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The generalized inverses of the products of two elements in a ring

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Abstract: In this paper, new extensions of Jacobson's lemma and Cline's formula for Drazin inverses, generalized Drazin inverses, and (m,n)-pseudo-inverses are given. As applications, we provide a formula for the powers of the products of two elements and establish connections between two B-Fredholm elements through the canonical map.

Key words: Drazin inverses, generalized Drazin inverses, (m,n)-pseudo-inverses, Jacobson's lemma, Cline's formula

1. Introduction

Let $\mathcal{R}$ be any associative ring with unit 1. The notation $\mathcal{R}^{inv}$ denotes the group of all the invertible elements in $\mathcal{R}$. The commutant and double commutant of an element $a \in \mathcal{R}$ are defined by

$$\mathrm{comm}(a) = \{x \in \mathcal{R} : ax = xa\}$$

and

$$\mathrm{comm}^2(a) = \{x \in \mathcal{R} : xy = yx \text{ for all } y \in \mathrm{comm}(a)\},$$

respectively. In 1958, the famous Drazin inverse in semigroup was firstly introduced in [8]. The notion of the Drazin inverse plays an important role in operator theory, singular differential equations, Markov chains, etc.

Definition 1.1 ([8]) An element $a \in \mathcal{R}$ is Drazin invertible if there exists an element $x \in \mathcal{R}$ such that

$$x \in \mathrm{comm}(a),\ \ xax = x \text{ and } a^k xa = a^k \text{ for some integer } k \geq 0.$$

In this case, x is called the Drazin inverse of a and denoted by a^D.

If such x exists, it is unique. The notation $\mathcal{R}^D$ denotes the set of all Drazin invertible elements in $\mathcal{R}$. The smallest nonnegative integer k is called the Drazin index of a, and is denoted by $\mathrm{ind}(a)$. If $\mathrm{ind}(a) = 1$, then a is called the group invertibility and the group inverse of a is denoted by $a^\sharp$. The notation $\mathcal{R}^\sharp$ denotes the set of all group invertible elements in $\mathcal{R}$. In [11], Koliha introduced the generalized notion of the Drazin invertibility in Banach algebras. Later, Koliha and Patrício [12] extended this concept from the category of Banach algebras to the category of rings. Recall that an element $a \in \mathcal{R}$ is said to be quasinilpotent if $1 + ax \in \mathcal{R}^{inv}$ for all $x \in \mathrm{comm}(a)$. The set of all quasinilpotent elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{qnil}$.

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Definition 1.2 ([11]) An element $a \in \mathcal{R}$ is generalized Drazin invertible if there exists an element $x \in \mathcal{R}$ such that

$$x \in \mathrm{comm}^2(a),\ xax = x \text{ and } axa - a \in \mathcal{R}^{qnil}.$$

In this case, x is called the generalized Drazin inverse of a and designated by a^{gD}.

If such x exists, it is unique. The set of all generalized Drazin invertible elements in $\mathcal{R}$ is denoted by $\mathcal{R}^{gD}$. Koliha and Patrício presented a characterization of the generalized Drazin invertibility in [12].

Lemma 1.3 ([12]) *Suppose that $a \in R$, then $a \in \mathcal{R}^{gD}$ if and only if there exists $p \in \mathcal{R}$ such that*

$$p = p^2,\ p \in comm^2(a),\ ap \in \mathcal{R}^{qnil} \text{ and } a + p \in \mathcal{R}^{inv}.$$

In this case, $p = 1 - aa^{gD}$ is called a spectral idempotent of a and denoted by a^{π}.

Lemma 1.4 ([10]) *Let $a \in \mathcal{R}$ and $n \geq 1$ be an integer. If $a^n \in \mathcal{R}^{gD}$, then $a \in \mathcal{R}^{gD}$ and $a^{gD} = (a^n)^{gD}a^{n-1}$.*

In the case of Lemma 1.4, we also have

$$(a^{gD})^n = (a^n)^{gD}. \tag{1.1}$$

In fact, if $a^n \in \mathcal{R}^{gD}$, the Lemma 1.4 implies that $a^{gD} = (a^n)^{gD}a^{n-1}$. Thus,

$$\begin{aligned}(a^{gD})^n &= (a^n)^{gD}a^{n-1} \cdot (a^n)^{gD}a^{n-1} \cdot \ldots \cdot (a^n)^{gD}a^{n-1} \\ &= (a^n)^{gD}a^n \cdot (a^n)^{gD}a^n \cdot \ldots \cdot a^n(a^n)^{gD} \\ &= (a^n)^{gD}.\end{aligned}$$

Hence, the equation (1.1) holds.

For any $a, c \in \mathcal{R}$, the Jacobson's lemma states that

$$1 - ac \in \mathcal{R}^{inv} \Longleftrightarrow 1 - ca \in \mathcal{R}^{inv}.$$

It has been realized that Jacobson's lemma has many suitable analogues for operator properties [1, 2, 16] and generalized inverses [5, 7, 20]. In [5, 7], Patrício-Hartwig and Cvetković-Ilić-Harte proved that the Drazin invertibility of $1 - ac$ implies that of $1 - ca$, respectively. In 2012, Zhuang and Chen [20] extended these results from Drazin invertibility to the generalized Drazin invertibility. As a further extension of Jacobson's lemma for generalized inverses, several researchers investigated the Drazin (resp. generalized Drazin) invertibility of $1 - ac$ and $1 - bd$ in rings under the conditions $acd = dbd$ and $dba = aca$ (see [13, 17]). Corresponding to the Jacobson's lemma, Cline [6] found a fundamental relation between the Drazin inverses of ac and ca, i.e. $(ac)^D = a(ca)^D c$. This result is known as Cline's formula, and generalizations of it for different kinds of generalized inverses were presented in [9, 15]. As a further extension of Cline's formula, Zeng et al. [10] investigated the formula between the Drazin inverses of ac and bd under the conditions $acd = dbd$ and $dba = aca$.

Recently, Yan et al. [18] proved that the Drazin (resp. generalized Drazin) spectrum of AC coincides with that of BD under the new conditions $BAC = BDB$ and $CDB = CAC$, where the objectives are bounded linear operators acting on a Banach space. This leads to the question whether these results can somehow be extended to the objectives of a ring. In this paper, we answer this question by showing new extensions of Jacobson's lemma and Cline's formula. In Section 2, new extensions of Cline's formula for Drazin inverses, generalized Drazin inverses and (m,n)-pseudo-inverses are given. As an application, an answer to the question posed by Mosić in [14] is provided. In Section 3, new extensions of Jacobson's lemma for Drazin inverses and generalized inverses are given. In Section 4, some applications to B-Fredholm elements in Banach algebras are provided.

2. New extensions of Cline's formula for generalized inverses

In this section, new extensions of Cline's formula for Drazin inverses, generalized Drazin inverses, and (m,n)-pseudo-inverses are given. We now begin with the following lemma, which plays a significant role in studying the extension of Cline's formula for generalized Drazin inverses.

Lemma 2.1 *Let $a,b,c,d\in\mathcal{R}$ satisfy $bac=bdb$ and $cdb=cac$, then*

$$ac\in\mathcal{R}^{qnil}\Longleftrightarrow bd\in\mathcal{R}^{qnil}.$$

Proof Suppose that $ac\in\mathcal{R}^{qnil}$, i.e. for arbitrary $g\in\text{comm}(ac)$, $1+g(ac)\in\mathcal{R}^{inv}$. To verify $bd\in\mathcal{R}^{qnil}$, it will suffice to show that $1+h(bd)\in\mathcal{R}^{inv}$ for all $h\in\text{comm}(bd)$. Indeed,

$$(acdh^3b)(ac)=acdh^3bdb=acdbdh^3b=acacdh^3b=(ac)(acdh^3b).$$

Therefore, $1+(acdh^3b)(ac)\in\mathcal{R}^{inv}$. From Jacobson's Lemma, it follows that $1+bacacdh^3\in\mathcal{R}^{inv}$. Moreover,

$$1+bacacdh^3=1+(bd)^3h^3=[1+h(bd)][1-h(bd)+h^2(bd)^2].$$

Thus, $1+h(bd)\in\mathcal{R}^{inv}$ because $1+h(bd)$ commutes with $1-h(bd)+h^2(bd)^2$. Consequently, $bd\in\mathcal{R}^{qnil}$. For the special case $a=d$ and $b=c$, the previous paragraph shows that $ac\in\mathcal{R}^{qnil}$ implies $ca\in\mathcal{R}^{qnil}$. Then, by interchanging a and c,

$$ac\in\mathcal{R}^{qnil}\Leftrightarrow ca\in\mathcal{R}^{qnil}.$$

To the opposite implication, suppose that $bd\in\mathcal{R}^{qnil}$, which implies $db\in\mathcal{R}^{qnil}$. Thus, by a similar argument as above, we have $ca\in\mathcal{R}^{qnil}$. This implies $ac\in\mathcal{R}^{qnil}$. □

Theorem 2.2 *Let $a,b,c,d\in\mathcal{R}$ satisfy $bac=bdb$ and $cdb=cac$, then*

$$ac\in\mathcal{R}^{gD}\Longleftrightarrow bd\in\mathcal{R}^{gD}.$$

Moreover,

$$(ac)^{gD}=acd[(bd)^{gD}]^3b\quad\text{and}\quad(bd)^{gD}=b[(ac)^{gD}]^2d.$$

Proof Suppose that $bd\in\mathcal{R}^{gD}$ and let $g=(bd)^{gD}$. Set $h=acdg^3b$. Thus, we are given $g\in\text{comm}^2(bd)$, $g(bd)g=g$ and $(bd)^2g-bd\in\mathcal{R}^{qnil}$, and must conclude from these that $h\in\text{comm}^2(ac)$, $h(ac)h=h$ and $(ac)^2h-ac\in\mathcal{R}^{qnil}$.

First, let $r(ac)=(ac)r$. Then,

$$(bracd)(bd)=bracacd=bacracd=(bd)(bracd),$$

so that $g(bracd) = (bracd)g$, whence also

$$\begin{aligned} rh &= racdg^3b = racdbdbdg^5b = racacacdg^5b = acacracdg^5b \\ &= acd(bracd)g^5b = acdg^5(bracd)b = acdg^5bracac \\ &= acdg^5bacacr = acdg^5bdbbdbr = acdg^3br = hr. \end{aligned}$$

Therefore, $h \in \mathrm{comm}^2(ac)$.

Next,

$$h(ac)h = acdg^3bacacdg^3b = acdg^3bdbdbdg^3b = acdg^3b = h.$$

Finally, define $c' = cdgb - c$ and $d' = dgbd - d$. Then

$$bac' = bacdgb - bac = bdbdgb - bac = bdgbdb - bdb = bd'b$$

and

$$\begin{aligned} c'd'b &= (cdgb - c)(cdgb - c)b = cdb - cdgbdb = cac - cdbdgb \\ &= cac - cacdgb = (cdgb - c)a(cdgb - c) = c'ac'. \end{aligned}$$

Thus, $b'd = (bd)^{2n}g - (bd)^n \in \mathcal{R}^{qnil}$ implies that $a'c \in \mathcal{R}^{qnil}$ by Lemma 2.1. That is $(ac)^2h - ac \in \mathcal{R}^{qnil}$. Hence, $ac \in \mathcal{R}^{gD}$.

Conversely, by a similar argument, we can show that if $ac \in \mathcal{R}^{gD}$, then so is bd and the equality $(bd)^{gD} = b[(ac)^{gD}]^2d$ holds. □

From Theorem 2.2, we get a series of useful corollaries about the powers of element products for generalized inverses.

Corollary 2.3 *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

$$(ac)^n \in \mathcal{R}^{gD} \Longleftrightarrow (bd)^n \in \mathcal{R}^{gD}.$$

Moreover,

$$[(ac)^n]^{gD} = (ac)^{2n-1}d\{[(bd)^n]^{gD}\}^3b$$

and

$$[(bd)^n]^{gD} = b\{[(ac)^n]^{gD}\}^2d(bd)^{n-1}.$$

Proof Let $a' = (ac)^{n-1}a$ and $d' = d(bd)^{n-1}$. Since $ba'c = bd'b$ and $cd'b = ca'c$, it follows that

$$(ac)^n = a'c \in \mathcal{R}^{gD} \Longleftrightarrow (bd)^n = bd' \in \mathcal{R}^{gD},$$

by Theorem 2.2. Also,

$$[(ac)^n]^{gD} = (a'c)^{gD} = a'cd'[(bd')^{gD}]^3b = (ac)^{2n-1}d\{[(bd)^n]^{gD}\}^3b$$

and

$$[(bd)^n]^{gD} = (bd')^{gD} = b[(a'c)^{gD}]^2d' = b\{[(ac)^n]^{gD}\}^2d(bd)^{n-1},$$

as required. □

Corollary 2.4 *Given any a, $c \in \mathcal{R}$, if $(ac)^n \in \mathcal{R}^{gD}$ for some integer $n \geq 1$, then so are $(ca)^n$, ca, and ac.*

Proof In the case $a = d$ and $b = c$, Corollary 2.3 and Lemma 1.4 combine to say that $(ca)^n$, ca, and $ac \in \mathcal{R}^{gD}$, when $(ac)^n \in \mathcal{R}^{gD}$. □

Corollary 2.5 *Let a, $c \in \mathcal{R}$. If $(ac)^{n+1} \in \mathcal{R}^{gD}$ and $(ca)^n \in \mathcal{R}^{gD}$ for some integer $n \geq 1$, then $[(ca)^n]^{gD} = c[(ac)^{n+1}]^{gD}a$.*

Proof On taking $a = d$ and $b = c$, Corollary 2.3 tells us that

$$[(ca)^n]^{gD} = c\{[(ac)^n]^{gD}\}^2 a(ca)^{n-1}.$$

Hence,

$$\begin{aligned}((ca)^n)^{gD} &= c\{[(ac)^n]^{gD}\}^2 a(ca)^{n-1} = c[(ac)^{gD}]^{2n} a(ca)^{n-1}\\ &= c[(ac)^{gD}]^{2n}(ac)^{n-1}a = c[(ac)^{gD}]^{n+1}a\\ &= c[(ac)^{n+1}]^{gD}a,\end{aligned}$$

the second and the last equalities hold due to the equality (1.1) in Section 1. □

Remark 2.6 *For a Banach algebra $\mathcal{A}$ and $a, c \in \mathcal{A}$, Mosić [14] proved that*

$$(ac)^{n+1} \in \mathcal{A}^{gD} \Longrightarrow (ca)^n \in \mathcal{A}^{gD}.$$

Moreover, the "Cline's formula" to the generalized Drazin invertibility of the powers of the products of elements a and c is given, i.e. $[(ca)^n]^{gD} = c[(ac)^{n+1}]^{gD}a$. In page 4 of [14], Mosić asked whether this "Cline's formula" for the powers of the element products can be extended from the category of Banach algebras to the category of rings. In Corollary 2.5, we provide an answer to Mosić's question and show the "Cline's formula" for the powers of the element products in rings under the additional assumption $(ca)^n \in \mathcal{R}^{gD}$. Although there is an additional assumption, our Corollary 2.5 still covers Theorem 2.1 of [14]. Indeed, given any $a \in \mathcal{A}$ and some integer $n \geq 1$, we have

$$a \in \mathcal{A}^{gD} \Longleftrightarrow a^n \in \mathcal{A}^{gD}$$

by Lemma 1.4 and Theorem 5.5 of [11]. Thus, it is easy to show that if $(ac)^{n+1} \in \mathcal{A}^{gD}$ then $(ac)^n \in \mathcal{A}^{gD}$, so that $(ca)^n \in \mathcal{A}^{gD}$. Hence, the equality $[(ca)^n]^{gD} = c[(ac)^{n+1}]^{gD}a$ holds in Banach algebra $\mathcal{A}$ by Corollary 2.5.

From Theorem 2.2, it is not hard to obtain the following equivalence about Drazin inverse.

Corollary 2.7 *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

$$ac \in \mathcal{R}^D \iff bd \in \mathcal{R}^D.$$

Moreover,

$$(ac)^D = acd[(bd)^D]^3 b \quad \text{and} \quad (bd)^D = b[(ac)^D]^2 d.$$

with $|ind(bd) - ind(ac)| \leq 2$.

Proof Suppose that $bd \in \mathcal{R}^D$ with $\operatorname{ind}(bd) = k$ and let $g = (bd)^D$. In view of Theorem 2.2, we have $ac \in \mathcal{R}^{gD}$ and $(ac)^{gD} = acd[(bd)^D]^3 b$. Hence,

$$(ac)^{k+2}(ac)^{gD}(ac) = (ac)^{k+2}acd[(bd)^D]^3 bac = acd(bd)^{k+2}[(bd)^D]^3 bdb = (ac)^{k+2},$$

which implies $(ac)^{gD} = (ac)^D$ and $\operatorname{ind}(ac) \leq k+2 = \operatorname{ind}(bd)+2$. By a similar argument, the second equality holds and $\operatorname{ind}(bd) \leq \operatorname{ind}(ac)+2$. □

Recall that the Jacobson radical of $\mathcal{R}$ is the two-sided ideal

$$J(\mathcal{R}) = \{a \in \mathcal{R} : 1 + ax \in \mathcal{R}^{inv} \text{ for all } x \in \mathcal{R}\}.$$

Generalizing ideas of Koliha and Patrício [12] in 2002 and of Wang and Chen [15] in 2012, Drazin [9] introduced a new outer generalized inverse, which is called (m,n)-pseudo-inverse. Given any ring $\mathcal{R}$ and any elements $a, m, n \in \mathcal{R}$, we say that a is (m,n)-pseudo-invertible if there exists an element $x \in \mathcal{R}$ such that

$$xax = x \in \operatorname{comm}^2(a),$$

$$m - xam,\ n - nax \in J(\mathcal{R})$$

and

$$x \in (m\mathcal{R} + J(\mathcal{R})) \cap (\mathcal{R}n + J(\mathcal{R})).$$

In this case, x is called the (m,n)-pseudo-inverse of a and denoted by $a^{(m,n)}$. If such x exists, it is unique (see [9]). We use $\mathcal{R}^{(m,n)}$ to denote the set of all (m,n)-pseudo-invertible elements in $\mathcal{R}$. In [9], Drazin showed the connections between the (m,n)-pseudo-polar and (m,n)-pseudo-invertible properties and established the Cline's formula for (m,n)-pseudo-inverses. The natural question is: Does there exist a suitable analogue of Theorem 2.2 for (m,n)-pseudo-inverses? We now consider this question.

Theorem 2.8 *Let $a, b, c, d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

(1) $ac \in \mathcal{R}^{(m,n)} \Longrightarrow bd \in \mathcal{R}^{(bm,nd)}$. *Moreover,*

$$(bd)^{(bm,nd)} = b[(ac)^{(m,n)}]^2 d.$$

(2) $bd \in \mathcal{R}^{(m,n)} \Longrightarrow ac \in \mathcal{R}^{(acdm,nb)}$. *Moreover,*

$$(ac)^{(acdm,nb)} = acd[(bd)^{(m,n)}]^3 b.$$

Proof (1) Suppose that $ac \in \mathcal{R}^{(m,n)}$ and let $x = (ac)^{(m,n)}$. Then we are given that

$$x(ac)x = x \in \operatorname{comm}^2(ac),$$

$$m - x(ac)m,\ n - n(ac)x \in J(\mathcal{R})$$

and

$$x \in (m\mathcal{R} + J(R)) \cap (\mathcal{R}n + J(\mathcal{R})).$$

Set $y = bx^2 d$.

First, we have

$$y(bd)y = bx^2d(bd)bx^2d = bx^3acdbdbx^2d = bx^3(ac)^3x^2d = bx^2d = y.$$

To prove that $y \in \mathrm{comm}^2(bd)$, let $r(bd) = (bd)r$. From

$$(acdrb)ac = acdrbdb = acdbdrb = ac(acdrb),$$

we obtain $(acdrb)x = x(acdrb)$, whence

$$\begin{aligned} yr &= bx^2dr = bx^4acacdr = bx^4acdbdr = bx^4(acdrb)d \\ &= b(acdrb)x^4d = bdbdrbx^4d = rbdbdbx^4d \\ &= rbacacx^4d = rbx^2d = ry. \end{aligned}$$

Thus, $y \in \mathrm{comm}^2(bd)$.

Next, since $J(\mathcal{R})$ is an ideal of $\mathcal{R}$, we have

$$y = bx^2d \in b(m\mathcal{R} + J(\mathcal{R}))xd \subseteq b(m\mathcal{R} + J(\mathcal{R})) \subseteq bm\mathcal{R} + J(\mathcal{R}),$$

and dually $y \in \mathcal{R}nd + J(\mathcal{R})$.

Finally,

$$\begin{aligned} bm - y(bd)(bm) &= bm - bx^2dbdbm = bm - bx^3acdbdbm \\ &= bm - bx^3acacacm = bm - bx(ac)m = b(m - x(ac)m) \in J(\mathcal{R}), \end{aligned}$$

and dually $nd - nd(bd)y \in J(\mathcal{R})$.

(2) By a similar argument as above, we can show that if $bd \in \mathcal{R}^{(m,n)}$ then so is bd and the equality $(ac)^{(acdm,nb)} = acd[(bd)^{(m,n)}]^3b$ holds. □

Remark 2.9 *For the special case $a = d$ and $b = c$ in (1) of Theorem 2.8, we obtain the Drazin's elegant result (Theorem 4.1 of [9]) as a corollary, i.e. $ac \in \mathcal{R}^{(m,n)} \Longrightarrow ca \in \mathcal{R}^{(cm,na)}$. Moreover, from (2) of Theorem 2.8, we can also obtain $ca \in \mathcal{R}^{(cacm,na)}$ with the assumption $ac \in \mathcal{R}^{(m,n)}$.*

3. New extensions of Jacobson's lemma for generalized inverses

In this section, new extensions of Jacobson's lemma for group inverses, Drazin inverses, and generalized Drazin inverses are given. The proof of the following theorem about generalized Drazin inverses is strongly inspired by that of Theorem 3.3 in [17].

Theorem 3.1 *Let a, b, c, $d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

$$\alpha = 1 - bd \in \mathcal{R}^{gD} \Longleftrightarrow \beta = 1 - ac \in \mathcal{R}^{gD}.$$

Moreover,

$$\beta^{gD} = \{1 - acd\alpha^{\pi}[1 - \alpha^{\pi}\alpha(1 + bd)]^{-1}b\}(1 + ac) + acd\alpha^{gD}b,$$

and

$$\alpha^{gD} = \{1 - b\beta^{\pi}[1 - \beta^{\pi}\beta(1+bd)]^{-1}acd\}(1+bd) + b\beta^{gD}acd.$$

where $\alpha^{\pi} = 1 - \alpha\alpha^{gD}$ *and* $\beta^{\pi} = 1 - \beta\beta^{gD}$.

Proof In view of Lemma 1.3, we have that $1 - p\alpha(1+bd)$ is invertible. Set $p = \alpha^{\pi}$, $x = \alpha^{gD}$ and $y = \{1 - acdp[1 - p\alpha(1+bd)]^{-1}b\}(1+ac) + acdxb$. We need to show that the following three conditions hold: (i) $y \in \mathrm{comm}^2(\beta)$; (ii) $y\beta y = y$; (iii) $\beta - \beta y\beta \in \mathcal{R}^{qnil}$.

(i) To prove that $y \in \mathrm{comm}^2(\beta)$, let $z_1 \in \mathcal{R}$ satisfy $z_1\beta = \beta z_1$. Then bz_1acd commutes with α, which implies that bz_1acd commutes with x, p and $[1 - p\alpha(1+bd)]^{-1}$.

First, by equality

$$p = p[1 - p\alpha(1+bd)]^{-1}(bd)^2, \tag{3.1}$$

we obtain

$$\begin{aligned} z_1(acdpb) &= z_1acdp[1 - p\alpha(1+bd)][1 - p\alpha(1+bd)]^{-1}b \\ &= z_1acd(bd)^2p[1 - p\alpha(1+bd)]^{-1}b = acdbz_1acdp[1 - p\alpha(1+bd)]^{-1}b \\ &= acdp[1 - p\alpha(1+bd)]^{-1}bz_1acdb = acdp[1 - p\alpha(1+bd)]^{-1}bz_1acac \\ &= acdp[1 - p\alpha(1+bd)]^{-1}bacacz_1 = acdp[1 - p\alpha(1+bd)]^{-1}(bd)^2bz_1 \\ &= (acdpb)z_1. \end{aligned}$$

From this, it follows that $acd\alpha xbz_1 = z_1acd\alpha xb$, which implies $acdbd\alpha xbz_1 = z_1acdbd\alpha xb$. Thus, $acd(1+bd)\alpha xbz_1 = z_1acd(1+bd)\alpha xb$, so that

$$acdxbz_1 - acd(bd)^2xbz_1 = z_1acdxb - z_1acd(bd)^2xb.$$

Since

$$acd(bd)^2xbz_1 = acdx(bd)^2bz_1 = acdxbz_1acdb = acdbz_1acdxb = z_1acd(bd)^2xbac,$$

then

$$acdxbz_1 = z_1acdxb. \tag{3.2}$$

Next, write $c_1 = acdp[1 - p\alpha(1+bd)]^{-1}b(1+ac)$. Then, having the identity (3.1) in mind,

$$\begin{aligned} z_1c_1 &= z_1dp[1 - p\alpha(1+bd)]^{-1}bac(1+ac) \\ &= z_1dp[1 - p\alpha(1+bd)][1 - p\alpha(1+bd)]^{-2}bac(1+ac) \\ &= z_1d(bd)^2p[1 - p\alpha(1+bd)]^{-2}bac(1+ac) \\ &= dbacz_1dp[1 - p\alpha(1+bd)]^{-2}bac(1+ac) \\ &= dp[1 - p\alpha(1+bd)]^{-2}bacz_1dbac(1+ac). \end{aligned}$$

and

$$\begin{aligned}c_1z_1 &= dp[1-p\alpha(1+bd)]^{-1}bac(1+ac)z_1\\ &= dp[1-p\alpha(1+bd)][1-p\alpha(1+bd)]^{-2}bac(1+ac)z_1\\ &= dp[1-p\alpha(1+bd)]^{-2}(bd)^2bac(1+ac)z_1\\ &= dp[1-p\alpha(1+bd)]^{-2}bacz_1dbac(1+ac).\end{aligned}$$

These two equalities together imply that $z_1c_1 = c_1z_1$, that is

$$z_1acdp[1-p\alpha(1+bd)]^{-1}b(1+ac) = acdp[1-p\alpha(1+bd)]^{-1}b(1+ac)z_1. \tag{3.3}$$

Hence (3.2), (3.3), and $(ac)z_1 = z_1(ac)$ combine to yield $yz_1 = z_1y$, which means $y \in \mathrm{comm}^2(\beta)$.

(ii) Now

$$\begin{aligned}y\beta &= 1-(ac)^2 - acdp[1-p\alpha(1+bd)]^{-1}b(1+ac)(1-ac) + acdxb(1-ac)\\ &= 1-[acdb - acdxb(1-ac)] - acdp[1-p\alpha(1+bd)]^{-1}(1-bd)(1+bd)b\\ &= 1-acdpb - acdp[1-p\alpha(1+bd)]^{-1}\alpha(1+bd)b\\ &= 1-acdp\{1+[1-p\alpha(1+bd)]^{-1}p\alpha(1+bd)\}b\\ &= 1-acdp[1-p\alpha(1+bd)]^{-1}b.\end{aligned}$$

Since $px = x - x\alpha x = 0$, it follows that

$$acdp[1-p\alpha(1+bd)]^{-1}bacdxb = acdp[1-\alpha(1+bd)]^{-1}px(bd)^2b = 0.$$

Hence,

$$\begin{aligned}y\beta y &= y - acdp[1-p\alpha(1+bd)]^{-1}by\\ &= y - acdp[1-p\alpha(1+bd)]^{-1}(1+bd)b + acdp[1-p\alpha(1+bd)]^{-2}(bd)^2\cdot\\ &\quad (1+bd)b\\ &= y - acdp[1-p\alpha(1+bd)]^{-2}[p-p\alpha(1+bd)+p(bd)^2](1+bd)b\\ &= y.\end{aligned}$$

(iii) Let $z \in \mathcal{R}$ with $z(\beta-\beta y\beta) = (\beta-\beta y\beta)z$, that is,

$$zacd\alpha p[1-p\alpha(1+bd)]^{-1}b = acd\alpha p[1-p\alpha(1+bd)]^{-1}bz.$$

Then we need to show that

$$1+acd\alpha p[1-p\alpha(1+bd)]^{-1}bz \in \mathcal{R}^{inv}.$$

From equality (3.1), it follows that

$$\begin{aligned}(baczd)\alpha p &= (bzacd)\alpha p[1-p\alpha(1+bd)]^{-1}(bd)^2\\ &= bzacd\alpha p[1-p\alpha(1+bd)]^{-1}bacd\\ &= bacd\alpha p[1-p\alpha(1+bd)]^{-1}(bzacd)\\ &= \alpha p(baczd),\end{aligned}$$

whence

$$[1-p\alpha(1+bd)]^{-1}baczd(\alpha p)=(\alpha p)[1-p\alpha(1+bd)]^{-1}baczd.$$

Since $\alpha p\in\mathcal{R}^{qnil}$, it follows that $1+[1-p\alpha(1+bd)]^{-1}bzacd\alpha p\in\mathcal{R}^{inv}$. Hence, by Jacobson's lemma, $1+acd\alpha p[1-p\alpha(1+bd)]^{-1}bz\in\mathcal{R}^{inv}$ as required.

The proof of the opposite implication is parallel to that above. □

Remark 3.2 *In Theorem 3.1, it is routine to compute the spectral idempotent of* β*, that is*

$$\beta^{\pi}=1-\beta\beta^{gD}=(ac)^2+acd\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]^{-1}\alpha(1+bd)b-acd\alpha^{gD}\alpha b.$$

To consider the Drazin invertible case in next theorem, we need to establish the connection between the power of $\beta\beta^{\pi}$ *and that of* $\alpha\alpha^{\pi}$*. Indeed, since the commutativity of* bd*,* α*,* α^{π} *and* $[1-\alpha^{\pi}\alpha(1+bd)]^{-1}$*, then*

$$\begin{aligned}\beta\beta^{\pi}&=\beta(ac)^2+\beta acd\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]^{-1}\alpha(1+bd)b-\beta acd\alpha^{gD}\alpha b\\&=acd\alpha b+acd\alpha\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]^{-1}\alpha(1+bd)b-acd\alpha\alpha^{gD}\alpha b\\&=acd\alpha\{1+\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]^{-1}\alpha(1+bd)-\alpha^{gD}\alpha\}b\\&=acd\alpha\{\alpha^{\pi}+\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]^{-1}\alpha(1+bd)\}b\\&=acd\alpha[1-\alpha^{\pi}\alpha(1+bd)]^{-1}\{\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]+\alpha^{\pi}\alpha(1+bd)\}b\\&=acd\alpha\alpha^{\pi}[1-\alpha^{\pi}\alpha(1+bd)]^{-1}b.\end{aligned}$$

Hence, the equality

$$(\beta\beta^{\pi})^k=acd(\alpha\alpha^{\pi})^k[1-\alpha^{\pi}\alpha(1+bd)]^{-k}(bd)^{2(k-1)}b$$

holds for any integer $k\geq 1$*.*

Theorem 3.3 *Let* $a,\ b,\ c,\ d\in\mathcal{R}$ *satisfy* $bac=bdb$ *and* $cdb=cac$*, then*

$$\alpha=1-bd\in\mathcal{R}^D\Longleftrightarrow\beta=1-ac\in\mathcal{R}^D$$

with $ind(1-bd)=ind(1-ac)$*. Moreover,*

$$\beta^D=(1-acd\alpha^{\pi}rb)(1+ac)+acd\alpha^Db,$$

and

$$\alpha^D=(1-b\alpha^{\pi}r'acd)(1+bd)+b\beta^Dacd,$$

where $\alpha^{\pi}=1-\alpha\alpha^D$*,* $\beta^{\pi}=1-\beta\beta^D$*,* $r=\sum_{i=0}^{k-1}[\alpha(1+bd)]^j$ *and* $r'=\sum_{i=0}^{k-1}[\beta(1+ac)]^j$*.*

Proof Suppose that $\alpha=1-bd\in\mathcal{R}^D$ with $\mathrm{ind}(1-bd)=k$. By Theorem 3.1, $\beta=1-ac\in R^{gD}$. In order to prove $\beta=1-ac\in\mathcal{R}^D$, we need only to show that there exists an integer $n\geq 1$ such that $(\beta\beta^{\pi})^n=0$. In fact by the previous remark, it follows that $(\beta\beta^{\pi})^k=acd(\alpha\alpha^{\pi})^k[1-\alpha^{\pi}\alpha(1+bd)]^{-k}(bd)^{2(k-1)}b=0$. Thus, $\beta=1-ac\in\mathcal{R}^D$ and $\mathrm{ind}(1-ac)\leq k$. By symmetry,

$$\mathrm{ind}(1-ac)=\mathrm{ind}(1-ca)\geq\mathrm{ind}(1-db)=\mathrm{ind}(1-bd)=k.$$

Hence, $\operatorname{ind}(1-ac) = \operatorname{ind}(1-bd) = k$. Moreover,

$$\alpha^\pi[1-\alpha^\pi\alpha(1+bd)]^{-1} = \alpha^\pi\sum_{i=0}^{k-1}[\alpha^\pi\alpha(1+bd)]^i = \alpha^\pi\sum_{i=0}^{k-1}[\alpha(1+bd)]^i.$$

This implies that the formula is true by Theorem 3.1.

The proof of the opposite implication is parallel to that above. □

On taking $k=1$ in Theorem 3.3, we deduce a result for the group inverse.

Corollary 3.4 *Let $a,\ b,\ c,\ d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

$$1-bd \in \mathcal{R}^\sharp \Longleftrightarrow 1-ac \in \mathcal{R}^\sharp.$$

The Theorem 3.1, 3.3, and Corollary 3.4 generalize a series of results in [5, 7, 20] in a new way. These extensions are different from that in [13, 17]. One can check the Example 4.1 in Section 4 for more details.

Remark 3.5 *In Section 2, the authors establish a new extension of Cline's formula for the (m,n)-pseudo-inverse. It is natural to ask whether our extension of Jacobson's lemma is suitable for the (m,n)-pseudo-inverse. Unfortunately, it seems to be much more difficult to consider Jacobson's lemma for generalized inverses than Cline's formula for them. In Section 6 of [9], Drazin listed a series of open questions about generalized inverses. One of these questions is whether Jacobson's lemma linking the (m,n)-pseudo-invertibility of $1-ac$ and $1-ca$ holds. Thus, it is meaningless to consider the abovementioned question until Drazin's questions in [9] are solved.*

4. Examples and further results

The following example is utilized to illustrate that the main results of this paper can deal with some cases which are different from that in [13, 17, 19].

Example 4.1 For a Banach space X, let $S_1, S_2, T_1, T_2 \in \mathcal{B}(X)$ be arbitrary nonzero operators satisfying $S_1 = T_2S_2$. We consider $A, B, C, D \in \mathcal{B}(X \oplus X)$ as follows

$$B = \begin{pmatrix} 0 & 0 \\ 0 & S_1 \end{pmatrix},\ C = \begin{pmatrix} 0 & S_2 \\ 0 & 0 \end{pmatrix} \text{ and } A = D = \begin{pmatrix} I & T_1 \\ T_2 & I \end{pmatrix},$$

respectively. Then $BAC = BDB$ and $CDB = CAC$. Hence, Theorem 3.1 implies that the operator

$$I - AC = \begin{pmatrix} I & -S_2 \\ 0 & I-S_1 \end{pmatrix}$$

is generalized Drazin invertible if and only if

$$I - BD = \begin{pmatrix} I & 0 \\ -S_1T_2 & I-S_1 \end{pmatrix}$$

is generalized Drazin invertible. On the other hand,

$$ACD = \begin{pmatrix} S_1 & S_2 \\ T_2S_1 & T_2S_2 \end{pmatrix} \text{ and } DBD = \begin{pmatrix} T_1S_1T_2 & T_1S_1 \\ T_2S_1T_2 & S_1 \end{pmatrix}.$$

Thus, the operator product ACD may not coincide with DBD, and so Theorem 3.3 of [17] is no longer valid.

The Fredholm theory plays an important role in operator theory. In [3], Barnes extended the classical Fredholm theory from the bounded linear operator category to the ring category. Inspired by Barnes' idea, Berkani [4] introduced B-Fredholm elements, which encompass his well-known B-Fredholm operators. Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and π be the canonical map from $\mathcal{R}$ to $\mathcal{R}/\mathcal{I}$. According to Berkani [4], an element $a \in \mathcal{R}$ is called a B-Fredholm element relative to $\mathcal{I}$ if $\pi(a)$ is Drazin invertible in quotient ring $\mathcal{R}/\mathcal{I}$. We use $B\Phi(\mathcal{R},\mathcal{I})$ to denote the set of all B-Fredholm elements relative to $\mathcal{I}$. It is natural to ask whether our new extensions of Jacobson's lemma and Cline's formula are suitable for B-Fredholm elements? The answer is affirmative.

Theorem 4.2 *Let $\mathcal{I}$ be an ideal of $\mathcal{R}$. Suppose that $a,b,c,d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

$$1 - ac \in B\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1 - bd \in B\Phi(\mathcal{R},\mathcal{I})$$

and

$$ac \in B\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow bd \in B\Phi(\mathcal{R},\mathcal{I}).$$

Proof Let π be the canonical map from $\mathcal{R}$ to $\mathcal{R}/\mathcal{I}$. Since $bac = bdb$ and $cdb = cac$, it follows that $\pi(b)\pi(a)\pi(c) = \pi(b)\pi(d)\pi(b)$ and $\pi(c)\pi(d)\pi(b) = \pi(c)\pi(a)\pi(c)$. Thus, $\pi(1-ac) = 1-\pi(a)\pi(c)$ is Drazin invertible if and only if $\pi(1-bd) = 1-\pi(b)\pi(c)$ is Drazin invertible by Theorem 3.3. Hence, $1-ac \in B\Phi(\mathcal{R},\mathcal{I})$ if and only if $1-bd \in B\Phi(\mathcal{R},\mathcal{I})$. Similarly, we can deduce that $ac \in B\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow bd \in B\Phi(\mathcal{R},\mathcal{I})$ from Corollary 2.7. $\square$

Taking into account the definition of B-Fredholm elements given by Berkani, we now define a new class of Fredholm elements which is called generalized B-Fredholm element.

Definition 4.3 *Let $\mathcal{I}$ be an ideal of $\mathcal{R}$ and π be the canonical map from $\mathcal{R}$ to $\mathcal{R}/\mathcal{I}$. An element $a \in \mathcal{R}$ is called a generalized B-Fredholm element relative to $\mathcal{I}$ if $\pi(a)$ is generalized Drazin invertible in quotient ring $\mathcal{R}/\mathcal{I}$.*

The notation $gB\Phi(\mathcal{R},\mathcal{I})$ denotes the set of all generalized B-Fredholm elements relative to $\mathcal{I}$. It is easy to see that $B\Phi(\mathcal{R},\mathcal{I}) \subset gB\Phi(\mathcal{R},\mathcal{I})$. From Theorem 2.2 and Theorem 3.1, we can obtain the following result immediately.

Theorem 4.4 *Let $\mathcal{I}$ be an ideal of $\mathcal{R}$. Suppose that $a,b,c,d \in \mathcal{R}$ satisfy $bac = bdb$ and $cdb = cac$, then*

$$1 - ac \in gB\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow 1 - bd \in gB\Phi(\mathcal{R},\mathcal{I})$$

and

$$ac \in gB\Phi(\mathcal{R},\mathcal{I}) \Longleftrightarrow bd \in gB\Phi(\mathcal{R},\mathcal{I}).$$

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