

On Hausdorff-Young inequalities in generalized Lebesgue spaces

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Abstract: Lebesgue spaces with the variable rate of summability are considered in this work. Generalizations of Riesz and Paley theorems are proved in these spaces. The obtained results are applied, in particular, to a classical exponential system.

Key words: Riesz and Paley theorems, Fourier series, Lebesgue spaces with variable exponent, exponential system

1. Introduction

In the context of applications to various branches of mathematics, for example, such as theory of partial differential equations, theory of approximations, harmonic analysis, etc., there arose great interest in nonstandard function spaces. As examples of such spaces, we can mention Lebesgue space with variable summability index, Morrey space, grand Lebesgue space, etc. These spaces have been considered by many authors such as Adams [1], Bilalov, Gasymov, Guliyeva [5], Bilalov, Guseynov [6, 7], Capone, Fiorenza [11], Castilo, Rafeiro [12], Cruz-Uribe, Fiorenza [13], Cruz-Uribe, Fiorenza, Neugebauer [14], Fiorenza, Karadzhov [20], Fiorenza, Krbec [21], Israfilov, Tozman [27, 28], Kokilashvili, Meskhi, Rafeiro, Samko [30], Morrey [31], Samko [33], Samko, Umarchadzhiev [34], Sharapudinov [35], Xianling, Dun [37], Zorko [39], etc. Note the theory Lebesgue spaces with variable exponent got a boost in 1931 when Orlicz published his seminal paper [32]. Orlicz considered the sequences of real numbers $\{x_n\}_{n \in \mathbb{N}}$ for which the series $\sum_{n=1}^{+\infty} |x_n|^{p_n}$ is convergent for $p_n \geq 1$. A lot of research has been later dedicated to this theory and any classical facts about harmonic analysis have been extended to these spaces. More details on these facts can be found, for example, in [13–19, 22, 23, 37]. One of the important directions of this theory is the study of Lebesgue space with variable summability index and with mixed norms. Lebesgue spaces with variable exponents with and mixed norms were considered, for example, in [2, 24, 25].

In problems where the solutions are sought in the form of Fourier series with respect to the eigenfunctions of some differential operators, it is always interesting to study the space of coefficients of this eigenfunction expansion. The relationship between Fourier coefficients and function space has been expressed by the theorems of Riesz, Paley, Hardy-Littlewood, etc. (see [3, 29, 38]). For analogues and generalizations of these results we refer the readers to [8, 10, 36]. Vector-valued analogues can be found in [4, 9, 26].

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This work deals with the Riesz and Paley theorems in Lebesgue space with variable summability index and with mixed norm. In the present work, we generalize Riesz and Paley theorems for the functions with vector-valued Fourier coefficients with respect to the orthonormal and uniformly bounded system in Lebesgue spaces with variable exponent and mixed norm. We introduce some spaces and establish a relationship between a function and the sequence of its coefficients in these spaces. The obtained results are applied to a classical exponential system.

2. Preliminary

Let $\Omega \subset R^n$, be a measurable set, $p(x)$ be a measurable function on Ω such that $p(x) \geq 1$. For $E \subset \Omega$ we denote

$$p_+(E) = \operatorname{ess\,sup}_{x \in E} p(x), p_-(E) = \operatorname{ess\,inf}_{x \in E} p(x),$$

$$p_+ = p_+(\Omega), p_- = p_-(\Omega), \Omega_\infty = \{x \in \Omega : p(x) = \infty\}.$$

Definition 2.1 *The number*

$$\rho_{p(\cdot), \Omega}(f) = \int_{\Omega/\Omega_\infty} |f(x)|^{p(x)} dx + \|f\|_{L_\infty(\Omega_\infty)}$$

is called the modular of the measurable function $f : \Omega \rightarrow R$ with respect to $p(x)$, where $\|f\|_{L_\infty(\Omega_\infty)} = \operatorname{ess\,sup}_{x \in \Omega_\infty} |f(x)|$.

Denote by $L_{p(\cdot)}(\Omega)$ the set of all measurable functions $f : \Omega \rightarrow R$ such that $\rho_{p(\cdot), \Omega}(f/\lambda) < +\infty$ for some $\lambda > 0$. $L_{p(\cdot)}(\Omega)$ is a Banach space with respect to the norm

$$\|f\|_{L_{p(\cdot)}(\Omega)} = \inf \{ \lambda > 0 : \rho_{p(\cdot), \Omega}(f/\lambda) \leq 1 \}.$$

If $p(x) = p$, then $L_{p(\cdot)}(\Omega)$ coincides with classical Lebesgue space $L_p(\Omega)$, i. e. if $p < +\infty$

$$\|f\|_{L_p(\Omega)} = \inf \{ \lambda > 0 : \rho_{p, \Omega}(f/\lambda) \leq 1 \}$$

and

$$\|f\|_{L_\infty(\Omega)} = \inf \{ \lambda > 0 : \rho_{\infty, \Omega}(f/\lambda) \leq 1 \}.$$

Let $q(x)$, $q(x) \geq 1$ be a measurable functions on Ω , and $T \subset R^n$ be a measurable set. By $L_{q(\cdot)}(\Omega, L_{p(\cdot)}(T))$ denote the space of measurable functions on $\Omega \times T$ such that for almost all $x \in \Omega$ $f(x, \cdot) \in L_{p(x)}(T)$ and $\|f(x; \cdot)\|_{L_{p(x)}(T)} \in L_{q(\cdot)}(\Omega)$.

Denote by $l_{p(\cdot)}(\Omega)$ the set of all sequences $\{c_k(x)\}_{k \in N}$ of measurable functions $c_k(x)$ on Ω such that

$$\sum_{k=1}^{\infty} \int_{\Omega} |c_k(x)|^{p(x)} dx < +\infty.$$

Let $l_{p(\cdot), p(\cdot)-2}(\Omega)$ be the set of sequences $\{c_k(x)\}_{k \in N}$ of measurable functions on Ω for which

$$\sum_{k=1}^{\infty} \int_{\Omega} k^{p(x)-2} |c_k(x)|^{p(x)} dx < +\infty.$$

Equipped with

$$\|\{c_k\}\|_{l_{p(\cdot)}(\Omega)} = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \int_{\Omega} \left(\frac{|c_k(x)|}{\lambda} \right)^{p(x)} \leq 1 \right\}$$

and

$$\|\{c_k\}\|_{l_{p(\cdot), p(\cdot)-2}(\Omega)} = \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \int_{\Omega} k^{p(x)-2} \left(\frac{|c_k(x)|}{\lambda} \right)^{p(x)} \leq 1 \right\}.$$

$l_{p(\cdot)}(\Omega)$ and $l_{p(\cdot), p(\cdot)-2}(\Omega)$ are Banach spaces.

Let $\{\varphi_n(t)\}_{n \in \mathbb{N}}$ be an orthonormal system of functions, defined in $[c, d]$ such that for each n $|\varphi_n(t)| \leq M$, a.e. in $[c, d]$, here M is a constant. We will need the following theorems (see [29, 38]).

Theorem 2.2 [Riesz] Let $1 < p \leq 2$.

1) Let $f \in L_p(c, d)$, and $c_k = \int_c^d f(t)\varphi_k(t)dt$. Then

$$\|\{c_k\}\|_{l_q} \leq M^{\frac{2}{p}-1} \|f\|_{L_p}, q = \frac{p}{p-1}.$$

2) Let $\{c_k\} \in l_p$. Then there exists $f \in L_q(c, d)$, $q = \frac{p}{p-1}$, for which $c_k = \int_c^d f(t)\varphi_k(t)dt$, and

$$\|f\|_{L_q} \leq M^{\frac{2}{p}-1} \|\{c_k\}\|_{l_p}.$$

Theorem 2.3 [Paley] 1) Let $f \in L_p(c, d)$, $1 < p \leq 2$, and $c_k = \int_c^d f(t)\varphi_k(t)dt$. Then

$$\left(\sum_{k=1}^{\infty} |c_k|^p k^{p-2} \right)^{\frac{1}{p}} \leq \frac{A}{p-1} M^{\frac{2-p}{p}} \|f\|_{L_p}.$$

2) Let $\{c_k\}$ be a sequence, such that: $\sum_{k=1}^{\infty} |c_k|^q k^{q-2} < +\infty$, $2 \leq q$. Then there exists $f \in L_q(c, d)$ for

which $c_k = \int_c^d f(t)\varphi_k(t)dt$, and

$$\|f\|_{L_q} \leq AqM^{\frac{q-2}{q}} \left(\sum_{k=1}^{\infty} |c_k|^q k^{q-2} \right)^{\frac{1}{q}}$$

(the constant A is independent on f).

3. A generalization of Riesz and Paley’s theorems

Consider the space $L_{q(\cdot)}(\Omega, L_{p(\cdot)}(T))$. The following results generalize Theorem 2.2 and Theorem 2.3.

Theorem 3.1 1) If $f \in L_{q(\cdot)}((a, b), L_{p(\cdot)}(c, d))$, $1 < p_- \leq p(x) \leq 2$, and $c_k(x) = \int_c^d f(x; t)\varphi_k(t)dt$, $k \in N$.

Then $\{c_k\} \in l_{q(\cdot)}(a, b)$, $q(x) = \frac{p(x)}{p(x)-1}$ and

$$\|\{c_k\}\|_{l_{q(\cdot)}(a;b)} \leq M_1(p) \|f\|_{L_{q(\cdot)}((a,b),L_{p(\cdot)}(c,d))}.$$

2) Let $\{c_k\} \in l_{p(x)}(a, b)$, $1 < p(x) \leq 2$. Then there exists $f \in L_{p(\cdot)}((a, b), L_{q(\cdot)}(c, d))$, $q(x) = \frac{p(x)}{p(x)-1}$, for which $c_k(x) = \int_c^d f(x; t)\varphi_k(t)dt$, $k \in N$, and

$$\|f\|_{L_{p(\cdot)}((a,b),L_{q(x)}(c,d))} \leq M_1(p) \|\{c_k\}\|_{l_{p(\cdot)}(a,b)},$$

where $M_1(p) = \max \left\{ M^{\frac{2}{p_-}-1}, M^{\frac{2}{p_+}-1} \right\}$.

Proof 1) Let $f \in L_{q(\cdot)}((a, b), L_{p(\cdot)}(c, d))$. Then for almost all $x \in [a, b]$ $f(x; \cdot) \in L_{p(x)}(c, d)$. By Theorem 2.2, for almost all $x \in [a, b]$, $\sum_{k=1}^{\infty} |c_k(x)|^{q(x)} < +\infty$ and

$$\left(\sum_{k=1}^{\infty} |c_k(x)|^{q(x)} \right)^{\frac{1}{q(x)}} \leq M^{\frac{2}{p(x)}-1} \left(\int_c^d |f(x; t)|^{p(x)} dt \right)^{\frac{1}{p(x)}}. \tag{3.1}$$

From (3.1) we get

$$\sum_{k=1}^{\infty} \left(\frac{|c_k(x)|}{\lambda} \right)^{q(x)} \leq \left(\frac{M_1(p) \|f(x; \cdot)\|_{L_{p(x)}(c,d)}}{\lambda} \right)^{q(x)}, \tag{3.2}$$

for $\forall \lambda > 0$.

Integrating (3.2) over $[a, b]$, we get

$$\sum_{k=1}^{\infty} \int_a^b \left(\frac{|c_k(x)|}{\lambda} \right)^{q(x)} dx \leq \int_a^b \left(\frac{M_1(p) \|f(x; \cdot)\|_{L_{p(x)}(c,d)}}{\lambda} \right)^{q(x)} dx.$$

Therefore,

$$\begin{aligned} & \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \int_a^b \left(\frac{|c_k(x)|}{\lambda} \right)^{q(x)} dx \leq 1 \right\} \leq \\ & \leq \inf \left\{ \lambda > 0 : \int_a^b \left(\frac{M_1(p) \|f(x; \cdot)\|_{L_{p(x)}(c,d)}}{\lambda} \right)^{q(x)} dx \leq 1 \right\}. \end{aligned}$$

Consequently,

$$\|\{c_k\}\|_{L_{q(\cdot)}(a;b)} \leq \|M_1(p)f\|_{L_{q(\cdot)}((a,b),L_{p(\cdot)}(c,d))} = M_1(p) \|f\|_{L_{q(\cdot)}((a,b),L_{p(\cdot)}(c,d))}.$$

2) Let $\sum_{k=1}^{\infty} \int_a^b |c_k(x)|^{p(x)} dx < +\infty$. By the Monotone convergence theorem we have $\sum_{k=1}^{\infty} |c_k(x)|^{p(x)} < +\infty$, for almost all $x \in [a, b]$. Then by Theorem 2.2, for almost all $x \in [a, b]$ there exists a function $f(x; \cdot) \in L_{q(x)}(c, d)$, for which $c_k(x) = \int_c^d f(x; t)\varphi_k(t)dt$, $k \in N$, and

$$\|f(x; \cdot)\|_{L_{q(x)}(c,d)} \leq M^{\frac{2}{p(x)}-1} \left(\sum_{k=1}^{\infty} |c_k(x)|^{p(x)} \right)^{\frac{1}{p(x)}} \leq M_1(p) \left(\sum_{k=1}^{\infty} |c_k(x)|^{p(x)} \right)^{\frac{1}{p(x)}}.$$

In what follows, we get

$$\|f(x, \cdot)\|_{L_{q(x)}(c,d)}^{p(x)} \leq \sum_{k=1}^{\infty} (M_1(p) |c_k(x)|)^{p(x)}. \tag{3.3}$$

Since $\left\| f(x; \cdot) - \sum_{k=1}^n c_k(x)\varphi_k(\cdot) \right\|_{L_{q(x)}(c,d)} \rightarrow 0$, as $n \rightarrow \infty$ the function f is measurable on $[a, b] \times [c, d]$.

From (3.3) for $\forall \lambda > 0$ we get

$$\left(\frac{\|f(x, \cdot)\|_{L_{q(x)}(c,d)}}{\lambda} \right)^{p(x)} \leq \sum_{k=1}^{\infty} \left(\frac{M_1(p) |c_k(x)|}{\lambda} \right)^{p(x)}. \tag{3.4}$$

Integrating of (3.4) over the segment $[a, b]$ gives us

$$\int_a^b \left(\frac{\|f(x, \cdot)\|_{L_{q(x)}(c,d)}}{\lambda} \right)^{p(x)} dx \leq \sum_{k=1}^{\infty} \int_a^b \left(\frac{M_1(p) |c_k(x)|}{\lambda} \right)^{p(x)} dx.$$

Then

$$\begin{aligned} & \inf \left\{ \lambda > 0 : \int_a^b \left(\frac{\|f(x, \cdot)\|_{L_{q(x)}(c,d)}}{\lambda} \right)^{p(x)} dx \leq 1 \right\} \leq \\ & \leq \inf \left\{ \lambda > 0 : \sum_{k=1}^{\infty} \int_a^b \left(\frac{M_1(p) |c_k(x)|}{\lambda} \right)^{p(x)} dx \leq 1 \right\}. \end{aligned}$$

Therefore

$$\|f\|_{L_{p(\cdot)}((a;b),L_{q(\cdot)}(c,d))} \leq \|M_1(p) \{c_k\}\|_{L_{p(\cdot)}(a,b)} = M_1(p) \|\{c_k\}\|_{L_{p(\cdot)}(a,b)}.$$

□

Theorem 3.2 1) Let $f \in L_{p(\cdot)}((a, b), L_{p(\cdot)}(c, d))$, $1 < p_- \leq p(x) \leq 2$, and $c_k(x) = \int_c^d f(x; t)\varphi_k(t)dt$, $k \in N$.

Then $\{c_k\} \in l_{p(\cdot), p(\cdot)-2}(a, b)$ and

$$\|\{c_k\}\|_{l_{p(\cdot), p(\cdot)-2}(a, b)} \leq \frac{AM_1(p)}{p_- - 1} \|f\|_{L_{p(\cdot)}((a, b), L_{p(\cdot)}(c, d))}.$$

2) Let $\{c_k\} \in l_{q(\cdot), q(\cdot)-2}(a, b)$, $2 \leq q(\cdot) \leq q_+ < \infty$. Then there exists $f \in L_{q(\cdot)}((a, b), L_{q(\cdot)}(c, d))$ for which $c_k(x) = \int_c^d f(x; t)\varphi_k(t)dt$, $k \in N$, and

$$\|f\|_{L_{q(\cdot)}((a, b), L_{q(\cdot)}(c, d))} \leq Aq_+M_2(q) \|\{c_k\}\|_{l_{q(\cdot), q(\cdot)-2}(a, b)}, \tag{3.5}$$

where $M_2(q) = \max \left\{ M^{1-\frac{2}{q_-}}, M^{1-\frac{2}{q_+}} \right\}$.

Proof 1) If $f \in L_{p(\cdot)}((a, b), L_{p(\cdot)}(c, d))$, then for almost all $x \in [a, b]$ $f(x; \cdot) \in L_{p(\cdot)}(c, d)$. By Theorem 2.3,

$\sum_{k=1}^{\infty} k^{p(x)-2} |c_k(x)|^{p(x)} < +\infty$ and

$$\begin{aligned} \sum_{k=1}^{\infty} k^{p(x)-2} |c_k(x)|^{p(x)} &\leq \left(\frac{A}{p(x) - 1} M^{\frac{2}{p(x)} - 1} \right)^{p(x)} \|f\|_{L_{p(x)}(c, d)}^{p(x)} \leq \\ &\leq \left(\frac{A}{p_- - 1} M_1(p) \right)^{p(x)} \|f\|_{L_{p(x)}(c, d)}^{p(x)}. \end{aligned} \tag{3.6}$$

Take arbitrary $\lambda > 0$. Then from (3.6) it follows that,

$$\sum_{k=1}^{\infty} \int_a^b k^{p(x)-2} \left(\frac{|c_k(x)|}{\lambda} \right)^{p(x)} dx \leq \int_a^b \left(\frac{AM_1(p) \|f\|_{L_{p(x)}(c, d)}}{\lambda(p_- - 1)} \right)^{p(x)} dx.$$

Thus, we obtain

$$\begin{aligned} \|\{c_k\}\|_{l_{p(\cdot), p(\cdot)-2}(a, b)} &\leq \left\| \frac{AM_1(p)}{p_- - 1} f \right\|_{L_{p(\cdot)}((a, b), L_{p(\cdot)}(c, d))} = \\ &= \frac{AM_1(p)}{p_- - 1} \|f\|_{L_{p(\cdot)}((a, b), L_{p(\cdot)}(c, d))}. \end{aligned}$$

2) Let $\{c_k(x)\}_{k \in N}$ is a sequence such that: $\sum_{k=1}^{\infty} \int_a^b k^{q(x)-2} |c_k(x)|^{q(x)} dx < +\infty$. By the monotone convergence for almost all $x \in [a, b]$, $\sum_{k=1}^{\infty} k^{q(x)-2} |c_k(x)|^{q(x)} < +\infty$. Therefore by Theorem 2.3, for almost

all $x \in [a, b]$, there exists a function $f(x; \cdot) \in L_{q(x)}(c, d)$ such that $f(x; t) = \sum_{k=1}^{\infty} c_k(x)\varphi_k(t)$ and

$$\|f\|_{L_{q(x)}(c, d)}^{q(x)} \leq (Aq(x))^{q(x)} M^{q(x)-2} \sum_{k=1}^{\infty} k^{q(x)-2} |c_k(x)|^{q(x)}$$

$$\leq (Aq_+M_2(q))^{q(x)} \sum_{k=1}^{\infty} k^{q(x)-2} |c_k(x)|^{q(x)}. \tag{3.7}$$

Thus, for arbitrary $\lambda > 0$ dividing the both-hand sides of (3.7) by $\lambda^{q(x)}$, and integrating over the segment $[a, b]$ we get

$$\int_a^b \left(\frac{\|f\|_{L_{q(x)}(c,d)}}{\lambda} \right)^{q(x)} dx \leq \sum_{k=1}^{\infty} \int_a^b k^{q(x)-2} \left(Aq_+M_2(q) \frac{|c_k(x)|}{\lambda} \right)^{q(x)} dx,$$

i.e. (3.5) is valid. □

Now consider the case of the system of exponents $\{e^{int}\}_{n \in Z}, t \in [-\pi, \pi]$.

Theorem 3.3 *Let $f \in L_{q(\cdot)}((a, b), W_{p(\cdot)}^{m+1}(-\pi, \pi))$, where $1 < p_- \leq p(x) \leq 2$, $q(x) = \frac{p(x)}{p(x)-1}$, and $\frac{\partial^k f(x; -\pi)}{\partial t^k} = \frac{\partial^k f(x; \pi)}{\partial t^k}$ ($k = 0, \dots, m$) for almost all $x \in [a, b]$ and $c_n(x) = \int_{-\pi}^{\pi} f(x; t)e^{-int} dt, n \in Z$. Then $\varphi(x) =$*

$$\sum_{n=-\infty}^{+\infty} |n^m c_n(x)| \in L_{q(\cdot)}(a, b) \text{ and}$$

$$\|\varphi\|_{L_{q(\cdot)}(a,b)} \leq \alpha(p) \left\| \frac{\partial^{m+1} f}{\partial t^{m+1}} \right\|_{L_{q(\cdot)}((a,b), L_{p(\cdot)}(-\pi, \pi))}, \tag{3.8}$$

where $\alpha^{p-}(p) = \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{|n|^{p-}}$.

Proof We have

$$\begin{aligned} c_n(x) &= \int_{-\pi}^{\pi} f(x; t)e^{-int} dt = \\ &= -\frac{1}{in} f(x; t)e^{-int} \Big|_{t=-\pi}^{t=\pi} + \frac{1}{in} \int_{-\pi}^{\pi} \frac{\partial f(x; t)}{\partial t} e^{-int} dt = \\ &= \frac{1}{in} \int_{-\pi}^{\pi} \frac{\partial f(x; t)}{\partial t} e^{-int} dt = -\frac{1}{(in)^2} \frac{\partial f}{\partial t} e^{-int} \Big|_{t=-\pi}^{t=\pi} + \frac{1}{(in)^2} \int_{-\pi}^{\pi} \frac{\partial^2 f(x; t)}{\partial t^2} e^{-int} dt = \\ &= \dots = \frac{1}{(in)^{m+1}} \int_{-\pi}^{\pi} \frac{\partial^{m+1} f(x; t)}{\partial t^{m+1}} e^{-int} dt. \end{aligned}$$

So, $c_n(x) = \frac{c_{n,m+1}(x)}{(in)^{m+1}}$, where

$$c_{n,m+1}(x) = \int_{-\pi}^{\pi} \frac{\partial^{m+1} f(x; t)}{\partial t^{m+1}} e^{-int} dt,$$

i.e. $\{c_{n,m+1}(x)\}_{n \in \mathbb{Z}}$ is the sequence of the Fourier coefficients of the function $\frac{\partial^k f(x,t)}{\partial t^k}$ by $\{e^{int}\}_{n \in \mathbb{Z}}$. By Theorem 3.1 $\{c_{n,m+1}\} \in l_{q(\cdot)}(a, b)$ and

$$\|\{c_{n,m+1}\}\|_{l_{q(\cdot)}(a,b)} \leq \left\| \frac{\partial^{m+1} f}{\partial t^{m+1}} \right\|_{L_{q(\cdot)}((a,b), L_{p(\cdot)}(-\pi, \pi))}. \tag{3.9}$$

Thus, using the Hölder’s inequality, we get

$$\begin{aligned} \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} |n^m c_n(x)| &= \sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{|c_{n,m+1}(x)|}{|n|} \leq \\ &\leq \left(\sum_{\substack{n=-\infty \\ n \neq 0}}^{+\infty} \frac{1}{|n|^{p(x)}} \right)^{\frac{1}{p(x)}} \left(\sum_{n=-\infty}^{+\infty} |c_{n,m+1}(x)|^{q(x)} \right)^{\frac{1}{q(x)}} \leq \\ &\leq \alpha(p) \left(\sum_{n=-\infty}^{+\infty} |c_{n,m+1}(x)|^{q(x)} \right)^{\frac{1}{q(x)}}. \end{aligned}$$

Hence, we have

$$\varphi(x)^{q(x)} = \left(\sum_{n=-\infty}^{+\infty} |n^m c_n(x)| \right)^{q(x)} \leq \sum_{k=1}^{\infty} (\alpha(p) |c_{n,m+1}(x)|)^{q(x)}.$$

We get

$$\int_a^b \left(\frac{\varphi(x)}{\lambda} \right)^{q(x)} dx \leq \sum_{k=1}^{\infty} \int_a^b \left(\frac{\alpha(p) |c_{n,m+1}(x)|}{\lambda} \right)^{q(x)} dx,$$

for $\forall \lambda > 0$. Therefore $\|\varphi\|_{L_{q(\cdot)}(a,b)} \leq \alpha(p) \|\{c_{n,m+1}\}\|_{l_{q(\cdot)}(a,b)}$. Taking into account (3.9), we get (3.8). \square

Theorem 3.4 Let $f \in L_{q(\cdot)}((a, b), W_{p(\cdot)}^m(-\pi, \pi))$, where $1 < p_- \leq p(x) \leq 2$, $m \in \mathbb{N}$, $q(x) = \frac{p(x)}{p(x)-1}$ and for almost all $x \in [a, b]$ $\frac{\partial^k f(x; -\pi)}{\partial t^k} = \frac{\partial^k f(x; \pi)}{\partial t^k}$, $k = 0, \dots, m - 1$, and $c_n(x) = \int_{-\pi}^{\pi} f(x; t) e^{-int} dt$, $n \in \mathbb{Z}$. Then $\{n^m c_n\} \in l_{q(\cdot)}(a, b)$ and

$$\|\{n^m c_n\}\|_{l_{q(\cdot)}(a,b)} \leq \left\| \frac{\partial^m f}{\partial t^m} \right\|_{L_{q(\cdot)}((a,b), L_{p(\cdot)}(-\pi, \pi))}.$$

Proof Taking into account $\frac{\partial^k f(x; -\pi)}{\partial t^k} = \frac{\partial^k f(x; \pi)}{\partial t^k}$, $k = 0, \dots, m - 1$, almost everywhere on $[a, b]$, we get

$$c_n(x) = \int_{-\pi}^{\pi} f(x; t) e^{-int} dt =$$

$$\begin{aligned}
&= -\frac{1}{in} f(x; t) e^{-int} \Big|_{t=-\pi}^{t=\pi} + \frac{1}{in} \int_{-\pi}^{\pi} \frac{\partial f(x; t)}{\partial t} e^{-int} dt = \\
&= \frac{1}{in} \int_{-\pi}^{\pi} \frac{\partial f(x; t)}{\partial t} e^{-int} dt = -\frac{1}{(in)^2} \frac{\partial f}{\partial t} e^{-int} \Big|_{t=-\pi}^{t=\pi} + \\
&+ \frac{1}{(in)^2} \int_{-\pi}^{\pi} \frac{\partial^2 f(x; t)}{\partial t^2} e^{-int} dt = \dots = \frac{1}{(in)^m} \int_{-\pi}^{\pi} \frac{\partial^m f(x; t)}{\partial t^m} e^{-int} dt = \frac{c_{n,m}(x)}{(in)^m}.
\end{aligned}$$

Since $\frac{\partial^m f}{\partial t^m} \in L_{q(\cdot)}((a, b), L_{p(\cdot)}(-\pi, \pi))$, then by Theorem 3.1 we get

$$\|\{c_{n,m}\}\|_{l_{q(\cdot)}(a,b)} \leq \left\| \frac{\partial^m f}{\partial t^m} \right\|_{L_{q(\cdot)}((a,b), L_{p(\cdot)}(-\pi,\pi))}.$$

From the relation $c_{n,m}(x) = (in)^m c_n(x)$ we have

$$\|\{n^m c_n\}\|_{l_{q(\cdot)}(a,b)} = \|\{c_{n,m}\}\|_{l_{q(\cdot)}(a,b)}.$$

□

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