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On $\mathcal{X}$-Gorenstein projective dimensions and precovers

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Abstract: For a class of R-modules $\mathcal{X}$ containing all projective R-modules, the $\mathcal{X}$-Gorenstein projective R-modules vary from projective to Gorenstein projective R-modules. We characterize the rings over which the left global $\mathcal{X}$-Gorenstein projective dimensions are finite. If further $\mathcal{Y}$ contains all injective R-modules, we show the existence of a new left global Gorenstein dimension of R with respect to $\mathcal{X}$ and $\mathcal{Y}$ satisfying proper conditions. As an application we characterize Ding-Chen rings by this new global Gorenstein dimension and show the existence of Ding-Chen rings with infinite global Gorenstein dimension. We also show the existence of $\mathcal{X}$-Gorenstein projective precovers for a large class of rings.

Key words: Ding-Chen rings, Ding projective (injective) modules, global Gorenstein dimensions, precovers, $\mathcal{X}$-Gorenstein projective modules

1. Introduction

Throughout this paper, R denotes a unitary associative ring and all modules are left R-modules if not specified otherwise. As usual, we use $\mathcal{P}$, $\mathcal{I}$ and $\mathcal{F}$ to denote respectively the classes of all projective, injective and flat R-modules, and we use $pd(M)$, $id(M)$ and $fd(M)$ to denote respectively the projective, injective and flat dimension of an R-module M. Let $\mathcal{X}$ be a class of R-modules that contains $\mathcal{P}$ and $\mathcal{Y}$ a class of R-modules containing $\mathcal{I}$. To provide a unified approach to the study of projective (injective) and Gorenstein projective (injective) R-modules (please cf. [2, 6] for the original definition) and their homological dimension theory, the authors of [1] defined and studied the modules given by the following definition:

Definition 1.1 An R-module M is called $\mathcal{X}$*-Gorenstein projective*, if there exists an exact sequence of projective R-modules $\mathbb{P} = \cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$ such that $M = \mathrm{Im}(P_0 \to P^0)$ and $\mathrm{Hom}_R(\mathbb{P}, F)$ is exact whenever $F \in \mathcal{X}$. The sequence $\mathbb{P}$ is called an $\mathcal{X}$*-complete projective resolution*. We denote by $\mathcal{GP}_{\mathcal{X}}(R)$ the class of all $\mathcal{X}$-Gorenstein projective R-modules. Dually we can define the $\mathcal{Y}$*-Gorenstein injective* R-modules and $\mathcal{GI}_{\mathcal{Y}}(R)$.

In fact, let $\mathcal{X} = \mathcal{Y}$ be the class ${}_R\mathrm{Mod}$ of all left R-modules, then $\mathcal{GP}_{\mathcal{X}}(R) = \mathcal{P}$ and $\mathcal{GI}_{\mathcal{Y}}(R) = \mathcal{I}$. Let $\mathcal{X} = \mathcal{P}$ and $\mathcal{Y} = \mathcal{I}$, then $\mathcal{GP}_{\mathcal{X}}(R)$ and $\mathcal{GI}_{\mathcal{Y}}(R)$ become the classes of Gorenstein projective modules (denoted by $\mathcal{GP}(R)$) and Gorenstein injective R-modules (denoted by $\mathcal{GI}(R)$). Furthermore, Let $\mathcal{X} = \mathcal{F}$ and $\mathcal{Y} = \mathcal{FI}$, i.e., the class of all FP-injective R-modules (see Definition 2.17), then $\mathcal{GP}_{\mathcal{X}}(R)$ and $\mathcal{GI}_{\mathcal{Y}}(R)$ become respectively

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the classes of Ding projective and Ding injective R-modules defined in [9]. Plenty of works on these modules can be found in [2, 4, 9, 11–14]. As one can see that, various topics on these relative Gorenstein modules, such as homological dimension theory, (pre)covering and (pre)enveloping theory are studied. However, it is seemingly lacked of a universal approach to carry out all the homological discussions once and for all.

Regarding this, the paper is dedicating to a systematical study of the global homological dimension theory and precovering (preenveloping) theory of these modules. In details, let us denote by $\mathcal{X}$-Gpd(M) the $\mathcal{X}$-Gorenstein projective dimension of M (see the definition at the beginning of section 2). Furthermore we define the *(left) global $\mathcal{X}$-Gorenstein projective dimension* $l.\mathcal{X}$-GPD(R) of R by $l.\mathcal{X}$- GPD(R)=supf{$\mathcal{X}$-Gpd(M)$|M$ is a (left) R-module.}. Also we have the dual definitions (see Notation 2.6). Moreover, given a class $\mathcal{F}$ of R-modules, recall that, a homomorphism $\varphi : F \to M$ is called an *$\mathcal{F}$-precover* of M if $\mathrm{Hom}(F', F) \to \mathrm{Hom}(F', M)$ is surjective for all $F' \in \mathcal{F}$. An *$\mathcal{F}$-preenvelope* of M can be defined dually.

The first main result of Section 2 (see Theorem 2.8) is a characterization of the rings over which the (left) global $\mathcal{X}$-Gorenstein projective dimensions are finite. Partial results of this theorem (for the unexplained notations and definitions, please see the words above Theorem 2.8) can be found in [4, 6, 14].

Theorem 1.2 *(Theorem 2.8) Let R be a ring and $\mathcal{X}$ be a class of R-modules that contains all projective R-modules. Then the following statements are equivalent:*

(1) $l.\mathcal{X}$-GPD$(R) \leq n$.

(2) *Each m-th ($m \geq n$) syzygy in any projective resolution of any module is in $\mathcal{GP}_{\mathcal{X}}(R)$.*

(3) *$id(\mathcal{X}) \leq n$ and $pd(\mathcal{I}) \leq n$.*

(4) *$(\mathcal{GP}_{\mathcal{X}}(R), \check{\mathcal{I}}_n)$ is a hereditary complete cotorsion pair.*

(5) *$(\mathcal{GP}_{\mathcal{X}}(R), \widehat{\mathcal{P}}_n)$ is a hereditary complete cotorsion pair and $\widehat{\mathcal{P}}_n = \check{\mathcal{I}}_n$.*

Note that $id(\mathcal{X})$ and $pd(\mathcal{I})$ are defined respectively as $\sup\{id(X)|X \in \mathcal{X}\}$ and $\sup\{pd(\mathcal{I})|I \in \mathcal{I}\}$. For two classes $\mathcal{X}$ and $\mathcal{Y}$ which satisfy certain conditions, the theorem above and its dual version give rise to the existence of a new global dimension of a ring R via the following result:

Theorem 1.3 *(see Theorem 2.13) Let $\mathcal{X}$ and $\mathcal{Y}$ be respectively projectively resolving and injectively coresolving classes of R-modules. Then $l.\mathcal{X}$-GPD(R) = $l.\mathcal{Y}$-GID(R) =max$\{id(\mathcal{X}), pd(\mathcal{Y})\}$ (interpreted as ∞ if either $id(\mathcal{X})$ or $pd(\mathcal{Y})$ is infinite) whenever one of the following conditions is satisfied:*

(1) *$id(\mathcal{X}) = id(\mathcal{P})$ and $pd(\mathcal{Y}) = pd(\mathcal{I})$.*

(2) *$id(\mathcal{X}) = pd(\mathcal{Y})$.*

(3) *$\mathcal{X}$-$pd(\mathcal{Y}) = \mathcal{Y}$-$id(\mathcal{X})$.*

The common value of the quantities in this theorem is denoted as $l.\mathrm{Ggldim}_{\mathcal{X},\mathcal{Y}}(R)$, and is called the *(left) global Gorenstein dimension with respect to $\mathcal{X}$ and $\mathcal{Y}$* of R. We point out that this dimension becomes the left global dimension or the left global Gorenstein dimension of a ring R (cf. [2] for the definition) by taking different $\mathcal{X}$ and $\mathcal{Y}$ (cf. Remark 2.16). They further imply a sufficient condition for that the functor $\mathrm{Hom}_R(-,-)$ is right balanced by $\mathcal{GP}_{\mathcal{X}}(R) \times \mathcal{GI}_{\mathcal{Y}}(R)$ (Corollary 2.11).

As an important application, we study the global dimensions related to Ding projective and injective modules. We prove the existence of $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R)$ for certain ring R (Proposition 2.22), and we show that if it exists, then it coincides with the left global Gorenstein dimension of R (Theorem 2.19). At last we give a characterization of a Ding-Chen ring (see Definition 2.18) or commutative coherent ring by $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R)$. This discussion gives many of the results in [4, 11, 13]. As an important conclusion we get the following result:

Corollary 1.4 *(see Corollary 2.25) There exists a Ding-Chen ring with infinite global Gorenstein dimension.*

This shows a difference between Ding-Chen rings and n-Gorenstein rings, since for any n-Gorenstein ring, we always have $l.\mathrm{Ggldim}(R) = id({}_RR) = n$, but for a Ding-Chen ring R it may hold $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R) > \mathcal{FI}$-$id({}_RR)$.

The main result of Section 3 is the following result concerning the existence of the $\mathcal{GP}_{\mathcal{X}}(R)$-precovers (see Theorem 3.2 and the dual result Theorem 3.3). It derives many well-known results as corollaries when applying to the context of [4, 6, 13].

Theorem 1.5 *Let R be a ring, $\mathcal{X}$ a class of R-modules that contains all projective R-modules. If every module in $\mathcal{X}$ has finite injective dimension, then every R-module has a $\mathcal{GP}_{\mathcal{X}}(R)$-precover.*

2. $\mathcal{X}$-Gorenstein projective dimensions

Let R be a ring and M any R-module, for a given class $\mathcal{W}$ of R-modules, M is said to have a left $\mathcal{W}$*-resolution* if there exists an exact sequence $\cdots \to W_n \to \cdots \to W_1 \to W_0 \to M \to 0$ with each $W_i \in \mathcal{W}$ for $i \geq 0$, and we call each $K_i = \mathrm{Ker}(W_i \to W_{i-1})$ the i-th *syzygy* of this left $\mathcal{W}$-resolution. We further define $\mathcal{W}$-pd(M) as the minimum number n (if it exists) such that there exists a left $\mathcal{W}$-resolution for M with all $W_i = 0$ for all $i > n$, otherwise we set it equal to ∞. Similarly we can define the *right* $\mathcal{W}$*-resolution* of M, the i-th *cosyzygy* and the dimension $\mathcal{W}$-id(M). For convenience, we use $\widehat{\mathcal{W}}$ (resp. $\check{\mathcal{W}}$) to denote the class of R-modules with finite left (resp. right) $\mathcal{W}$-resolutions. In particular, for given two certain classes $\mathcal{X}$ and $\mathcal{Y}$, when $\mathcal{W} = \mathcal{GP}_{\mathcal{X}}(R)$ (resp. $\mathcal{W} = \mathcal{GI}_{\mathcal{Y}}(R)$), for clarity, we write $\mathcal{W}$-pd(M) (resp. $\mathcal{W}$-id(M)) by $\mathcal{X}$-Gpd(M) (resp. $\mathcal{Y}$-Gid(M)). For a class of R-modules $\mathcal{X}$, it is also convenient to use $\mathcal{W}$-pd$(\mathcal{X})$ to denote $\sup\{\mathcal{W}\text{-pd}(X) | X \in \mathcal{X}\}$, similarly the meaning of $\mathcal{W}$-id$(\mathcal{X})$ is clear. In particular, the notations $pd(\mathcal{X})$, $id(\mathcal{X})$ and $fd(\mathcal{X})$ are also clear.

Let $\mathcal{X}$ be a class of R-modules and $\mathcal{H}$ some subclass of $\mathcal{X}$. We recall that, if for any R-module $X \in \mathcal{X}$ there exists a short exact sequence $0 \to X \to H \to X' \to 0$ with $H \in \mathcal{H}$ and $X' \in \mathcal{X}$, and it holds that $\mathrm{Ext}^1_R(\mathcal{X}, \mathcal{H}) = 0$, then the class $\mathcal{H}$ is called an Ext-*injective cogenerator* of $\mathcal{X}$ (see also, eg., [16]). Note that by definition $\mathcal{P}$ is an Ext-injective cogenerator of $\mathcal{GP}_{\mathcal{X}}(R)$.

The following is a direct result of [16, Theorem 3.1] and [1, Theorem 2.3], which will be frequently used in the sequel.

Proposition 2.1 *Let R be a ring and $\mathcal{X}$ a class of R-modules that contains all projective R-modules, M an R-module with $\mathcal{X}$-Gpd$(M) < \infty$. Then the following statements are equivalent:*

(1) $\mathcal{X}$-Gpd$(M) \leq n$.

(2) *Each m-th $(m \geq n)$ syzygy in any projective resolution of M is in $\mathcal{GP}_{\mathcal{X}}(R)$.*

(3) *Each m-th $(m \geq n)$ syzygy in any $\mathcal{GP}_{\mathcal{X}}(R)$ resolution of M is in $\mathcal{GP}_{\mathcal{X}}(R)$.*

(4) $\mathrm{Ext}^m_R(M, X) = 0$ *for all $m > n$ and all $X \in \mathcal{P}$.*

(5) $\mathrm{Ext}^m_R(M, H) = 0$ *for all $m > n$ and all $H \in \widehat{\mathcal{P}}(R)$.*

(6) *There exists a short exact sequence* $0 \to M \to W \to X \to 0$ *with* $pd(W) \leq n$ *and* $X \in \mathcal{GP}_{\mathcal{X}}(R)$.

(7) *There exists a short exact sequence* $0 \to K \to X \to M \to 0$ *with* $pd(K) = n-1$ *and* $X \in \mathcal{GP}_{\mathcal{X}}(R)$.

(8) M *admits a surjective* $\mathcal{GP}_{\mathcal{X}}(R)$*-precover* $\varphi : X \to M$ *with* $K = ker\varphi$ *satisfying* $pd(K) \leq n-1$.

We also have:

Proposition 2.2 *Let* R *be a ring,* M *an* R*-module, and let* $\mathcal{X}$ *be a class of* R*-modules containing all projective* R*-modules. If it holds either* $\mathcal{X}$*-pd*$(M) < \infty$ *or* $id(M) < \infty$*, then* $\mathcal{X}$-Gpd$(M) = pd(M)$.

Proof Suppose first $\mathcal{X}$-$pd(M) = n < \infty$. Apparently we have $\mathcal{X}$-Gpd$(M) \leq pd(M)$. For the inverse inequality, suppose that $\mathcal{X}$-Gpd$(M) = n < \infty$, we shall show that $pd(M) \leq n$. By Proposition 2.1(7) we have a short exact sequence $0 \to K \to X \to M \to 0$ with $pd(K) = n-1$ and $X \in \mathcal{GP}_{\mathcal{X}}(R)$. Thus we obtain another short exact sequence: $0 \to X \to P \to H \to 0$ with P projective and $H \in \mathcal{GP}_{\mathcal{X}}(R)$. Consider the push out diagram of $X \to M$ and $X \to P$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & K \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & K \arrow[r] & P \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & H \arrow[r, equal] \arrow[d] & H \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

It follows that $pd(L) \leq n$ from the second row of the diagram, while the last column splits follows from $\mathrm{Ext}^1_R(H, M) = 0$ since $\mathcal{X}$-pd$(M) = n$, one has $pd(M) \leq n$, as needed.

Now suppose $id(M) = n < \infty$. If we still have $pd(M) < \infty$, then we are through by the first part of the proof. So assume $pd(M) = \infty$, and we need to show $\mathcal{X}$-Gpd$(M) = \infty$. Suppose it is not the case, say $\mathcal{X}$-Gpd$(M) = m$, then as above once again one gets a short exact sequence $0 \to M \to L \to H \to 0$ with $pd(L) = n$ and $H \in \mathcal{GP}_{\mathcal{X}}(R)$. Thus we have a complete projective resolution of H: $\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$. Set $N = Ker(P^k \to P^{k+1})$ for some $k \geq n$, we have $\mathrm{Ext}^1_R(H, M) \cong \mathrm{Ext}^{k+1}_R(N, M) = 0$ since $id(M) = n$. It follows that M is a direct summand of L, thus we have $pd(M) \leq pd(L) \leq m$, and it contradicts. So we have $\mathcal{X}$-Gpd$(M) = pd(M) = \infty$, as desired. □

For an R-module M, it is interesting to ask how the two dimensions $\mathcal{X}$-Gpd(M) and $\mathcal{X}$-$pd(M)$ are related. The following result gives a partial answer. Let us recall that, a class $\mathcal{X}$ of R-modules is called *projectively resolving* if $\mathcal{X}$ contains all projective R-modules, and for every short exact sequence $0 \to X' \to X \to X'' \to 0$ with $X'' \in \mathcal{X}$ the conditions $X' \in \mathcal{X}$ and $X \in \mathcal{X}$ are equivalent. The *injectively coresolving* class of R-modules can be defined dually. It is easy to see that the class $\mathcal{F}$ of flat R-modules is projectively resolving. Furthermore, if $\mathcal{X}$ is closed under direct sums then the class $\mathcal{GP}_{\mathcal{X}}(R)$ is projectively resolving and closed under arbitrary direct sums and under direct summands by an almost the same proof as that of [12, Theorem 2.5].

Corollary 2.3 *Let M be an R-module and $\mathcal{X}$ a projectively resolving class of R-modules. If either $pd(M) < \infty$ or $id(M) < \infty$ holds, then we have $\mathcal{X}$-$pd(M) \leq \mathcal{X}$-$\mathrm{Gpd}(M)$ and the equality holds if and only if $\widehat{\mathcal{P}} \cap \mathcal{X} = \mathcal{P}$.*

Proof The first assertion follows from the obvious inequality $\mathcal{X}$-$pd(M) \leq pd(M)$ and Proposition 2.2, while the second assertion follows by [16, Propositon 2.3(2)]. □

It is obvious that the class $\mathcal{GP}_{\mathcal{X}}(R)$ is usually a subclass of $\mathcal{GP}(R)$ by our definition. On the other hand, let $\mathcal{X} = \mathcal{GP}(R)$, M be an R-module such that $pd(M) < \infty$, by Corollary 2.3 we get that $pd(M) = \mathcal{GP}(R)$-$\mathrm{Gpd}(M)$. This particularly implies that $\mathcal{GP}_{\mathcal{X}}(R) = \mathcal{P}$. In fact, more generally we have

Proposition 2.4 *Let R be a ring, and let $\mathcal{X} \subseteq \mathcal{X}'$ be two classes of R-modules which both contain all projective R-modules. If $\mathcal{GP}_{\mathcal{X}}(R) \subseteq \mathcal{X}'$, then $\mathcal{GP}_{\mathcal{X}'}(R) = \mathcal{P}$.*

Proof Only the inclusion $\mathcal{GP}_{\mathcal{X}'}(R) \subseteq \mathcal{P}$ needs to be shown. Let $M \in \mathcal{GP}_{\mathcal{X}'}(R)$ be an R-module. By definition we have a short exact sequence: $0 \to M' \to P \to M \to 0$ with P some projective R-module and $M' \in \mathcal{GP}_{\mathcal{X}'}$. In fact we have $M' \in \mathcal{GP}_{\mathcal{X}}(R) \subseteq \mathcal{X}'$ since $\mathcal{X} \subseteq \mathcal{X}'$. Now it follows from $\mathrm{Ext}^1_R(M, M') = 0$ that the above sequence splits, and the result follows. □

Remark 2.5 By Proposition 2.4 one may ask, for any two class $\mathcal{X}$ and $\mathcal{X}'$ of R-modules which both contain all projective R-modules, in what conditions do we have $\mathcal{GP}_{\mathcal{X}}(R) = \mathcal{GP}_{\mathcal{X}'}(R)$? In the case where $l.\mathcal{X}$-$\mathrm{GPD}(R) \leq n$ or $l.\mathcal{X}'$-$\mathrm{GPD}(R) \leq n$ holds, we will show that the equality holds true if and only if $\widehat{\mathcal{X}}_n = \widehat{\mathcal{X}}'_n = \widehat{\mathcal{P}}_n$ (see Corollary 2.9 below).

We introduce some notations for later use.

Notation 2.6 The usual and well-investigated *(left) finite projective dimension* $l.\mathrm{FPD}(\mathrm{R})$ of R is defined by $l.\mathrm{FPD}(R)$=sup{pd$(M)|M$ is a (left) R-module with finite left projective dimension}, and we define the *(left) global $\mathcal{X}$-Gorenstein projective dimension* $l.\mathcal{X}$-$\mathrm{GPD}(\mathrm{R})$ of R by $l.\mathcal{X}$-$\mathrm{GPD}(R)$=sup{$\mathcal{X}$-$\mathrm{Gpd}(M)|M$ is a (left) R-module}. Similarly we define the *(left) finitistic $\mathcal{X}$-Gorenstein projective dimension* of R by $l.\mathcal{X}$-$\mathrm{FGPD}(R)$=sup{$\mathcal{X}$-$\mathrm{Gpd}(M)|M$ is a (left) R-module with finite left $\mathcal{X}$-Gorenstein projective dimension}. Note that the *(left) finite injective dimension* $\mathrm{FID}(R)$, the *(left) global $\mathcal{Y}$-Gorenstein injective dimension* $l.\mathcal{Y}$-$\mathrm{GID}(R)$ and the *(left) finitistic $\mathcal{Y}$-Gorenstein injective dimension* $l.\mathcal{Y}$-$\mathrm{FGID}(R)$ of R can all be defined dually.

The next result extends [12, Proposition 2.28].

Proposition 2.7 *For any ring R and any class $\mathcal{X}$ of R-modules that contains all projective R-modules, there is an equality $\mathcal{X}$-$\mathrm{FGPD}(R) = \mathrm{FPD}(R)$.*

Proof Clearly we have $l.\mathrm{FPD}(R) \leq l.\mathcal{X}$-$\mathrm{FGPD}(R)$. For inverse inequality, note that for any R-module M with $\mathcal{X}$-$\mathrm{Gpd}(M) = n$, there exists a short exact sequence $0 \to M \to W \to X \to 0$ with $pd(W) = n$ and $X \in \mathcal{GP}_{\mathcal{X}}(R)$ by Proposition 2.1(6). This shows that $l.\mathrm{FPD}(R) \geq l.\mathcal{X}$-$\mathrm{FGPD}(R)$, and it completes the proof. □

Now we are in a position to state the first main reult of this section. But before doing this, we need some notations and definitions that can be found in [10]. Given a class $\mathcal{C}$ of R-modules, we denote by ${}^{\perp}C$ (resp. $C^{\perp}$) the class of R-modules F such that $\mathrm{Ext}^1_R(F,C)=0$ (resp. $\mathrm{Ext}^1_R(C,F)=0$) for all $C\in\mathcal{C}$. ${}^{\perp}C$ ($C^{\perp}$) is called the *left (right) orthogonal class* of $\mathcal{C}$. Recall that a pair $(\mathcal{F},\mathcal{C})$ of classes of R-modules is called a *cotorsion pair* if $\mathcal{F}^{\perp}=\mathcal{C}$ and ${}^{\perp}\mathcal{C}=\mathcal{F}$. Further, a cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be *complete* if for every R-module X there exist two exact sequences $0\to B\to A\to X\to 0$ and $0\to X\to B'\to A'\to 0$ with $A,A'\in\mathcal{A}$ and $B,B'\in\mathcal{B}$. Meanwhile a cotorsion pair $(\mathcal{A},\mathcal{B})$ is said to be *hereditary* if for every short exact sequence $0\to A'\to A\to A''\to 0$ with A, $A''\in\mathcal{A}$, then $A'\in\mathcal{A}$, or equivalently, if $0\to B'\to B\to B''\to 0$ is exact with B', $B\in\mathcal{B}$, then $B''\in\mathcal{B}$. We use $\widehat{\mathcal{X}}_n$ (resp. $\check{\mathcal{Y}}_n$) to denote the class of R-modules M with $\mathcal{X}$-$pd(M)\leq n$ (resp. $\mathcal{Y}$-$id(M)\leq n$) for some $n\geq 0$. The following theorem is a generalization of [4, Theorems 4.1 and 4.2] and many results in [14], it reveals that the left global $\mathcal{X}$-Gorenstein projective dimension of a ring R is strictly controlled by the classes $\mathcal{X}$, $\mathcal{I}$ and special cotorsion pairs.

Theorem 2.8 *Let R be a ring and $\mathcal{X}$ be a class of R-modules that contains all projective R-modules. Then the following statements are equivalent:*

(1) $l.\mathcal{X}$-GPD$(R)\leq n$.

(2) *Each m-th $(m\geq n)$ syzygy in any projective resolution of any module is in $\mathcal{GP}_{\mathcal{X}}(R)$.*

(3) *$id(\mathcal{X})\leq n$ and $pd(\mathcal{I})\leq n$.*

(4) *$(\mathcal{GP}_{\mathcal{X}}(R),\check{\mathcal{I}}_n)$ is a hereditary complete cotorsion pair.*

(5) *$(\mathcal{GP}_{\mathcal{X}}(R),\widehat{\mathcal{P}}_n)$ is a hereditary complete cotorsion pair and $\widehat{\mathcal{P}}_n=\check{\mathcal{I}}_n$.*

Proof

(5) $\Rightarrow$ (4) is obvious. (1) $\Leftrightarrow$ (2) is also obvious by Proposition 2.1.

(1) $\Rightarrow$ (3). Suppose $l.\mathcal{X}$-GPD$(R)\leq n$. We first show that id$(\mathcal{X})\leq n$. Indeed, otherwise we will have id$(\mathcal{X})> n$, then at least for some two R-modules M and $X\in\mathcal{X}$, it holds that $\mathrm{Ext}^{n+1}_R(M,X)\neq 0$. But since $\mathcal{X}$-GPD$(R)\leq n$, so by dimension shifting we get that $\mathrm{Ext}^{n+1}_R(M,X)=0$, hence a contradiction. We still need to show $pd(\mathcal{I})\leq n$. Assume the contrary, then $pd(\mathcal{I})=m\geq n+1$. It follows that there exists an injective R-module I and an R-module B such that $\mathrm{Ext}^m_R(I,B)\neq 0$. Choose a short exact sequence $0\to K\to P\to B\to 0$ and we have an associated long exact sequence:

$$\cdots\to\mathrm{Ext}^m_R(I,P)\to\mathrm{Ext}^m_R(I,B)\to\mathrm{Ext}^{m+1}_R(I,K)\to\cdots.$$

Hence we obviously have $\mathrm{Ext}^m_R(I,P)\neq 0$, but by Proposition 2.1 this contradicts with the fact that $l.\mathcal{X}$-GPD$(R)\leq n$ and $m\geq n+1$.

(3) $\Rightarrow$ (1). We first show that, any exact complex of projective R-modules $\mathbb{P}=\cdots\to P^{-2}\to P^{-1}\to P^0\to P^1\to\cdots$ is a complete $\mathcal{X}$-Gorenstein projective resolution. So let A be any R-module in $\mathcal{X}$, by hypothesis there is a finite injective resolution of A: $0\to A\to I_0\to I_1\to\cdots\to I_{n-1}\to I_n\to 0$ with n some positive integer. Set $K_{n-1}=\mathrm{Ker}(I_{n-1}\to I_n)$, it follows from the short exact sequence of complexes of R-modules:

$$0\to\mathrm{Hom}_R(\mathbb{P},K_{n-1})\to\mathrm{Hom}_R(\mathbb{P},I_{n-1})\to\mathrm{Hom}_R(\mathbb{P},I_n)\to 0.$$

that $\mathrm{Hom}_R(\mathbb{P},K_{n-1})$ is exact by the exactness of the other two complexes. Iterating this procedure we get that $\mathrm{Hom}_R(\mathbb{P},A)$ is exact, as wanted.

Now we have to show that every R-module has a $\mathcal{GP}_{\mathcal{X}}(R)$-resolution of length no greater than n. So let M be any R-module, we shall construct such a resolution. The construction is essentially contained in the proof of [8, Theorem 4.1], but for the sake of completeness, we shall give it in details. Let us choose an injective resolution of $M: 0 \to M \to I^0 \to I^1 \to I^2 \to \cdots$. Then for each I^i we can choose a projective resolution $\mathbb{P}^i = \cdots \to P_1^i \to P_0^i \to I^i \to 0$. Set $C_n^i = \mathrm{Ker}(P_{n-1}^i \to P_{n-2}^i)$, then C_n^i is projective for all $i \geq 0$ since $pd(\mathcal{I}) \leq n$. Let J_k be the kernel of $P_k^0 \to P_k^1$ for all $0 \leq k \leq n-1$, then we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C & \longrightarrow & C_n^0 & \longrightarrow & C_n^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_{n-1} & \longrightarrow & P_{n-1}^0 & \longrightarrow & P_{n-1}^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_0 & \longrightarrow & P_0^0 & \longrightarrow & P_0^1 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots
\end{array}$$

where $C = \mathrm{Ker}(C_n^0 \to C_n^1)$. Thus we get that all these R-modules J_k and C have complete $\mathcal{X}$-Gorenstein projective resolutions by the above discussion, just note that any module naturally admits a left projective resolution. So the following resolution of M appeared in the above diagram is a desired $\mathcal{GP}_{\mathcal{X}}(R)$-resolution of length n:

$$0 \to C \to J_{n-1} \to \cdots \to J_0 \to M \to 0.$$

(1) $\Rightarrow$ (5). Before showing this implication, we first claim that if (1) holds, then $(\mathcal{GP}_{\mathcal{X}}(R), \widehat{\mathcal{X}}_n)$ is a cotorsion pair and $\widehat{\mathcal{X}}_n = \widehat{\mathcal{P}}_n$. To do this we shall show that $\widehat{\mathcal{X}}_n = \mathcal{GP}_{\mathcal{X}}(R)^{\perp}$ and $\mathcal{GP}_{\mathcal{X}}(R) = {}^{\perp}\widehat{\mathcal{X}}_n$. The inclusions

$$\widehat{\mathcal{P}}_n \subseteq \widehat{\mathcal{X}}_n \subseteq \mathcal{GP}_{\mathcal{X}}(R)^{\perp}$$

follow directly by dimension shifting and the definition. To show $\mathcal{GP}_{\mathcal{X}}(R)^{\perp} \subseteq \widehat{\mathcal{X}}_n$, let $M \in \mathcal{GP}_{\mathcal{X}}(R)^{\perp}$, then we have a short exact sequences: $0 \to M \to I \to L \to 0$ with I injective. Furthermore, by (1) and Proposition 2.1, there exists another short exact sequence $0 \to K \to G \to L \to 0$ with $pd(K) \leq (n-1)$ and $G \in \mathcal{GP}_{\mathcal{X}}(R)$.

Consider the pullback diagram of $I \to L$ and $G \to L$:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & K \arrow[r, equal] \arrow[d] & K \arrow[d] & \\
0 \arrow[r] & M \arrow[r] \arrow[d, equal] & D \arrow[r] \arrow[d] & G \arrow[r] \arrow[d] & 0. \\
0 \arrow[r] & M \arrow[r] & I \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

It follows from $pd(I) \leq n$ and $pd(K) \leq (n-1)$ that $pd(D) \leq n$. The middle row in the above diagram splits since $G \in \mathcal{GP}_{\mathcal{X}}(R)$, hence $pd(M) \leq n$, so we have

$$\mathcal{GP}_{\mathcal{X}}(R)^{\perp} \subseteq \widehat{\mathcal{P}}_n \subseteq \widehat{\mathcal{X}}_n.$$

Combining it with the inclusions at the beginning we get $\mathcal{GP}_{\mathcal{X}}(R)^{\perp} = \widehat{\mathcal{P}}_n = \widehat{\mathcal{X}}_n$. To conclude our claim, it still needs to show that $\mathcal{GP}_{\mathcal{X}}(R) = ^{\perp}(\widehat{\mathcal{X}}_n)$. Obviously only the implication $^{\perp}(\widehat{\mathcal{X}}_n) \subseteq \mathcal{GP}_{\mathcal{X}}(R)$ is nontrivial. So let $H \in ^{\perp}(\widehat{\mathcal{X}}_n)$ be any R-module, by Proposition 2.1 we have a short exact sequence $0 \to L \to B \to H \to 0$ with $pd(L) \leq (n-1)$ and $B \in \mathcal{GP}_{\mathcal{X}}(R)$. This sequence splits since $L \in \widehat{\mathcal{X}}_n$. It yields that H is a direct summand of B, hence is in $\mathcal{GP}_{\mathcal{X}}(R)$, as desired.

The fact that the cotorsion pair $(\mathcal{GP}_{\mathcal{X}}(R), \widehat{\mathcal{X}}_n)$ or $(\mathcal{GP}_{\mathcal{X}}(R), \widehat{\mathcal{P}}_n)$ is hereditary complete follows from Proposition 2.1 and that the class $\mathcal{GP}_{\mathcal{X}}(R)$ is projectively resolving. Also we deduce that $(1) \Rightarrow (5)$, since (1) implies (3), for any projective resolution of $M \in \widehat{\mathcal{P}}_n$ with length n, (3) and a discussion of dimension shifting give that $M \in \check{\mathcal{I}}_n$ and vice versa.

$(4) \Rightarrow (1)$. Let A be any R-module. Consider the long exact sequence obtained from a projective resolution of $A : 0 \to J_n \to P_{n-1} \to \cdots \to P_0 \to A \to 0$ with all P_i projective. For any R-module L, we see that $\mathrm{Ext}^1_R(J_n, L) \cong \mathrm{Ext}^{n+1}_R(A, L)$. Let L vary in $\check{\mathcal{I}}_n$, (4) yields that $J_n \in ^{\perp}(\check{\mathcal{I}}_n) = \mathcal{GP}_{\mathcal{X}}(R)$, and (1) follows. □

Note that some well-known results in [6] are particular cases of Theorem 2.8, also note that the assumption $l.\mathcal{X}$-GPD$(R) < \infty$ in [14, Proposition 3.14] can be taken off and the cotorsion pair in [14, Theorem 3.19] can be explicitly given. The following result gives a partial answer to the question asked in Remark 2.5.

Corollary 2.9 *For any two classes $\mathcal{X}$ and $\mathcal{X}'$ of R-modules which both contain all projective R-modules, assume that either $l.\mathcal{X}$-GPD$(R) \leq n$ or $l.\mathcal{X}'$-GPD$(R) \leq n$ holds. Then the two conditions are equivalent:* (1) $\mathcal{GP}_{\mathcal{X}}(R) = \mathcal{GP}_{\mathcal{X}'}(R)$; (2) $\widehat{\mathcal{X}}_n = \widehat{\mathcal{X}'}_n = \widehat{\mathcal{P}}_n$.

Proof Suppose that $l.\mathcal{X}$-GPD$(R) \leq n$, the case where $l.\mathcal{X}'$-GPD$(R) \leq n$ holds can be proved similarly. We first show the implication $(1) \Rightarrow (2)$. It follows from the proof of Theorem 2.8 that we have $\widehat{\mathcal{X}}_n = \widehat{\mathcal{P}}_n = \widehat{\mathcal{X}'}_n$

Hence (2) holds. Now assume that (2) holds, then $\mathcal{X} \subseteq \widehat{\mathcal{X}}_n = \widehat{\mathcal{X}'}_n$ gives that $\mathcal{GP}_{\mathcal{X}}(R) \supseteq \mathcal{GP}_{\mathcal{X}'}(R)$. A similar argument shows $\mathcal{GP}_{\mathcal{X}}(R) \subseteq \mathcal{GP}_{\mathcal{X}'}(R)$, as desired. □

It is easy to formulate he dual version of Theorem 2.8:

Theorem 2.10 *Let R be a ring and $\mathcal{Y}$ be a class of R-modules that contains all injective R-modules. Then the following statements are equivalent:*

(1) *$l.\mathcal{Y}$-GID$(R) \leq n$.*

(2) *Each m-th $(m \geq n)$ cosyzygy in any injective resolution of any module is in $\mathcal{GI}_{\mathcal{Y}}(R)$.*

(3) *$pd(\mathcal{Y}) \leq n$ and $id(\mathcal{P}) \leq n$.*

(4) *$(\widehat{\mathcal{P}}_n, \mathcal{GI}_{\mathcal{Y}}(R))$ is a hereditary complete cotorsion pair.*

(5) *$(\check{\mathcal{I}}_n, \mathcal{GI}_{\mathcal{Y}}(R))$ is a hereditary complete cotorsion pair and $\check{\mathcal{I}}_n = \widehat{\mathcal{P}}_n$.*

To state the next result we recall from [5] that, for two classes of R-modules $\mathcal{C}$ and $\mathcal{D}$, the functor $\mathrm{Hom}(-,-)$ is said to be right $\mathcal{C} \times \mathcal{D}$ balanced if for any two R-modules M and N, there exists a left $\mathcal{C}$-resolution $\mathbb{M}$ of M and a right $\mathcal{D}$-resolution $\mathbb{N}$ of N such that $\mathrm{Hom}(\mathbb{M}, D)$ and $\mathrm{Hom}(C, \mathbb{N})$ are always exact whenever $C \in \mathcal{C}$ and $D \in \mathcal{D}$. As a direct consequence of Theorem 2.8 and 2.10, we have the following corollary, which extends [6, Theorem 12.1.4].

Corollary 2.11 *Let R be a ring, $\mathcal{X}$ and $\mathcal{Y}$ be the classes of R-modules that contains respectively all projective and all injective R-modules. Assume that $id(\mathcal{X}) \leq n$ and $pd(\mathcal{Y}) \leq n$, then $\mathrm{Hom}(-,-)$ is right balanced by $\mathcal{GP}_{\mathcal{X}}(R) \times \mathcal{GI}_{\mathcal{Y}}(R)$.*

Proof By Theorems 2.8 and 2.10, we get that both of the cotorsion pairs $(\mathcal{GP}_{\mathcal{X}}(R), \check{\mathcal{I}}_n)$ and $(\check{\mathcal{I}}_n, \mathcal{GI}_{\mathcal{Y}}(R))$ are hereditary complete, then [5, Lemma 4.1] gives the desired result. □

Theorem 2.12 *Let $\mathcal{X}' \subseteq \mathcal{X}$ be two classes of R-modules such that $\mathcal{X}'$ is projectively resolving. If $\mathcal{X}'$-$pd(\mathcal{X}) < \infty$, then $l.\mathcal{X}'$-GPD(R)=$l.\mathcal{X}$-GPD(R). The dual result also holds.*

Proof By Definition we have $l.\mathcal{X}'$-GPD$(R) \leq l.\mathcal{X}$-GPD(R), so it suffices to show the inverse inequality. If $l.\mathcal{X}'$-GPD$(R) = \infty$ then we are done, so suppose $l.\mathcal{X}'$-GPD$(R) = n < \infty$, and we shall show that $l.\mathcal{X}$-GPD$(R) \leq n$. By Theorem 2.8, it suffices to show that $id(\mathcal{X}) \leq n$. Let $X \in \mathcal{X}$, then a left projective resolution of X gives an exact sequence: $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to X \to 0$ where each $P_i \in \mathcal{P}$ $(0 \leq i \leq n-1)$ and K_n is the n-th syzygy. It follows from [16, Proposition 1.2] we get that $\mathcal{X}'$-$pd(K_n) < \infty$ since $\mathcal{X}'$-$pd(X) < \infty$ by assumption. On the other hand by Theorem 2.8(2) we get $K_n \in \mathcal{GP}_{\mathcal{X}'}(R)$. So Proposition 2.2 implies that K_n is projective. Now the above exact sequence implies $id(X) \leq n$ since by Theorem 2.8(3) and the assumption at the beginning of the proof we have $id(\mathcal{P}) \leq id(\mathcal{X}') \leq n$. □

Theorem 2.13 *Let $\mathcal{X}$ and $\mathcal{Y}$ be respectively projectively resolving and injectively coresolving classes of R-modules. Then $l.\mathcal{X}$-GPD$(R) = l.\mathcal{Y}$-GID(R)=max$\{id(\mathcal{X}), pd(\mathcal{Y})\}$ (interpreted as ∞ if either $id(\mathcal{X})$ or $pd(\mathcal{Y})$ is infinite) whenever one of the following conditions is satisfied:*

(1) *$id(\mathcal{X}) = id(\mathcal{P})$ and $pd(\mathcal{Y}) = pd(\mathcal{I})$.*

(2) *$id(\mathcal{X}) = pd(\mathcal{Y})$.*

(3) $\mathcal{X}$-$pd(\mathcal{Y}) = \mathcal{Y}$-$id(\mathcal{X})$.

Proof It is obvious that if $l.\mathcal{X}$-GPD$(R) = l.\mathcal{Y}$-GID(R), then it naturally equals to $\max\{id(\mathcal{X}), pd(\mathcal{Y})\}$ by Theorem 2.8(3), Theorem 2.10(3) and the inequalities $id(\mathcal{P}) \leq id(\mathcal{X})$, $pd(\mathcal{I}) \leq pd(\mathcal{Y})$. To show that $l.\mathcal{X}$-GPD$(R) = l.\mathcal{Y}$-GID(R), it needs only to show that the two inequalities $l.\mathcal{X}$-GPD$(R) \leq n$ and $l.\mathcal{Y}$-GID$(R) \leq n$ imply each other for any positive integer n. However, this is obvious by Theorem 2.8(3) and Theorem 2.10(3) whenever the condition (1) or (2) is satisfied.

Now suppose that (3) holds, we shall show that $l.\mathcal{X}$-GPD$(R) \leq n$ implies $l.\mathcal{Y}$-GID$(R) \leq n$, for the inverse implication the proof is similar. First note that $id(\mathcal{X}) \leq n$ and $\mathcal{X}$-$pd(\mathcal{Y}) = \mathcal{Y}$-$id(\mathcal{X}) \leq id(\mathcal{X}) \leq n$ follows from $l.\mathcal{X}$-GPD$(R) \leq n$ by Theorem 2.8(3), so it needs only to show $pd(\mathcal{Y}) \leq n$. For this take any $Y \in \mathcal{Y}$, and it follows from a left projective resolution of Y we get an exact sequence $0 \to K_n \to P_{n-1} \to \cdots \to P_0 \to Y \to 0$ where each $P_i \in \mathcal{P}$ $(0 \leq i \leq n-1)$ and $K_n \in \mathcal{GP}_{\mathcal{X}}(R)$ by Theorem 2.8(2). Therefore, by [16, Proposition 2.2] and the assumption we have $\mathcal{X}$-$pd(K_n) < \infty$. Proposition 2.2 implies that K_n is projective and so $pd(Y) \leq n$ for any $Y \in \mathcal{Y}$, as desired. □

We call the common value of the quantities in Theorem 2.13 the *left global Gorenstein dimension of* R *with respect to* $\mathcal{X}$ *and* $\mathcal{Y}$ if one of the conditions is satisfied, and denote it by $l.\mathrm{Ggldim}_{\mathcal{X},\mathcal{Y}}(R)$.

We conclude this section with a sequel of examples and applications of the above results.

Example 2.14 Let $\mathcal{X} = \mathcal{Y} = {}_R\mathrm{Mod}$, then $\mathcal{GP}_{\mathcal{X}}(R) = \mathcal{P}$, $\mathcal{GI}_{\mathcal{Y}}(R) = \mathcal{I}$ and the value of the quantities $pd(R) = l.\mathcal{X}$-GPD$(R) = l.\mathcal{Y}$-GID$(R) = id(R)$ in Theorem 2.13 becomes the usual left global dimension of the ring R.

Next for $\mathcal{X} = \mathcal{P}$ and $\mathcal{Y} = \mathcal{I}$, we carry out the main result of [2] as a corollary of Theorem 2.13 (cf. [2, Theorem 1.1]). For convenience, we rewrite for short the notations $l.\mathcal{X}$-GPD(R) and $l.\mathcal{Y}$-GID(R) as $l.$GPD(R) and $l.$GID(R) respectively when $\mathcal{X} = \mathcal{P}$ and $\mathcal{Y} = \mathcal{I}$.

Corollary 2.15 ([2]) *Let* R *be a ring, then it holds the equality:* $l.$GPD$(R) = l.$GID(R).

Remark 2.16 The common value of the quantities in Corollary 2.15 is called the *left global Gorenstein dimension* of R in [2]. In contrast to the proof of [2, Theorem 1.1], we do not need the notion of strongly Gorenstein projective R-modules here. This observation shows that the left global dimension and the left global Gorenstein dimension of a ring R can be viewed as two different types of $l.\mathrm{Ggldim}_{\mathcal{X},\mathcal{Y}}(R)$ by taking proper $\mathcal{X}$ and $\mathcal{Y}$.

Other types of $\mathcal{X}$-Gorenstein projective and $\mathcal{Y}$-Gorenstein injective modules are studied in [4, 13]. Let us recall some definitions.

Definition 2.17 An R-module N is called FP-*injective* provided $\mathrm{Ext}_R^{n\geq 1}(M, N) = 0$ for any finite presented module M.

We use $\mathcal{FI}$ to denote the class of all FP-injective R-modules, and it is easy to check that $\mathcal{FI}$ is injectively coresolving. If we set $\mathcal{X} = \mathcal{F}$ and $\mathcal{Y} = \mathcal{FI}$, following [10], the corresponding modules in $\mathcal{GP}_{\mathcal{X}}(R)$ and $\mathcal{GI}_{\mathcal{Y}}(R)$ are called respectively *Ding projective* and *Ding injective modules* (which are first called *strongly Gorensein flat* and *Gorenstein* FP-*injective modules* in [4, 13]).

The following notion of Ding-Chen rings can be viewed as a generalization of that of n-Gorenstein rings (i.e. left and right Noetherian rings with self injective dimensions at most n on both sides).

Definition 2.18 ([9]) A ring R is said to be a *Ding-Chen ring* if it is an n*-FC ring* for some integer $n \geq 0$, where an n-FC ring is a two-sided (left and right) coherent ring with $\mathcal{FI}$-$id({}_RR) \leq n$ and $\mathcal{FI}$-$id(R_R) \leq n$.

Our next aim is to show that, for any two-sided coherent ring R with $\mathcal{FI}$-$id({}_RR) = \mathcal{FI}$-$id(R_R)$ the dimension $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R)$ exists, and it coincides with $l.\mathrm{Ggldim}(R)$ (see Proposition 2.22 below). For consistence of the notation, we write for short respectively $l.\mathrm{SGFP}(R)$ and $l.\mathrm{GFID}(R)$ instead of $l.\mathcal{X}$-$\mathrm{GPD}(R)$ and $l.\mathcal{Y}$-$\mathrm{GID}(R)$ when $\mathcal{X} = \mathcal{F}$ and $\mathcal{Y} = \mathcal{FI}$, as used in [4, 13].

Theorem 2.19 *Let R be a ring, then $l.\mathrm{SGFP}(R) = l.\mathrm{GPD}(R)$.*

Proof Obviously we have $l.\mathrm{GPD}(R) \leq l.\mathrm{SGFP}(R)$. To show the inverse inequality, suppose that $l.\mathrm{GPD}(R) = n < \infty$. Thus by [2, Corollary 2.7] we get $pd(\mathcal{F}) \leq n$, and Theorem 2.12 gives the result. □

In the sequel we shall use the notation M^+ to denote the character module of an R-module M, which is defined as $\mathrm{Hom}_R(M, \mathbb{Q}/\mathbb{Z})$.

Lemma 2.20 *Let R be a two-sided coherent ring, then $fd(\mathcal{FI}) = \mathcal{FI}$-$id(\mathcal{F}_R) = \mathcal{FI}$-$id(R_R)$. The similar result also holds if we switch the left and right R-modules.*

Proof ($\leq$) First note that [3, , Theorem 3.8] implies $\mathcal{FI}$-$id(\mathcal{F}_R) = \mathcal{FI}$-$id(R_R)$. For any $M \in \mathcal{FI}$, by [3, Lemmas 2.1 and 2.4] we have $fd(M) = \mathcal{FI}$-$id(M^+) \leq \mathcal{FI}$-$id(\mathcal{F}_R)$.

($\geq$) Now for any $F \in \mathcal{F}_R$, by [3, Lemma 2.3] we have $\mathcal{FI}$-$id(F) = fd(F^+) \leq fd(\mathcal{I}) \leq fd(\mathcal{FI})$, as needed. □

Lemma 2.21 *Let R be a two-sided coherent ring, then $id(\mathcal{F}) = \mathcal{FI}$-$id({}_RR) \leq pd(\mathcal{I}_R) \leq pd(\mathcal{FI}_R)$.*

Proof For any $F \in \mathcal{F}$, by [6, Proposition 5.3.9] we have a pure exact sequence: $0 \to F \to F^{++} \to F^{++}/F \to 0$. Since R is right coherent, F^{++} is left flat. Now the above short pure exact sequence yields that F^{++}/F is left flat. Hence F is a direct summand of F^{++}, and by [3, , Lemmas 2.1 and 2.4] we have $id(F) \leq id(F^{++}) = fd(F^+) \leq fd(\mathcal{I}_R)$, on the other hand we get $\mathcal{FI}$-$id({}_RR) = fd(\mathcal{I}_R) \leq id(\mathcal{F})$ by [3, Lemma 2.1 and Theorem 3.8], this gives the desired inequality $id(\mathcal{F}) = \mathcal{FI}$-$id({}_RR) = fd(\mathcal{I}_R) \leq pd(\mathcal{I}_R) \leq pd(\mathcal{FI}_R)$. □

Proposition 2.22 *Let R be a two-sided coherent ring such that $\mathcal{FI}$-$id({}_RR) = \mathcal{FI}$-$id(R_R)$, then $\mathcal{FI}$-$id({}_RR) \leq l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R) = pd(\mathcal{I}) = pd(\mathcal{FI})$.*

Proof The existence of $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R)$ follows directly from Theorem 2.13 and Lemma 2.20. By Lemma 2.21, Theorems 2.13 and 2.19 we obtain

$$pd(\mathcal{FI}) = l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R) = l.\mathrm{GPD}(R) = \max\{id(\mathcal{P}), pd(\mathcal{I})\} = pd(\mathcal{I}),$$

thus $pd(\mathcal{FI}) = pd(\mathcal{I})$. □

We remark that for a two-sided coherent ring, $\mathcal{FI}\text{-}id({}_RR) = \mathcal{FI}\text{-}id(R_R)$ holds if and only if both $\mathcal{FI}\text{-}id({}_RR)$ and $\mathcal{FI}\text{-}id(R_R)$ are finite or infinite by [3, Corollary 3.18].

At last we shall use these results to characterize Ding-Chen rings and commutative coherent rings by $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R)$. For this we recall from [7, Definition 1.1] that, a ring R is said to be n-*perfect* provided that all flat R-modules have projective dimensions less or equal than n, i.e. $pd(\mathcal{F}) \leq n$. Note also that for a Ding-Chen ring, the following result could also be deduced from [11, Theorem 1.1].

Theorem 2.23 *Let R be a Ding-Chen ring or commutative coherent ring, if $\mathcal{FI}\text{-}id({}_RR)$ is finite, then for any integer $m \geq \mathcal{FI}\text{-}id({}_RR)$, the following statements are equivalent:*

(1) $l.\mathrm{Ggldim}_{\mathcal{F},\mathcal{FI}}(R) = m$.

(2) $l.\mathrm{Ggldim}\,(R) = m$.

(3) $pd(\mathcal{I}) = pd(\mathcal{FI}) = m$.

(4) *The m-th syzygy of a projective resolution of any R-module is Ding projective.*

(5) *The m-th cosyzygy of an injective resolution of any R-module is Ding injective.*

(6) *All Gorenstein projective modules are Ding projective and $(\mathcal{GP}(R), \check{\mathcal{I}}_m)$ is a hereditary complete cotorsion pair.*

(7) *All Gorenstein injective modules are Ding injective and $(\widehat{\mathcal{P}}_m, \mathcal{GI}(R))$ is a hereditary complete cotorsion pair.*

Furthermore, each of these conditions implies that R is m-perfect. Otherwise if $\mathcal{FI}\text{-}id({}_RR)$ is infinite, then the equivalent statements (1-3) still validates except that m is interpreted as ∞.

Proof The equivalences of (1-5) follow from Corollary 2.15, Theorem 2.19 and Proposition 2.22. (1) $\Leftrightarrow$ (6) and (1) $\Leftrightarrow$ (7) follow from Theorems 2.8, 2.12 and Corollary 2.9. As to the last statement, obeserve that $id(\mathcal{F}) = n$ and $pd(\mathcal{I}) = m$ imply $pd(\mathcal{F}) \leq m$. □

Remark 2.24 (1) First note that if R is an n-FC and left $(n'-n)$-perfect ring with $n' \geq n$, then all the equivalent statements in above theorem hold for some m such that $n'-n \leq m \leq n'$. To see this, let $m = pd(\mathcal{I})$ and observe that $n'-n \leq pd(\mathcal{I}) \leq n'$ follows from $pd(\mathcal{F}) \leq n'-n$ and $fd(\mathcal{I}) = n$ by the proof of Lemma 2.21.

(2) Also note that there may not exist an positive integer $m = n$ such that one of the conditions in Theorem 2.23 is satisfied. For example, let $R = \prod F$, an infinite product of a field F; then R is a commutative von Neumann regular ring and it is not semisimple (see [4]). Clearly, R is 0-FC but the global dimension m of R is strictly greater than 0, thus R is not left perfect as $pd(\mathcal{F}) = m$. Hence we conclude that the value m in Theorem 2.23 is strictly greater than 0 if it is finite (for example, this happens if F is the complex number field). Furthermore, for any commutative von Neumann regular ring with infinite global dimension, there exists no finite integer m such that it satisfies the equivalent conditions in the theorem above.

We restate the above remark as follows.

Corollary 2.25 *There exists a Ding-Chen ring with infinite global Gorenstein dimension.*

3. $\mathcal{X}$-Gorenstein projective precovers

In this section, we shall study the existence of $\mathcal{GP}_{\mathcal{X}}(R)$-precovers. We start with the following definition, remember that a subcategory $\mathcal{C}$ of an abelian category $\mathcal{A}$ is said to be *thick*, if for any short exact sequence $0 \to M \to M' \to M'' \to 0$ in $\mathcal{A}$, $M' \in \mathcal{C}$ if and only if M and M'' are in $\mathcal{C}$.

Definition 3.1 ([10]) *Given an abelian category $\mathcal{A}$, a complete cotorsion pair $(\mathcal{F}, \mathcal{C})$ is called an projective cotorsion pair whenever $\mathcal{C}$ is thick and $\mathcal{F} \cap \mathcal{C}$ is the class of projective objects in $\mathcal{A}$.*

For instance, the well-known cotorsion pair $(\mathcal{P}, {}_R\mathrm{Mod})$ is a canoical projective cotorsion pair in ${}_R\mathrm{Mod}$, in fact it is even more cogenerated by the 0 module, and for more examples one may refer to [10]. Now we turn to the main result of this section. Recall that Theorem 2.8 implies that for any ring R with $id(\mathcal{P})$ and $pd(\mathcal{I})$ finite, every module has a special $\mathcal{GP}(R)$-precover, the following result shows that the first condition is enough for the existence of $\mathcal{GP}(R)$-precovers.

Theorem 3.2 *Let R be a ring, $\mathcal{X}$ a class of R-modules that contains all projective R-modules. If every module in $\mathcal{X}$ has finite injective dimension, then every R-module has a $\mathcal{GP}_{\mathcal{X}}(R)$-precover.*

Proof Suppose $\mathcal{X}$ is such a class. First note that, By the discussion at the beginning of "(3) $\Rightarrow$ (1)" part of the proof of Theorem 2.8, any long exact complex of projective R-modules $\mathbb{P} = \cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to \cdots$ is a complete $\mathcal{X}$-Gorenstein projective resolution of some module $M = \mathrm{Ker}(P^0 \to P^1)$.

Denote $\widetilde{\mathcal{P}}$ the class of all exact degreewise projective complexes of R-modules. We now claim that every complex of R-modules has a $\widetilde{\mathcal{P}}$-precover. In fact, this is a direct consequence of [10, Proposition 7.3], since the cotorsion pair $(\mathcal{P}, {}_R\mathrm{Mod})$ is a projective cotorsion pair cogenerated by some set in ${}_R\mathrm{Mod}$ as we saw.

Take any R-module M, we can associate a complex $M[1] = \cdots \to 0 \to M \stackrel{1_M}{\to} M \to 0 \to \cdots$, which is concentrated at degrees -1 and 0 with the module M and whose only nonzero differential ∂_{-1} is the identity map of M. By what we have proved there exists a $\widetilde{\mathcal{P}}$-precover $g : \mathbb{P} \to M[1]$ where $\mathbb{P} = \cdots \to P^{-2} \to P^{-1} \to P^0 \to P^1 \to \cdots$ is an exact complex of projective R-modules. Denote $G = \mathrm{Ker}(P^0 \to P^1)$, thus g naturally induces a map $\tilde{g} : G \to M$. Now by a same argument as in the proof of [15, Theorem A], we can show that the map $\tilde{g} : G \to M$ is the desired $\mathcal{GP}_{\mathcal{X}}(R)$-precover of M, it then finishes the proof. □

Using [10, Proposition 7.2] and a dual argument to that of [15, Theorem A], one can get the dual result of Theorem 3.2 as follows.

Theorem 3.3 *Let R be a ring, $\mathcal{Y}$ a class of R-modules that contains all injective R-modules. If every module in $\mathcal{Y}$ has finite projective dimension, then every R-module has a $\mathcal{GI}_{\mathcal{Y}}(R)$-preenvelope.*

Let $\mathcal{X}$ be the class of projective R-modules, we then have the following results, and one can get their dual versions easily.

Corollary 3.4 *Let R be a ring such that all projective R-modules have finite injective dimensions, then every R-module has a Gorenstein projective precover.*

In particular we get the following known result ([6, Theorem 11.5.1]):

Corollary 3.5 *Let R be an n-Gorenstein ring, then every R-module has a Gorenstein projective precover.*

Proof Follows directly by Corollary 3.3 and [6, Theorem 9.1.11]. □

Let $\mathcal{X}$ be the class of flat R-modules, hence we have the following:

Corollary 3.6 *Let R be a ring such that all flat R-modules have finite injective dimensions, then every R-module has a Ding projetive precover.*

Thus by Lemma 2.21 we have the following result.

Corollary 3.7 *Let R be a Ding-Chen ring, then every R-module has a Ding projective precover.*

Similarly, take $\mathcal{Y}$ to be the class of FP-injective R-modules, then by the Theorem 3.3 we have the following result.

Corollary 3.8 *Let R be a ring such that all* FP*-injective R-modules have finite projective dimensions, then every R-module has a Ding injective preenvelope.*

For example, let R be a Ding-Chen ring with $l.\mathrm{Ggldim}(R) < \infty$, then by Theorem 2.23 all FP-injective R-modules have finite projective dimensions. Furthermore, if $l.\mathrm{GFI}(R)$ is finite, then by Corollary 3.7 or Theorem 2.8 every R-module has a Ding injective preenvelope.

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