5 (Thm 2.40, [2]) *For an OBS E, the following are equivalent:*

1. *E^+ is normal;*

2. *the dual wedge of E^+ is generating in E'.*

Throughout the paper, by an operator between two normed spaces we will mean a continuous linear mapping. The notation c_0 (resp. $C[0, 1]$) will stand for the BL of real null sequences (resp. real-valued continuous functions on $[0, 1]$), where the norm is the usual supremum norm and the ordering is the usual coordinatewise (resp. pointwise) one. Moreover, the space $\mathbb{R}^n$ will be endowed with the coordinatewise ordering.

For more details, we refer the reader to [2] for ordered Banach space theory and to [1, 10] for Banach lattice theory.

3. KB-operators on Banach lattices

Let us introduce the class of (weak) KB-operators in its general setting.

Definition 3.1 *An operator* $T : E \to X$, *from an ONS into a normed space, is said to be a KB-operator if for every increasing norm bounded sequence* $(x_n) \subset E^+$, (Tx_n) *has a convergent subsequence. If the latter sequence has a weakly convergent subsequence then* T *is called a weak KB-operator.*

Since the standard BL $L_1[0,1]$ of real-valued Lebesgue integrable functions on $[0,1]$ is a KB-space, the identity operator of $L_1[0,1]$ is an example of an operator which is both KB (hence weak KB) and bw-compact.

Our main result below gives a negative answer to a question in [5] by showing that the three classes of KB, weak KB and bw-compact operators on a BL are the same.

Theorem 3.2 *Let* E, X *be respectively a BL and a BS. Then, for an operator* $T : E \to X$ *the following are equivalent:*

1. *T is a bw-compact operator;*

2. *T is a KB-operator;*

3. *T is a weak KB-operator.*

The proof is based on the following well-known lemmas from the Banach lattice literature.

Lemma 3.3 (Thm 4.50, [1]) *For a BL* E, *there exists a lattice embedding* $i : c_0 \to E$ *iff there exists a disjoint sequence* $(x_n) \subset E^+$ *with*

1. $\|x_n\| \nrightarrow 0$ *as* $n \to \infty$, *and*

2. $\exists M > 0$, $\forall n \in \mathbb{N}$, $\|\sum_{k=1}^{n} x_k\| \leq M$.

Note that for the "if part" of the preceding lemma, by passing to a subsequence of (x_n), we may assume that the lattice embedding i satisfies $i(e_n) = x_n$ for all n, where e_n is the unit vector of c_0 with one on the nth coordinate and zero elsewhere.

Lemma 3.4 (Prop 2.11, [3]) *Let* E *be a BL with order continuous norm,* X *be a BS and* $T : E \to X$ *be an operator. Then,* T *is bw-compact iff* $T''(B(E)) \subset X$, *where* $B(E)$ *stands for the band generated by* E *in* E''.

Proof [of Theorem 3.2] $1 \Rightarrow 2$. Follows from $i \Leftrightarrow ii$ of Proposition 1 of [4].

$2 \Rightarrow 3$. Obvious.

$3 \Rightarrow 1$. Let $(x_n) \subset E^+$ be a disjoint b-order bounded sequence. We shall show that $\|Tx_n\| \to 0$. Otherwise, there exists a subsequence $(x_{\varphi(n)}) \subset (x_n)$ with

$$\left\| Tx_{\varphi(n)} \right\| > \varepsilon \text{ for each } n \text{ and for some } \varepsilon > 0. \tag{$*$}$$

Note that $(x_{\varphi(n)})$ is also a disjoint b-order bounded sequence, and hence the sequence of its partial sums is norm bounded. Since T is continuous, $(x_{\varphi(n)})$ does not converge to 0. So, it follows from Lemma 3.3 that there exists a lattice embedding $i : c_0 \to E$ and, by passing to a subsequence of $(x_{\varphi(n)})$, we may assume that $i(e_n) = x_{\varphi(n)}$ for each n. Taking into account this lattice embedding, we may assume that c_0 is a closed Riesz

subspace of E. Let the operator $T_{c_0} : c_0 \to X$ be the restriction of T to c_0. If $z = (z_n) \in (B(c_0))^+ = l_\infty^+$, then the sequence $(u_n) \subset c_0$ defined by $u_n = (z_1, z_2, ..., z_n, 0, 0, ...)$ satisfies $0 \leq u_n \uparrow z$ in $c_0'' = l_\infty$ and we have that $u_n \xrightarrow{w^*} z$ in c_0''. Since $T_{c_0}'' : c_0'' \to X''$ is weak* to weak* continuous we get that $T_{c_0}(u_n) = T_{c_0}''(u_n) \xrightarrow{w^*} T_{c_0}'' z$ in X''. Now, from the hypothesis, there exists a subsequence $(u_{\psi(n)}) \subset (u_n)$ with $T_{c_0}(u_{\psi(n)}) \xrightarrow{w} y$ in X. From the uniqueness of the weak* limit in X'', we get $T_{c_0}'' z = y$, that is, we have shown that $T_{c_0}''(B(c_0)) \subset X$. It follows from Lemma 3.4 that T_{c_0} is a bw-compact operator, and hence $\left\| Tx_{\varphi(n)} \right\| = \left\| T_{c_0}(e_n) \right\| \to 0$ as $n \to \infty$, contradicting $(*)$. This shows that T is bw-compact as desired. □

Now, we address the question whether the equivalence $(2) \Leftrightarrow (3)$ in Theorem 3.2 remains true in the general setting of OBSs. This is the subject of the next section.

4. KB-operators between ordered Banach spaces

In this section, we deal with operators satisfying some preserving conditions in relation with the order structure of the underlying spaces assumed to be only OBSs. Recall first that a subset B of a partially ordered set $(A, \leq)$ is said to be a chain if it is totally ordered i.e. every two elements of B are comparable, i.e. for every $x, y \in B$ one has $x \leq y$ or $y \leq x$. If each two distinct elements of B are incomparable, then B is called an antichain.

A mapping $T : A \to B$ between two partially ordered sets is said to be monotone if it is order-preserving, that is for every $x, y \in A$,

$$x \leq y \Rightarrow Tx \leq Ty.$$

A linear mapping $T : E \to F$ between two OVSs is monotone iff it is positive, that is $T(E^+) \subset F^+$.

We now introduce the following notions of preserving mappings.

Definition 4.1 *A mapping $T : A \to B$ between two partially ordered sets is said to be*

1. *chain-preserving, if it maps any chain of A into a chain of B.*

2. *weak chain-preserving, if for every chain $C \subset A$ the image $T(C)$ contains no infinite antichain.*

Clearly, the following implications hold true:

$$T \text{ is monotone} \Rightarrow T \text{ is chain-preserving} \Rightarrow T \text{ is weak chain-preserving.}$$

However, the converse of each one of the above implications is false in general. In fact, the operator $T : C[0,1] \to \mathbb{R}$ defined by $T(f) = f(1) - f(0)$ is a (weak) chain-preserving mapping which fails to be a positive operator. Moreover, if the mapping $T : C[0,1] \to \mathbb{R}^2$ is defined by

$$\begin{cases} T(\theta) = (0,0), \\ T(f) = (-1, f(1)), & f \neq \theta, \end{cases}$$

where θ is the null function of $C[0,1]$, then it is easily seen that T is weak chain-preserving which fails to be chain-preserving.

Note also that, for two partially ordered sets A and B, the set of weak chain-preserving mappings from A to B is strictly contained in the set B^A of all mappings from A to B. Indeed, the operator $T : C[0,1] \to \mathbb{R}^2$ defined by $T(f) = f(1)(-1,1)$ fails to be weak chain-preserving.

We need the following lemma.

Lemma 4.2 ([8]) *Let* $(A, \leq)$ *be a partially ordered set. Then, every sequence in* A *admits a monotone subsequence iff* A *contains no infinite antichain. In particular, every sequence in a chain of* A *admits a monotone subsequence.*

Since for every subsequence $(x_{\varphi(n)})$ of an increasing (resp. decreasing) sequence (x_n) of a partially ordered set $\sup\{x_n : n \in \mathbb{N}\} = x$ (resp. $\inf\{x_n : n \in \mathbb{N}\} = x$) iff $\sup\{x_{\varphi(n)} : n \in \mathbb{N}\} = x$ (resp. $\inf\{x_{\varphi(n)} : n \in \mathbb{N}\} = x$), and since the cone of an OBS is weakly closed, the following lemma is an immediate consequence of [2, Lem 2.3 (4)].

Lemma 4.3 *Let* E *be an OBS and* $(x_n) \subset E$ *be an increasing (resp. decreasing) sequence. Then, for every subsequence* $(x_{\varphi(n)})$ *of* (x_n), *if* $x_{\varphi(n)} \xrightarrow{w} x$ *then* $x = \sup\{x_n : n \in \mathbb{N}\}$ *(resp.* $x = \inf\{x_n : n \in \mathbb{N}\}$*).*

The details of the proof of the following lemma can be found in [7, Thm 2.2.2]. For a simple proof using the preceding lemma and [2, Lem 2.28], see the proof of [9, Lem 1.4].

Lemma 4.4 *Assume that an OBS* E *has a normal cone* E^+. *Then, a monotone sequence of* E *is convergent iff it admits a weakly convergent subsequence.*

When we restrict ourselves to positive operators between two OBSs, we get the following characterizations of KB-operators.

Proposition 4.5 *Let* $T : E \to F$ *be a positive operator between two OBSs such that* F^+ *is normal (in particular, if* F *is a BL). Then, the following are equivalent:*

1. *for every increasing bounded sequence* $(x_n) \subset E^+$, (Tx_n) *is convergent.*

2. T *is a KB-operator;*

3. T *is a weak KB-operator.*

Proof $1 \Rightarrow 2 \Rightarrow 3$ are obvious. Let $(x_n) \subset E^+$ be an increasing bounded sequence so that there exists a subsequence $(x_{\varphi(n)})$ of (x_n) with $Tx_{\varphi(n)} \xrightarrow{w} x \in F$. Hence, since (Tx_n) is increasing, we see by Lemma 4.4 that $Tx_n \to x$ as desired. □

Since the cone of a BL is normal, we immediately have the following corollary.

Corollary 4.6 *Let* E *be a BL. Then, the following are equivalent:*

1. E *is a KB-space;*

2. *the identity operator of* E *is KB;*

3. *the identity operator of* E *is weak KB.*

In the particular setting of BLs, the classes of KB and b-weakly compact operators coincide as we mentioned before. So, since the class of b-weakly compact operators between BLs satisfies the domination property (see [3, Cor 2.9]), this is also the case for KB-operators. Such a property can be stated as well in the setting of OBSs for KB-operators.

Proposition 4.7 *Let $T : E \to F$ be an operator between two OBSs such that F^+ is normal and the order intervals $[x, y]$ of F, $x \leq y$, are weakly compact. Assume that T is a weak chain-preserving operator dominated by some positive operator $S : E \to F$, that is, $-S \leq T \leq S$. Then, T is a KB-operator whenever S is one.*

Proof   Let $(x_n) \subset E^+$ be an increasing bounded sequence. There exists a subsequence $\left(x_{\alpha(n)}\right)$ of (x_n) with $Sx_{\alpha(n)} \to x \in F$. Since (Sx_n) is an increasing sequence, it follows by Lemma 4.3 that $x = \sup \left\{Sx_n : n \in \mathbb{N}\right\}$. In particular, we have

$$-x \leq -Sx_n \leq Tx_n \leq Sx_n \leq x$$

for each n. The hypothesis on the intervals of F shows that $Tx_{\varphi(n)} \xrightarrow{w} y \in F$ for some subsequence $\left(x_{\varphi(n)}\right)$ of (x_n). Since T is weak chain-preserving, the sequence $\left(Tx_{\varphi(n)}\right)$ admits by Lemma 4.2 a monotone subsequence $\left(Tx_{\varphi \circ \psi(n)}\right)$ which converges by Lemma 4.4, and hence T is KB. $\qquad \square$

Corollary 4.8 *Let E, F be two OBSs such that F^+ is normal and the intervals $[x, y]$ of F, $x \leq y$, are weakly compact. Assume that $T, S : E \to F$ are two positive operators such that T is dominated by S, that is, $0 \leq T \leq S$. Then, T is a KB-operator whenever S is one.*

Remark 4.9 *The hypothesis for the target space F in the preceding proposition is illustrated by the following two cases in which it is fulfilled.*

1. F is a reflexive OBS with a normal cone, in which case the order intervals are convex, closed and norm bounded (by Theorem 2.3), and hence weakly compact.

2. F is a BL such that its norm is order continuous (see [1, Thm 4.9]).

Our main result below shows that the equivalence $(2) \Leftrightarrow (3)$ in Theorem 3.2 remains true for weak chain-preserving operators between OBSs. It is an improvement of Proposition 4.5 if the operator $T : E \to F$ is defined on an OBS with a normal and generating cone.

Theorem 4.10 *Let E, F be two OBSs with normal cones such that E^+ is generating. Then, every weak chain-preserving operator $T : E \to F$ is KB iff it is weak KB.*

Proof   The "only if" part is obvious. For the "if part", let $(x_n) \subset E^+$ be an increasing bounded sequence. For $0 \leq f \in E'$, the sequence $(f(x_n))$ is clearly increasing such that

$$|f(x_n)| \leq M \|f\| \text{ for each } n \text{ and for some } M \geq 0. \qquad (**)$$

Hence, $f(x_n) \to \varphi(f)$ for every $f \in E'_+$, where $\varphi : E'_+ \to \mathbb{R}^+$ is the mapping defined by $\varphi(f) = \sup_{n \in \mathbb{N}} f(x_n)$. It is easily seen that φ is in fact an additive mapping. Since E^+ is normal and generating, Theorem 2.5 ensures that E'_+ is a generating cone in E'. Therefore, Lemma 2.1 shows that φ extends uniquely to a positive linear functional on E' (denoted again by φ) defined by

$$\varphi(f) = \varphi(f_1) - \varphi(f_2), \quad f = f_1 - f_2, \quad f_1, f_2 \in E'_+.$$

It follows that $f(x_n) \to \varphi(f)$ for every $f \in E'$. Now, if $f \in E'$, pick by Theorem 2.2 $f_1, f_2 \in E'_+$ with $f = f_1 - f_2$ and $\|f_i\| \leq N \|f\|$, $i = 1, 2$ for some constant $N > 0$. Hence, it follows from $(**)$ that

$$|\varphi(f)| \leq 2MN \|f\|, \quad f \in E'.$$

That is $\varphi \in E''_+$. Therefore, $x_n \xrightarrow{w^*} \varphi$ in E'' and hence since $T'' : E'' \to F''$ is weak* to weak* continuous, $T''(x_n) = T(x_n) \xrightarrow{w^*} T''(\varphi)$ in F''. Now, the hypothesis asserts that $T(x_{\alpha(n)}) \xrightarrow{w} y$ in F for some subsequence $(x_{\alpha(n)}) \subset (x_n)$. From the uniqueness of the weak* limit in F'', we see that $T''(\varphi) = y$. Therefore, $T(x_n) \xrightarrow{w^*} y$ in F'' or $T(x_n) \xrightarrow{w} y$ in F. Since T is weak chain-preserving, by Lemma 4.2 the sequence (Tx_n) has a monotone subsequence $(Tx_{\psi(n)})$. So, Lemma 4.4 shows that the subsequence $(Tx_{\psi(n)})$ converges and hence T is KB as required. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

We conclude this paper by the following question to which we hinted in the introduction.

Question 4.11 *Is the equivalence in Theorem 4.10 still true for all operators from E into F?*

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