Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1783 – 1791
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doi:10.3906/mat-2003-63

A note on simple trinomial units in $U_1(\mathbb{Z}C_p)$

Necat GÖRENTAŞ*
Department of Mathematics, Faculty of Science, Van Yüzüncü Yıl University, Van, Turkey

Received: 13.03.2020 • **Accepted/Published Online:** 11.07.2020 • **Final Version:** 21.09.2020

Abstract: In this paper, some new notions are defined about the unit group $U_1(\mathbb{Z}G)$ of a finite group G. Especially, notion of simple unit is defined by using the number of elements in its support and absolutely small coefficients of the unit. Units are classified as monomial, binomial, trinomial and k-nomial, level, unit with level l and simple unit. We have shown triviality of monomial units and nonexistence of binomial units in the unit group $U_1(\mathbb{Z}G)$ of an arbitrary finite group G. Some basic results and examples are posed about simple units and simple trinomial units in $U_1(\mathbb{Z}C_p)$ of a cyclic group C_p, where $p \geqslant 5$.

Key words: Monomial unit, binomial unit, trinomial unit, unit with level l, simple unit

1. Introduction

Let G be a finite group and $\mathbb{Z}G$ be its integral group ring. Let $U(\mathbb{Z}G)$ be the unit groups of the integral group ring. Let us denote its normalized units by $U_1(\mathbb{Z}G)$. Several results of algebraic topology (see Milnor [9], Theorem 12.8 and Corollary 12.9) suggest the significance of find out exactly what units exist with the integral group ring ZG. If G = A is an abelian group then by Higman[5], Ayoub and Ayoub[2] characterization of the unit group is reduced into characterization of torsion free part of the unit group:

$$U(\mathbb{Z}A) = \pm A \times F \tag{1.1}$$

where F is a finitely generated free abelian group with rank

$$r = \frac{1}{2}(n + 1 + n_2 - 2l).$$

Here n is the order of A, n_2 is the number of cyclic subgroup of order 2 and l is the number of distinct cyclic subgroup of A. Since $U(\mathbb{Z}A) = \pm U_1(\mathbb{Z}A)$, remain part we'll use normalized units $U_1(\mathbb{Z}A)$. Then the equation(1.1) can be written as:

$$U_1(\mathbb{Z}A) = A \times F. \tag{1.2}$$

That is, there exists a system of r units $u_1, u_2, ..., u_r$ in $U_1(\mathbb{Z}A)$ such that each unit of $\mathbb{Z}A$ is represented uniquely in the form

$$\pm g u_1^{n_1} u_2^{n_2} ... u_r^{n_r} (n_i \in \mathbb{Z}, g \in A). \tag{1.3}$$

*Correspondence: ngortas@yahoo.com
2010 *AMS Mathematics Subject Classification:* 16U60.

The set $\{u_1, u_2, ..., u_r\}$ is called a fundament system of units in $\mathbb{Z}A$ and each u_i is called fundamental unit. Problem can be stated in two different ways [6]:

Problem A: Given a finite abelian group A, find a specific fundamental system of units in $U_1(\mathbb{Z}A)$.

This problem was found exceptionally difficult problem and formidable task. For this reason, the following related problem was seemed more appropriate to attack.

Problem B: Given a finite abelian group A, find a specific system of r independent units of infinite order in $U_1(\mathbb{Z}A)$, which generates a subgroup of finite index.

Problem B was solved by Bass [3] for cyclic group by defining Bass cyclic units. But there is a problem. If A has a large order than expressing each bass cyclic unit explicitly is difficult. That is, while expressing bass cyclic unit, large numbers are obtained in the coefficients and more terms are used to write the unit. To avoid these two problems we handled to solve problem A. Let us start with listing exceptional solved problems:

Let $C_n =< a : a^n = 1 >$. By [7] for $n = 5, 8$ the unit groups $U_1(\mathbb{Z}A)$ are described:

$$U_1(\mathbb{Z}C_5) = C_5 \times < -1 + a^2 + a^{-2} > .$$

$$U_1(\mathbb{Z}C_8) = C_8 x < 2 + (a + a^{-1}) - (a^3 + a^{-3}) - a^4 > .$$

Aleev and Pannina [1] described the structures of $U_1(\mathbb{Z}C_n)$ for n=7,9 :

$$U_1(\mathbb{Z}C_7) = C_7 \times < -1 + a + a^{-1}, -1 + 2(a^2 + a^{-2}) - (a^3 + a^{-3}) > .$$

$$U_1(\mathbb{Z}C_9) = C_9 \times < -1 - (a + a^{-1}) - (a^2 + a^{-2}) + 2(a^4 + a^{-4}), -1 - (a + a^{-1}) + (a^2 + a^{-2}) > .$$

Bilgin [4] characterized the unit group for $n = 12$:

$$U_1(\mathbb{Z}C_{12}) = C_{12} \times < 3 + 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4}), -2(a^5 + a^{-5}) - 2a^6 > .$$

By using exact sequences Low [8] generalized these results to the direct product $C_n \times C_2$.

To express several generators for the unit group we must use less and absolutely small coefficients and less group elements in the support of each unit as much as possible. Before defining this notion, let us illustrate two examples:

Example-1. Let F be multiplicative free abelian group generated by u and v. Then the set of generators of F can be expressed infinitely many ways:

$$F =< u, v >=< u, v^{-1} >=< u^{-1}, v >=< u^{-1}, v^{-1} > \tag{1.4}$$

or

$$F =< uv^{-1}, v >=< uv^{-1}, vu^2 >=< u^3v^2, v^{-1}u^{-1} > \tag{1.5}$$

in the equation (1.2) powers are ± 1 and there is no product so we say that generator sets are simple. By the same reason none of the generator sets in equation (1.3) are simple.

Example-2. Let V be a vector space over the field $\mathbb{R}$ spanned by vectors v_1, v_2, v_3, v_4 and W be a subspace of V spanned by the vectors:

$$w_1 = 2v_1 - v_3 + 6v_4, w_2 = 3v_2 - 5v_3 - 13v_4, w_3 = 2v_1 + 8v_2 + 4v_4 \tag{1.6}$$

Question: How can we express a basis for W in ‘a simple way’?

We can express a basis for W in a simple way by

. using less vectors in each combinations first and then,
. using absolutely small coefficients in each combination, and
. using at least one coefficient in each combination as 1.

By using these three criteria above let us find a basis for W in a simple way. We claim that the vectors

$$u_1 = v_1 - 2v_3, u_2 = v_2 - v_3, u_3 = v_3 + 2v_4$$

are simple basis for W since each combination no more than 2 vectors are used and all coefficients in each combination are absolutely less than or equal to 2. Of course, we can also find several simple bases for W, but there are not infinitely many bases for W.

Now let us modify this analogy to the normalized units in integral unit groups and define several new concepts. After that let us state some basic results. First, monomials are trivial. Therefore, nothing to say more. Secondly, we have shown that there is no binomial units in integral group rings $\mathbb{Z}G$ for any group G. Later, we have demonstrated product of all simple trinomial units in the standard form is identity, which means that inverse of any simple trinomial unit can be expressed as a product of other simple trinomial units in the standard form. Since the number of simple units in the standard form is the same with number of generators of $U_1(\mathbb{Z}C_p)$ for prime $p \geq 5$, a conjecture is added.

2. Definitions and some basic results

Definition 3. Let G be a finite group of order n, $\gamma \in \mathbb{Z}G$ and $S = supp(\gamma)$.
(i) $|S|$ is called length of γ.
(ii) $\max U_{g\in S}\{|\gamma_g|\}$ is called level of γ.

Definition 4. Let G be a finite group of order n, $\gamma \in U_1(\mathbb{Z}G)$ and $S = supp(\gamma)$.
(i) If $|S| = k$, then γ is called k-nomial unit. Especially if $k = 1, 2$ or 3, then γ is called monomial, binomial or trinomial unit, respectively. **(ii)** If the length of γ is l , then γ is called k-nomial unit with level l.
(iii) If $l = 1$, then γ is called simple k-nomial unit.

Example 5. All monomial units of $\gamma \in U_1(\mathbb{Z}G)$ are trivial units. All of them are simple.

Example 6. If $n = 5, 8$ or 12, then torsion free part of $U_1(\mathbb{Z}C_n)$ is generated by a single unit as follows:
If $n = 5$, then $u = -1 + a + a^{-1}$ with $u^{-1} = -1 + a^2 + a^{-2}$.
If $n = 8$, then $u = 2 + (a + a^{-1}) - (a^3 + a^{-3}) - a^4$ with $u^{-1} = 2 - (a + a^{-1}) + (a^3 + a^{-3}) - a^4$.
If $n = 12$, then $u = 3 + 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4}) - 2(a^5 + a^{-5}) - 2a^6$ with
$u^{-1} = 3 - 2(a + a^{-1}) + (a^2 + a^{-2}) - (a^4 + a^{-4}) + 2(a^5 + a^{-5}) - 2a^6$.
We say that torsion free part of $U_1(\mathbb{Z}C_5)$ is generated by a simple trinomial unit and torsion free part of $U_1(\mathbb{Z}C_{12})$ is generated by simple 9-nomial unit with level 3.
For $n = 8$, we have

$$v = a^4u = -1 - (a + a^{-1}) + (a^3 + a^{-3}) + 2a^4 = -1 - (a + a^{-1}) + (a^3 + a^{-3}) + (a^4 + a^{-4}).$$

We say that torsion free part of $U_1(\mathbb{Z}G)$ is generated by simple 7-nomial unit.

Theorem 7. If G be a finite group, then $U_1(\mathbb{Z}G)$ has no binomial unit.

Proof. Let $\gamma = \gamma_0 g_0 + \gamma_1 g_1 \in U_1(\mathbb{Z}G)$ be a binomial unit. Let $\beta = g_0^{-1}\gamma$ and $g = g_0^{-1}g_1$ of order n.

Then $\beta = \gamma_0 + \gamma_1 g \in U_1(\mathbb{Z} < g >)$. Consider the inverse of β as $\beta^{-1} = \sum_{i=1}^{n-1} \beta_i g^i$. Hence

$$\begin{aligned} 1 &= \beta.\beta^{-1} \\ &= (\gamma_0 + \gamma_1 g) \sum_{i=0}^{n-1} \beta_i g^i \\ &= (\gamma_0\beta_0 + \gamma_1\beta_{n-1})1 + (\gamma_0\beta_1 + \gamma_1\beta_0)g + (\gamma_0\beta_2 + \gamma_1\beta_1)g^2 + ... + (\gamma_0\beta_{n-1} + \gamma_1\beta_{n-1})g^{n-1}. \end{aligned}$$

Then, we get
$\gamma_0\beta_0 + \gamma_1\beta_{n-1} = 1$ and $\gamma_0\beta_i + \gamma_1\beta_{i-1} = 0$ $(1 \le i < n-1)$.
Since β is normalized unit, then $\gamma_0 + \gamma_1 = 1$ so that $\gamma_0 = 1 - \gamma_1$. Hence, we decude

$$\frac{\beta_{i-1}}{\beta_i} = -\frac{1-\gamma_1}{\gamma_1} = \frac{\gamma_1 - 1}{\gamma_1}(1 \le i \le n-1) \ and \ \beta_0 = (\frac{\gamma_1 - 1}{\gamma_1})^{n-1}\beta_{n-1}.$$

Thus, it follows that

$$1 = \gamma_0\beta_0 + \gamma_1\beta_{n-1} = (1-\gamma_1)(\frac{\gamma_1 - 1}{\gamma_1})^{n-1}\beta_{n-1} + \gamma_1\beta_{n-1} = \beta_{n-1}[\frac{(\gamma_1 - 1)^n}{\gamma_1^{n-1}} + \gamma_1].$$

Hence

$$\beta_{n-1} = \frac{\gamma_1^{n-1}}{\gamma_1^n - (\gamma_1 - 1)^n} \ and \ \beta_i = \frac{(\gamma_1 - 1)^i \gamma_1^{n-i-1}}{\gamma_1^n - (\gamma_1 - 1)^n}, (1 \le i < n-1).$$

Here

$$\beta_{n-1} = \frac{\gamma_1^{n-1}}{\gamma_1^n - (\gamma_1 - 1)^n} = \frac{\gamma_1^{n-1}}{\gamma_1^{n-1} + (\gamma_1 - 1)^{n-2}\gamma_1 + ... + \gamma_1(\gamma_1 - 1)^{n-2} + \gamma_1^{n-1}} = \frac{\gamma_1^{n-1}}{\gamma_1^{n-1} + \Delta}.$$

Since β_{n-1} is an integer, then

$$|\beta_{n-1}| \le 1 \Rightarrow \beta_{n-1} = 0 or 1 \Rightarrow \gamma_1 = 0 \ or \ 1 \Rightarrow \beta = 1 \ or \ g \Rightarrow \gamma = g_0 \ or \ g_1.$$

This contradicts with being binomial unit of γ. Hence, there is no binomial unit in $U_1(\mathbb{Z}G)$ for arbitrary G.

Lemma 8. Let $5 \le p = 2k+1$ and $e^{2\pi i/p}$. Then

$$\prod_{j=1}^{k}(-1 + \varepsilon^j + \varepsilon^{-j}) = 1.$$

Proof. Let us consider following statement

$$-1 + +\varepsilon^j + \varepsilon^{-j} = -1 + 2cos(\frac{2\pi j}{p}) = -1 + 2[2cos^2(\frac{\pi j}{p}) - 1] = 4cos^2(\frac{\pi j}{p}) - 3.$$

Then, we have

$$\begin{aligned} \prod_{j=1}^{k}(-1 + \varepsilon^j + \varepsilon^{-j}) &= \prod_{j=1}^{k}(4cos^2(\frac{\pi j}{p}) - 3) \\ &= \prod_{j=1}^{k}(\frac{4cos^3(\frac{\pi j}{p}) - 3cos(\frac{\pi j}{p})}{cos(\frac{\pi j}{p})}) \end{aligned}$$

$$
\begin{aligned}
&= \prod_{j=1}^{k} \frac{cos(\frac{3\pi j}{p})}{cos(\frac{\pi j}{p})} \\
&= \left[\frac{sin(\frac{\pi}{p})}{sin(\frac{3\pi}{p})}\right]\left[\frac{sin(\frac{3\pi}{p})cos(\frac{3\pi}{p})cos(\frac{6\pi}{p})cos(\frac{9\pi}{p})...cos(\frac{3k\pi}{p})}{sin(\frac{\pi}{p})cos(\frac{\pi}{p})cos(\frac{2\pi}{p})cos(\frac{3\pi}{p})...cos(\frac{k\pi}{p})}\right] \\
&= \left[\frac{sin(\frac{\pi}{p})}{sin(\frac{3\pi}{p})}\right]\left[\frac{sin(\frac{6k\pi}{p})/2^k}{sin(\frac{2k\pi}{p})/2^k}\right] \\
&= \left[\frac{sin(\frac{\pi}{p})}{sin(\frac{3\pi}{p})}\right]\left[\frac{sin(\frac{3(2k+1)\pi}{p}-\frac{3\pi}{p})}{sin(\frac{(2k+1)\pi}{p}-\frac{\pi}{p})}\right] \\
&= \left[\frac{sin(\frac{\pi}{p})}{sin(\frac{3\pi}{p})}\right]\left[\frac{sin(2\pi+(\pi-\frac{3\pi}{p}))}{sin(\pi-\frac{\pi}{p})}\right] \\
&= \left[\frac{sin(\frac{\pi}{p})sin(\pi-\frac{3\pi}{p})}{sin(\frac{3\pi}{p})sin(\pi-\frac{\pi}{p})}\right] \\
&= 1.
\end{aligned}
$$

This result completes the proof of Lemma 8.
Theorem 9. Let $p = 2k+1$ and $C_p =< a : a^p = 1 >$. Then

$$\prod_{j=1}^{k}(-1 + a^j + a^{-j}) = 1.$$

Proof. Let $\varepsilon = e^{2\pi i/p}$. Consider the faithful representation G over $\mathbb{C}$ of degree p:

$$
\begin{aligned}
&\rho : G \to GL(p, \mathbb{C}) \\
&a \to \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & \varepsilon & 0 & & & \\ 0 & 0 & \varepsilon^2 & & & \\ & & & . & & \\ & & & & . & \\ & & & & & . \\ & & & & & \varepsilon^{p-1} \end{bmatrix}.
\end{aligned}
$$

By extending representation ρ over $\mathbb{Z}$ we get again a faithful representation:

$$\bar{\rho} : \mathbb{Z}G \to C^{p\times p}$$

$$\sum_{j=0}^{p-1} \gamma_j a^j \to \sum_{j=0}^{p-1} \gamma_j p(a^j)$$

$$\bar{\rho}(\prod_{j=1}^{k} -1 + a^j + a^{-j})$$

$$= \prod_{j=1}^{k} \bar{\rho}(-1 + a^j + a^{-j})$$

$$= \prod_{j=1}^{k} (-I + \rho(a^j) + \rho(a^{-j}))$$

$$= \prod_{j=1}^{k} (-I + \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & \varepsilon & 0 & & & \\ 0 & 0 & \varepsilon^2 & & & \\ & & & . & & \\ & & & & . & \\ & & & & & . \\ & & & & & \varepsilon^{p-1} \end{bmatrix}^{j} + \begin{bmatrix} 1 & 0 & 0 & & & \\ 0 & \varepsilon & 0 & & & \\ 0 & 0 & \varepsilon^2 & & & \\ & & & . & & \\ & & & & . & \\ & & & & & . \\ & & & & & \varepsilon^{p-1} \end{bmatrix}^{-j})$$

$$= \prod_{j=1}^{k} \begin{bmatrix} -1 + 1^j + 1^{-j} & 0 & 0 & & \\ 0 & -1 + \varepsilon^j + \varepsilon^{-j} & 0 & & \\ 0 & 0 & -1 + \varepsilon^{2j} + \varepsilon^{-2j} & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & -1 + \varepsilon^{(p-1)j} + \varepsilon^{-(p-1)j} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & & \\ 0 & \prod_{j=1}^{k} -1 + \varepsilon^j + \varepsilon^{-j} & 0 & & \\ 0 & 0 & \prod_{j=1}^{k} -1 + \varepsilon^{2j} + \varepsilon^{-2j} & & \\ & & . & & \\ & & & . & \\ & & & & . \\ & & & & \prod_{j=1}^{k} -1 + \varepsilon^{(p-1)j} + \varepsilon^{-(p-1)j} \end{bmatrix} = I.$$

Since $\bar{\rho}$ is a faithful representation, we obtain $\prod_{j=1}^{k} -1 + a^j + a^{-j} = 1$.This is the end of the proof.

Corollary 10. $-1 + a^i + a^{-i}$ is a simple trinomial unit with inverse

$$\prod_{i=1//i\neq j}^{k} -1 + a^i + a^{-i}.$$

Theorem 11. Let $\gamma \in U_1(\mathbb{Z}C_p)$ be a simple trinomial unit. Then γ has the following form:

$$\gamma = a^i(-1 + a^j + a^{-j}),$$

where $i, j \in \mathbb{Z}$ and $0 < i,\ j < p$.

Proof. Let $\gamma \in U_1(\mathbb{Z}C_p)$ be a simple trinomial unit. Then γ can be written as follows:

$$\gamma = -a^i + a^j + a^k.$$

For some distinct integers $0 < i,\ j, k < p$, choose $\beta = a^i\gamma$. Then

$$\beta = -1 + a^{j-i} + a^{k-i}$$

or equally we write

$$\beta = -1 + a^j + a^k.$$

Now let us demonstrate $a^k = a^{-j}$. Let $\varepsilon = e^{2\pi i/p}$ be primitive root of unit. Consider the multiplicative group isomorphism:

$$\psi : C_p \to <\varepsilon>,$$
$$a^i \mapsto \varepsilon^i.$$

By extending linearly over $\mathbb{Z}$ we get ring epimorphism:

$$\begin{aligned}
&\psi : \mathbb{Z}C_p \to \mathbb{Z}[\varepsilon]\\
&\sum_{i=0}^{p-1} \gamma_i a^i \to \sum_{i=0}^{p-1} \gamma_i \varepsilon^i\\
1 =&\bar{\psi}(\beta\beta^{-1})\\
=&\bar{\psi}(-1 + a^j + a^k)\bar{\psi}(\beta^{-1})\\
=&\bar{\psi}(-1 + \varepsilon^j + \varepsilon^k)\bar{\psi}(\beta^{-1})\\
=&(-1 + cos(\frac{2\pi j}{p}) + isin(\frac{2\pi j}{p}) + cos(\frac{2\pi k}{p}) + isin(\frac{2\pi k}{p}))\bar{\psi}(\beta)^{-1},
\end{aligned}$$

which implies that

$$\bar{\psi}(\beta)^{-1} = (-1 + cos(\frac{2\pi j}{p}) + cos(\frac{2\pi k}{p})) - i(sin(\frac{2\pi j}{p}) + sin(\frac{2\pi k}{p})).$$

Therefore, we get

$$\begin{aligned}
1 =&(-1+cos(\frac{2\pi j}{p})+cos(\frac{2\pi k}{p}))^2-(sin(\frac{2\pi j}{p})+sin(\frac{2\pi k}{p}))^2\\
=&1+cos^2(\frac{2\pi j}{p})+cos^2(\frac{2\pi k}{p})-2cos(\frac{2\pi j}{p})-2cos(\frac{2\pi k}{p})+2cos(\frac{2\pi j}{p})2cos(\frac{2\pi k}{p})\\
&+sin^2(\frac{2\pi j}{p})+sin^2(\frac{2\pi k}{p})+2sin(\frac{2\pi j}{p})2sin(\frac{2\pi k}{p})\\
=&3-2(cos(\frac{2\pi j}{p})+cos(\frac{2\pi k}{p})-cos(\frac{2\pi(j-k)}{p}))\\
=&cos(\frac{2\pi j}{p})+cos(\frac{2\pi k}{p})-cos(\frac{2\pi(j-k)}{p}). \qquad (2.1)
\end{aligned}$$

Now, by denoting $cos(\frac{2\pi j}{p})=x$ and $cos(\frac{2\pi j}{p})=y$, we can simplify equation (2.1) as follows:

$$\begin{aligned}
1=x+y-(xy-\sqrt{1-x^2}\sqrt{1-y^2}) \Rightarrow&(1-x-y+xy)^2=(1-x^2)(1-y^2)\\
\Rightarrow&(1-x)^2(1-y)^2=(1-x^2)(1-y^2)\\
\Rightarrow&1-x-y+xy=1+x+y+xy\\
\Rightarrow&y=-x
\end{aligned}$$

$$\begin{aligned}
\Rightarrow&cos(\frac{2\pi j}{p})=-cos(\frac{2\pi k}{p})\\
\Rightarrow&cos(\frac{2\pi j}{p})=cos(\pi-\frac{2\pi k}{p})\\
\Rightarrow&\frac{2\pi(j+k)}{p}=\pi\\
\Rightarrow&j=-k(modp)\\
\Rightarrow&a^k=a^{-j}.
\end{aligned}$$

Hence, it is derived that

$$\beta=-1+a^j+a^{-j} \ or \ \gamma=a^i(-1+a^j+a^{-j}). \qquad (2.2)$$

Remark 12. In the equation (2.2) $\beta=-1+a^j+a^(-j)$ is called standard form of simple trinomial unit. If $p=2k+1$, then there are only k distinct simple trinomial unit in the standard form.

Example 13. Consider the unit group $U_1(\mathbb{Z}C_5)$ and $U_1(\mathbb{Z}C_7)$. Then

$$U_1(\mathbb{Z}C_n)=C_5\times<-1+a+a^{-1}>.$$

The torsion free part of the unit group is generated by the simple trinomial unit. Similarly, we have

$$U_1(\mathbb{Z}C_7)=C_7\times<-1+a+a^{-1},\ -1+2(a^2+a^{-2})-(a^3+a^{-3})>.$$

Since

$$(-1+a+a^{-1})^2(-1+a^2+a^{-2})=-1+2(a^2+a^{-2})-(a^3+a^{-3}),$$

then we write

$$U_1(\mathbb{Z}C_7) = C_7 \times < -1 + a + a^{-1}, -1 + a^2 + a^{-2} > .$$

That is, torsion free part of the unit group is generated by the simple trinomial units in the standard form.

Now let us concentrate on unit group $U_1(\mathbb{Z}C_p)$ of prime order group C_p. For $p = 2k+1$, by computing the rank of torsion free part, we have

$$n = p = 2k+1, n_2 = 0, l = 2 \Rightarrow r = \frac{1}{2}((2k+1) + 1 - 0 - 2) = k.$$

Torsion free part of $U_1(ZC_p)$ is generated by k units and there are k simple trinomial units in the standard form for $U_1(ZC_p)$. We claim the following hypothesis.

Conjecture. Let $5 \leq p$ be a prime number. Then

$$U_1(ZC_p) = C_p \times F,$$

where F is generated by all simple trinomial units in the standard form.

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