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On the chromatic polynomial and the domination number of k-Fibonacci cubes

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Abstract: Fibonacci cubes are defined as subgraphs of hypercubes, where the vertices are those without two consecutive 1's in their binary string representation. k-Fibonacci cubes are in turn special subgraphs of Fibonacci cubes obtained by eliminating certain edges. This elimination is carried out at the step analogous to where the fundamental recursion is used to construct Fibonacci cubes themselves from the two previous cubes by link edges. In this work, we calculate the vertex chromatic polynomial of k-Fibonacci cubes for $k = 1, 2$. We also determine the domination number and the total domination number of k-Fibonacci cubes for $n, k \leq 12$ by using an integer programming formulation.

Key words: Hypercube, Fibonacci cube, Fibonacci number, k-Fibonacci cube, vertex coloring, domination

1. Introduction

The n-dimensional hypercube Q_n is the n-fold Cartesian product of the complete graph with two vertices, and can be decomposed into two copies of Q_{n-1} connected to each other by a perfect matching for $n \geq 1$. Vertices of Q_n are labeled with binary strings of length n. Two vertices are adjacent if the binary string representation of them differ in only one coordinate, that is, their Hamming distance is one.

Hsu defined the n-dimensional Fibonacci cube Γ_n as a special subgraph of Q_n in [5]. Γ_n is induced by vertices whose binary representation do not contain two consecutive 1's. For convenience, Γ_0 is defined as Q_0, the graph with a single vertex and no edges. Many interesting properties of Γ_n including representations, recursive construction, Hamiltonicity and degree sequences is given in the survey [8]. The induced d-dimensional hypercubes in Γ_n are studied in [9, 11, 17]. The distance polynomial called the q-cube polynomial is defined in [18], which keeps track of the number of subcubes that are at a given distance from the all zero vertex. By extending this idea, daisy cubes including the Fibonacci cubes are defined and generalized in [10, 20]. The boundary enumerator polynomial of the induced hypercubes in Γ_n is considered in [19].

Special subgraphs and generalizations of the Fibonacci cubes have also been studied. For instance, by removing some vertices in Γ_n, Lucas cubes are obtained [13]. Fibonacci (p, r)-cubes are presented in [3]. The generalized Fibonacci cube $Q_n(f)$ is defined in [7], as the graph obtained from Q_n by removing all vertices that contain some forbidden binary string f as a factor. With this formulation one has $\Gamma_n = Q_n(11)$. Pell graphs are defined on certain ternary strings and turn out to be subgraphs of Fibonacci cubes of odd index

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[12]. Recently, by eliminating certain edges from the Γ_n, a special subgraph family called k-Fibonacci cubes Γ_n^k have been introduced (see [4, Section 3]).

In this work, we construct the vertex chromatic polynomial of k-Fibonacci cubes for $k=1,2$. We also determine the domination number and the total domination number of k-Fibonacci cubes for $n,k \leq 12$ using an integer linear programming approach as considered in [1, 6].

2. Preliminaries

The vertex set and the edge set of the n-dimensional Fibonacci cube $\Gamma_n=(V,E)$ can be written as

$$\begin{aligned} V &= \{b_1 b_2 \ldots b_n \mid b_i \in \{0,1\},\ 1 \leq i \leq n-1 \text{ with } b_i \cdot b_{i+1} = 0\} \\ E &= \{\{u,v\} \mid u,v \in V(\Gamma_n) \text{ and } d_H(u,v)=1\}, \end{aligned}$$

where d_H denotes the Hamming distance, that is, the number of different coordinates. Note that the distance between two vertices u and v in a connected graph G is defined as the length of a shortest path between u and v in G.

It is known that the number of vertices of Γ_n is f_{n+2} where f_n is the n-th Fibonacci number. These are defined by the recursion $f_n = f_{n-1} + f_{n-2}$ for $n \geq 2$, with $f_0 = 0$ and $f_1 = 1$. Reflecting this numerical recursion, Γ_n has a useful decomposition called the fundamental decomposition [8], denoted symbolically by $\Gamma_n = 0\Gamma_{n-1} + 10\Gamma_{n-2}$. Here, Γ_n is decomposed into the subgraphs induced by the vertices that start with 0 and 10, respectively. The former constitute a graph isomorphic to Γ_{n-1} and the latter constitute a graph isomorphic to Γ_{n-2}. Furthermore, there is a perfect matching between $10\Gamma_{n-2}$ and $00\Gamma_{n-2} \subset 0\Gamma_{n-1}$. The edges in this matching are called *link* edges.

k-Fibonacci cubes Γ_n^k are defined as special subgraphs of Fibonacci cubes obtained by eliminating certain edges [4]. This elimination is carried out at the step analogous to where the fundamental recursion is used to construct Fibonacci cubes from the two previous cubes by link edges. The fundamental decomposition of Fibonacci cubes introduces f_n link edges. Let $n_0 = n_0(k)$ be the smallest integer for which $f_{n_0} > k$. Then for any nonnegative integer $n < n_0$ the k-Fibonacci cubes are defined as $\Gamma_n^k = \Gamma_n$. For any integer $n \geq n_0$, the graphs Γ_n^k are defined by the recursion $\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k$ where only the first k link edges between $10\Gamma_{n-2}^k$ and $00\Gamma_{n-2}^k \subset 0\Gamma_{n-1}^k$ are included. That is, the link edges past the first k pairs of vertices in Γ_{n-2}^k in the binary ordering of the vertices from the smallest to the largest are discarded. For illustrations, we present the first five 2-Fibonacci cubes in Figure 1 and the graphs Γ_5^2, Γ_5^3 and Γ_5^4 in Figure 2.

In Figure 3, we present the structure of the adjacency matrix A_n in terms of A_{n-1} and A_{n-2} and structure of the adjacency matrix A_n^k in terms of A_{n-1}^k and A_{n-2}^k of Γ_n and Γ_n^k, respectively.

By an admissible (vertex) coloring of a simple graph G, we mean an assignment of colors from a coloring kit with x colors to the vertices of G in such a way that no adjacent vertices are given the same color. Let $p(G,x)$ denote the number of such colorings of G. It is well known that $p(G,x)$ is a polynomial in x of degree equal to the number of vertices of G, called the *chromatic polynomial* of G.

A set $D \subseteq V$ is called a dominating set of G if every vertex in $V \setminus D$ is adjacent to some vertex in D. The domination number of G is defined as the minimum cardinality of a dominating set of G, denoted by $\gamma(G)$. Similarly, $D \subseteq V$ is called a total dominating set of an isolate-free graph G if every vertex in V is adjacent to some vertex in D. The total domination number of G is defined as the minimum cardinality

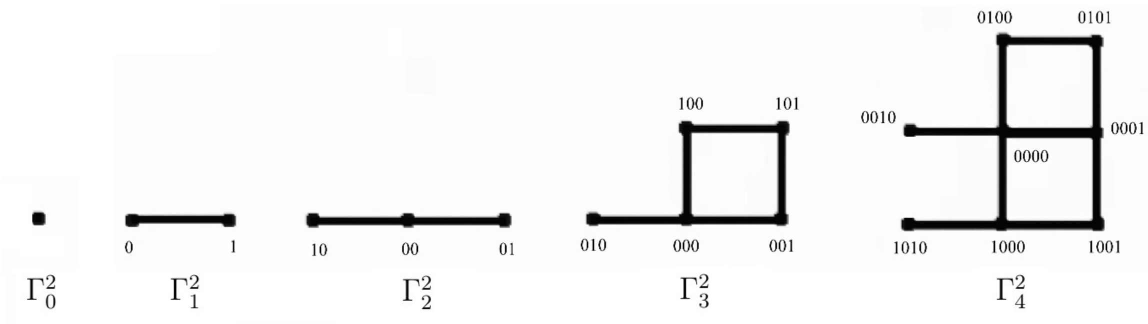

Figure 1. 2-Fibonacci cubes Γ_0^2 through Γ_4^2.

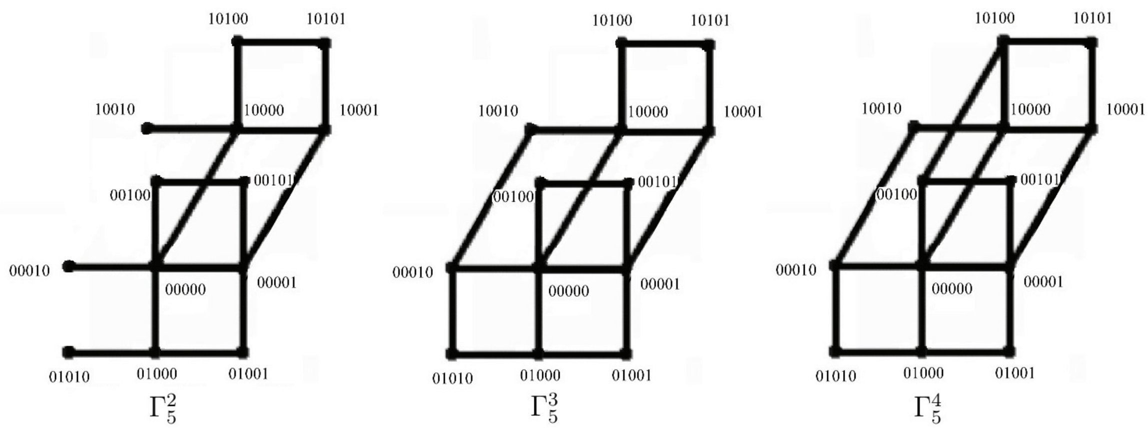

Figure 2. k-Fibonacci cubes Γ_5^2, Γ_5^3 and Γ_5^4.

$$A_n = \left(\begin{array}{c|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_{n-1}}} & \mathbf{I} \\ \cline{3-3} \multicolumn{2}{c|}{} & 0 \\ \hline \mathbf{I} & 0 & A_{n-2} \end{array}\right) \qquad A_n^k = \left(\begin{array}{c|c|c} \multicolumn{2}{c|}{\multirow{2}{*}{A_{n-1}^k}} & \mathbf{I}_k \;|\; 0 \\ \cline{3-3} \multicolumn{2}{c|}{} & 0 \\ \hline \begin{array}{c}\mathbf{I}_k \\ \hline 0\end{array} & 0 & A_{n-2}^k \end{array}\right)$$

Figure 3. Left: Adjacency matrix A_n of Γ_n where $\mathbf{I}$ is the $f_n \times f_n$ identity matrix, and the remaining elements are zeros. Right: Adjacency matrix A_n^k of Γ_n^k where $\mathbf{I_k}$ is the $k \times k$ identity matrix, and the remaining elements are zeros.

of a total dominating set of G, denoted by $\gamma_t(G)$. Bounds on the domination number and total domination number of Γ_n are obtained in [1, 2, 14, 15] by using the degree information and the decomposition of Γ_n. Some improvements appear in [16]. By an integer linear programming formulation, the exact values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ is calculated for small values of n in [6] and [1] respectively. We use a similar approach to determine $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$ for $n, k \leq 12$ in Section 4.

3. Chromatic polynomials

To construct the chromatic polynomials of k-Fibonacci cubes Γ_n^k for $k = 1, 2$, first, we present two basic results that we will use in our proof in Section 3.1.

There are two extreme classes of graphs for which $p(G, x)$ is easy to compute:

$$p(F_n, x) = x^n, \quad p(K_n, x) = (x)_n$$

where F_n is the graph on n vertices with no edges, K_n is the complete graph on n vertices and $(x)_n = x(x-1)\cdots(x-n+1)$ is the lower (or falling) factorial. One way to calculate the chromatic polynomial (in the power basis) is the basic recursion:

$$p(G, x) = p(G - e, x) - p(G/e, x) \ . \tag{3.1}$$

Here for $e \in E$, the graph $G - e$ in (3.1) is obtained from G by removing the edge e from E, and G/e is obtained from G by contracting the edge e. In contraction we progressively shrink e until the end points collapse into a single vertex. If multiple edges are created by this process, we collapse them into single edges.

Alternately, for an edge $e \notin E$,

$$p(G, x) = p(G + e, x) + p(G/e, x) \ . \tag{3.2}$$

where $G + e$ is the graph obtained by adding e to E. (3.2) is the recursion that can be used to express $p(G, x)$ in the lower factorial basis $\{(x)_n\}_{n \geq 1}$.

The chromatic polynomials of the Fibonacci cubes themselves for up to $n = 5$ are as follows*:

$$\begin{aligned}
p(\Gamma_0, x) &= x \\
p(\Gamma_1, x) &= x(x-1) \\
p(\Gamma_2, x) &= x(x-1)^2 \\
p(\Gamma_3, x) &= x(x-1)^2(x^2 - 3x + 3) \\
p(\Gamma_4, x) &= x(x-1)(x^2 - 3x + 3)^3 \\
p(\Gamma_5, x) &= x(x-1)(x^{11} - 19x^{10} + 171x^9 - 960x^8 + 3732x^7 - 10544x^6 \\
&\qquad +22088x^5 - 34314x^4 + 38774x^3 - 30408x^2 + 14942x - 3499).
\end{aligned}$$

There does not seem to be a nice expression for these polynomials for arbitrary n.

3.1. Chromatic polynomials of the k-Fibonacci cubes

We need the following lemma on chromatic polynomials.

Lemma 3.1 *Let $G = (V, E)$ be a simple graph, $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are subgraphs of G such that $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$. Suppose $|V_1 \cap V_2| = r > 0$ and the subgraph induced by $V_1 \cap V_2$ in G is the complete graph K_r (in other words G_1 and G_2 share the common subgraph K_r). Then we have*

$$p(G, x) = \frac{p(G_1, x)p(G_2, x)}{(x)_r} \ .$$

*As computed by the ChromaticPolynomial functionality of Mathematica.

Proof We prove this result in the form

$$(x)_r\ p(G,x) = p(G_1,x)p(G_2,x)$$

by constructing a bijection between a pair of admissible colorings (C_1, C_2) of the pair of graphs K_r and G, and the pair of admissible colorings (C'_1, C'_2) of the graphs G_1 and G_2. Given (C_1, C_2), C'_2 is defined as the restriction of C_2 to G_2. C'_1 is defined as the admissible coloring of G_1 obtained by renaming some of the colors (i.e. those assigned to the copy K_r in the coloring C_2) by the colors used in C_1. The colors that do not appear in C_1 are left as they are in C_2 in constructing C'_1. It is easy to see that C'_1 and C'_2 are admissible colorings of G_1 and G_2 respectively, and that this map has an inverse. □

So if G_1 and G_2 share only a common vertex, a common edge, or a common triangle, then

$$\begin{aligned} p(G,x) &= \frac{p(G_1,x)p(G_2,x)}{x}, \\ p(G,x) &= \frac{p(G_1,x)p(G_2,x)}{x(x-1)}, \\ p(G,x) &= \frac{p(G_1,x)p(G_2,x)}{x(x-1)(x-2)}, \end{aligned}$$

respectively.

For $k=1$, the graphs Γ^1_n are all trees. If we think of them as rooted at the all zero vertex, starting with the trees Γ^2_0 and Γ^2_1 on 1 and 2 vertices, respectively, the next tree in the sequence is obtained by making the previous tree a principal subtree of the current one. Since the chromatic polynomial of a tree with m nodes is $x(x-1)^{m-1}$, we have

$$p(\Gamma^1_n, x) = x(x-1)^{f_{n+2}-1}\ .$$

For $k=2$, $\Gamma^2_n = \Gamma_n$ for $n \le 2$ and for $n \ge 3$, Γ^2_n consists of $f_n - 1$ $(1, 2, 4, 7, 12, \ldots)$ squares (4-cycles) glued by their edges and f_{n-1} pendant vertices (see [4]). We note that $k=2$ of the k-Fibonacci cubes is a nontrivial case in which the chromatic polynomials can be explicitly constructed.

Theorem 3.2 *For $n \ge 1$, the chromatic polynomial of Γ^2_n is given by*

$$p(\Gamma^2_n, x) = x(x-1)^{f_{n-1}+1}(x^2-3x+3)^{f_n-1}\ . \tag{3.3}$$

Proof For $n = 1, 2$ the graphs Γ^2_n are trees on 2 and 3 vertices respectively, and therefore

$$p(\Gamma^2_1, x) = x(x-1), \quad p(\Gamma^2_2, x) = x(x-1)^2$$

which are of the form (3.3).

For $n \ge 3$, the graph Γ^2_n is constructed from Γ^2_{n-1} and Γ^2_{n-2} as symbolically indicated in Figure 4. The vertices labeled 0 and 1 are the vertices with labels $0\ldots00$ and $0\ldots01$ in Γ^2_{n-1} (and also in Γ^2_n), whereas the labels $0'$ and $1'$ are the vertices labeled $0\ldots00$ and $0\ldots01$ in Γ^2_{n-2}, which are labeled as $10\ldots00$ and $10\ldots01$ in Γ^2_n after the addition of the link edges. Let e denote the link edge from 1 to $1'$ as shown in Figure

4 and put $G = \Gamma_n^2$. $G - e$ consists of the union of two graphs G_1 and G_2, where G_1 is obtained from Γ_{n-1}^2 by adding the vertex $0'$ and the edge $\{0, 0'\}$; G_2 is obtained from Γ_{n-2}^2 by adding the vertex 0 and the edge $\{0, 0'\}$. Then

$$p(G_1, x) = p(\Gamma_{n-1}^2, x)(x-1), \qquad p(G_2, x) = p(\Gamma_{n-2}^2, x)(x-1),$$

and therefore by Lemma 3.1

$$p(G - e, x) = \frac{p(\Gamma_{n-1}^2, x)p(\Gamma_{n-2}^2, x)(x-1)^2}{x(x-1)} = p(\Gamma_{n-1}^2, x)p(\Gamma_{n-2}^2, x)\frac{(x-1)}{x} \; .$$

In G/e, denote the vertex obtained by the identification of the endpoints 1 and $1'$ of e by v. G/e consists of the union of two graphs H_1 and H_2, where H_1 is obtained from Γ_{n-1}^2 by adding the vertex $0'$ and the edges $\{0, 0'\}$ and $\{0', v\}$; H_2 is obtained from Γ_{n-2}^2 by adding the vertex 0 and the edges $\{0, 0'\}$ and $\{0, v\}$. H_1 and H_2 meet at the triangle with vertices $0, 0', v$. Therefore

$$p(H_1, x) = p(\Gamma_{n-1}^2, x)(x-2), \qquad p(H_2, x) = p(\Gamma_{n-2}^2, x)(x-2),$$

and by Lemma 3.1,

$$\begin{aligned} p(G/e, x) &= \frac{p(\Gamma_{n-1}^2, x)p(\Gamma_{n-2}^2, x)(x-2)^2}{x(x-1)(x-2)} \\ &= p(\Gamma_{n-1}^2, x)p(\Gamma_{n-2}^2, x)\frac{(x-2)}{x(x-1)} \; . \end{aligned}$$

By recursion (3.1),

$$\begin{aligned} p(G, x) &= p(\Gamma_{n-1}^2, x)p(\Gamma_{n-2}^2, x)\left(\frac{x-1}{x} - \frac{(x-2)}{x(x-1)}\right) \\ &= p(\Gamma_{n-1}^2, x)p(\Gamma_{n-2}^2, x)\left(\frac{x^2 - 3x + 3}{x(x-1)}\right) \end{aligned}$$

and the result follows by induction on n. $\square$

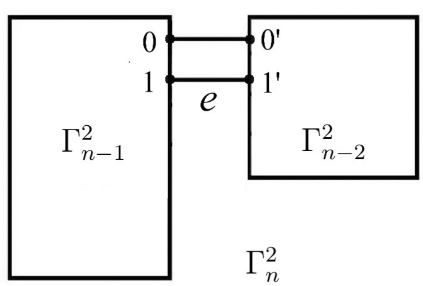

Figure 4. The construction of Γ_n^2 from Γ_{n-1}^2 and Γ_{n-2}^2 by adding $k = 2$ link edges.

Remark 3.3 *We have* $\Gamma_n^3 = \Gamma_n^4 = \Gamma_n$ *for* $n \leq 4$ *and therefore for these graphs the chromatic polynomials are the same as those of the Fibonacci cubes themselves. For* $n = 5$ *we have*

$$\begin{aligned} p(\Gamma_5^3, x) &= x(x-1)(x^2-3x+3)^2\left(x^7 - 11x^6 + 55x^5 - 161x^4 + 298x^3\right. \\ &\quad \left. -350x^2 + 244x - 79\right), \\ p(\Gamma_5^4, x) &= x(x-1)\left(x^{11} - 18x^{10} + 153x^9 - 809x^8 + 2955x^7 - 7830x^6\right. \\ &\quad \left. +15367x^5 - 22360x^4 + 23675x^3 - 17410x^2 + 8026x - 1763\right). \end{aligned}$$

Again, for $k \geq 3$, there does not seem to be a nice expression for these chromatic polynomials for arbitrary n.

4. Domination number and total domination number of k-Fibonacci cubes

In this section we first prove upper and lower bounds on $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$. Using the definition of Γ_n^k and the recursion $\Gamma_n^k = 0\Gamma_{n-1}^k + 10\Gamma_{n-2}^k$ we obtain the following result.

Theorem 4.1 *For any positive integer* n *and* k *we have*

$$\gamma(\Gamma_n^{k+1}) \leq \gamma(\Gamma_n^k) \leq \gamma(\Gamma_{n-1}^k) + \gamma(\Gamma_{n-2}^k) \textit{ and } \gamma_t(\Gamma_n^{k+1}) \leq \gamma_t(\Gamma_n^k) \leq \gamma_t(\Gamma_{n-1}^k) + \gamma_t(\Gamma_{n-2}^k) \; .$$

Proof By the definition of k-Fibonacci cubes, Γ_n^k can be obtained from Γ_n^{k+1} by removing certain edges. This means that a (total) dominating set for Γ_n^k is also a (total) dominating set for Γ_n^{k+1}, which gives

$$\gamma(\Gamma_n^{k+1}) \leq \gamma(\Gamma_n^k) \text{ and } \gamma_t(\Gamma_n^{k+1}) \leq \gamma_t(\Gamma_n^k) \; .$$

Consider the fundamental decomposition of Γ_n^k into the subgraphs induced by the vertices that start with 0 and 10, which are isomorphic to the graphs Γ_{n-1}^k and Γ_{n-2}^k, respectively. Then we have

$$\gamma(\Gamma_n^k) \leq \gamma(\Gamma_{n-1}^k) + \gamma(\Gamma_{n-2}^k) \text{ and } \gamma_t(\Gamma_n^k) \leq \gamma_t(\Gamma_{n-1}^k) + \gamma_t(\Gamma_{n-2}^k) \; .$$

□

Next we describe a general integer linear programming formulation used in [6] to find $\gamma(\Gamma_n)$. A similar approach is used in [1] to find $\gamma_t(\Gamma_n)$. We also use integer linear programming to obtain $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$ for $n, k \leq 12$.

Let $N(v)$ denote the set of vertices adjacent to v and $N[v] = N(v) \cup \{v\}$. Suppose each vertex $v \in V(\Gamma_n^k)$ is associated with a binary variable x_v. The problems of determining $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$ can be expressed as a problem of minimizing the objective function

$$\sum_{v \in V(\Gamma_n^k)} x_v \tag{4.1}$$

subject to the following constraints for every $v \in V(\Gamma_n^k)$:

$$\begin{aligned} \sum_{a \in N[v]} x_a &\geq 1 \text{ (for the domination number),} \\ \sum_{a \in N(v)} x_a &\geq 1 \text{ (for the total domination number).} \end{aligned}$$

The value of the objective function gives $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$ respectively. Note that this problem has $|V(\Gamma_n^k)| = f_{n+2}$ variables and f_{n+2} constraints.

Table 1. Values of $\gamma(\Gamma_n)$ and $\gamma(\Gamma_n^k)$ for $n, k \leq 12$.

n	2	3	4	5	6	7	8	9	10	11	12
$\lvert V(\Gamma_n^k)\rvert$	3	5	8	13	21	34	55	89	144	233	377
$\gamma(\Gamma_n)$	1	2	3	4	5	8	12	17	25	39	55-60
$\gamma(\Gamma_n^1)$	1	2	3	5	8	13	21	34	55	89	144
$\gamma(\Gamma_n^2)$	1	2	3	5	8	13	21	34	55	89	144
$\gamma(\Gamma_n^3)$	1	2	3	4	6	10	16	26	42	68	110
$\gamma(\Gamma_n^4)$	1	2	3	4	6	10	16	26	42	68	110
$\gamma(\Gamma_n^5)$	1	2	3	4	6	9	14	23	37	60	97
$\gamma(\Gamma_n^6)$	1	2	3	4	5	8	13	21	34	55	89
$\gamma(\Gamma_n^7)$	1	2	3	4	5	8	13	21	34	55	89
$\gamma(\Gamma_n^8)$	1	2	3	4	5	8	13	21	34	55	89
$\gamma(\Gamma_n^9)$	1	2	3	4	5	8	13	20	32	52	84
$\gamma(\Gamma_n^{10})$	1	2	3	4	5	8	13	20	32	52	84
$\gamma(\Gamma_n^{11})$	1	2	3	4	5	8	13	20	32	52	84
$\gamma(\Gamma_n^{12})$	1	2	3	4	5	8	12	19	31	50	81

We implemented the integer linear programming problem (4.1) using the Gurobi[†] optimization package and obtained the values of $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$ for $n, k \leq 12$. We collect the known values of $\gamma(\Gamma_n)$ and $\gamma_t(\Gamma_n)$ for $n \leq 12$ (see, [1, 6]) and the new values of $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$ for $n, k \leq 12$ in Table 4 and Table 2 respectively.

Here we note that $\Gamma_n^k = \Gamma_n$ for $f_n > k$, that is, $\gamma(\Gamma_n^k) = \gamma(\Gamma_n)$ for $f_n > k$ in Table 4 and $\gamma_t(\Gamma_n^k) = \gamma_t(\Gamma_n)$ for $f_n > k$ in Table 2. Furthermore, the old bounds for $\gamma(\Gamma_{12})$ were 54-61 [1]. Our calculations improve this slightly to $55 \leq \gamma(\Gamma_{12}) \leq 60$.

Using Theorem 4.1 and the results in Tables 4 and 2 we give the following upper bounds on $\gamma(\Gamma_n^k)$ and $\gamma_t(\Gamma_n^k)$.

Corollary 4.2 *As a function of n and k we have the following upper bounds on $\gamma(\Gamma_n^k)$:*

- *If $n \geq 13$ and $k \in \{1, 2\}$, then $\gamma(\Gamma_n^k) \leq f_n$.*
- *If $n \geq 13$ and $k \in \{3, 4\}$, then $\gamma(\Gamma_n^k) \leq 42f_{n-8} - 16f_{n-10}$.*
- *If $n \geq 13$ and $k = 5$, then $\gamma(\Gamma_n^k) \leq 37f_{n-8} - 14f_{n-10}$.*
- *If $n \geq 13$ and $k \in \{6, 7, 8, \}$, then $\gamma(\Gamma_n^k) \leq f_{n-1}$.*
- *If $n \geq 13$ and $k \in \{9, 10, 11\}$, then $\gamma(\Gamma_n^k) \leq 32f_{n-8} - 12f_{n-10}$.*
- *If $n \geq 13$ and $k \geq 12$, then $\gamma(\Gamma_n^k) \leq 31f_{n-8} - 12f_{n-10}$.*

[†]System Specification: Intel Core i7-4770K @3.50GHz, 12 GB RAM, 64-bit operating system.

Table 2. Values of $\gamma_t(\Gamma_n)$ and $\gamma_t(\Gamma_n^k)$ for $n, k \leq 12$.

n	2	3	4	5	6	7	8	9	10	11	12
$\lvert V(\Gamma_n^k)\rvert$	3	5	8	13	21	34	55	89	144	233	377
$\gamma_t(\Gamma_n)$	2	2	3	5	7	10	13	20	30	44	65
$\gamma_t(\Gamma_n^1)$	2	2	3	5	8	13	21	34	55	89	144
$\gamma_t(\Gamma_n^2)$	2	2	3	5	8	13	21	34	55	89	144
$\gamma_t(\Gamma_n^3)$	2	2	3	5	8	13	21	34	55	89	144
$\gamma_t(\Gamma_n^4)$	2	2	3	5	8	13	21	34	55	89	144
$\gamma_t(\Gamma_n^5)$	2	2	3	5	7	10	16	26	42	68	110
$\gamma_t(\Gamma_n^6)$	2	2	3	5	7	10	16	26	42	68	110
$\gamma_t(\Gamma_n^7)$	2	2	3	5	7	10	16	26	42	68	110
$\gamma_t(\Gamma_n^8)$	2	2	3	5	7	10	15	23	37	60	97
$\gamma_t(\Gamma_n^9)$	2	2	3	5	7	10	14	22	36	58	94
$\gamma_t(\Gamma_n^{10})$	2	2	3	5	7	10	14	22	36	58	94
$\gamma_t(\Gamma_n^{11})$	2	2	3	5	7	10	14	22	36	58	94
$\gamma_t(\Gamma_n^{12})$	2	2	3	5	7	10	14	22	35	57	92

Proof We give the proof only for the case $k \in \{1, 2\}$ and note that the same proof is valid for all of the other stated cases. From Table 4 we know that $\gamma(\Gamma_{11}^k) = f_{11}$ and $\gamma(\Gamma_{12}^k) = f_{12}$ where $k \in \{1, 2\}$. Then for $n \geq 13$, using Theorem 4.1 we have $\gamma(\Gamma_n^k) \leq \gamma(\Gamma_{n-1}^k) + \gamma(\Gamma_{n-2}^k) \leq f_n$. □

Corollary 4.3 *As a function of n and k we have the following upper bounds on $\gamma_t(\Gamma_n^k)$:*

- *If $n \geq 13$ and $k \in \{1, 2, 3, 4\}$, then $\gamma_t(\Gamma_n^k) \leq f_n$.*
- *If $n \geq 13$ and $k \in \{5, 6, 7\}$, then $\gamma_t(\Gamma_n^k) \leq 42f_{n-8} - 16f_{n-10}$.*
- *If $n \geq 13$ and $k = 8$, then $\gamma_t(\Gamma_n^k) \leq 37f_{n-8} - 14f_{n-10}$.*
- *If $n \geq 13$ and $k \in \{9, 10, 11\}$, then $\gamma_t(\Gamma_n^k) \leq 36f_{n-8} - 14f_{n-10}$.*
- *If $n \geq 13$ and $k \geq 12$, then $\gamma_t(\Gamma_n^k) \leq 35f_{n-8} - 13f_{n-10}$.*

We find the exact values of domination and total domination numbers of Γ_n^k for $k \in \{1, 2\}$.

Proposition 4.4 *For any positive integer $n \geq 2$ and $k \in \{1, 2\}$ we have $\gamma(\Gamma_n^k) = \gamma_t(\Gamma_n^k) = f_n$.*

Proof Assume that $k \in \{1, 2\}$. From Tables 4 and 2 the statement is clear for $n \leq 12$. For $n \geq 13$, using the definition of domination and total domination numbers, Corollaries 4.2 and 4.3 we know that $\gamma(\Gamma_n^k) \leq \gamma_t(\Gamma_n^k) \leq f_n$. Furthermore, Theorem 4.1 implies that $\gamma(\Gamma_n^2) \leq \gamma(\Gamma_n^1)$. So, it is enough to show that $\gamma(\Gamma_n^2) \geq f_n$.

Let α and β be any Fibonacci string of length $n-3$ and $n-4$ respectively, and u, v be vertices of Γ_n^2 whose string representations are $\alpha 010$ and $\beta 0101$ respectively. We know that the number of such $\alpha 010$'s are

f_{n-1} and the number of such $\beta0101$'s are f_{n-2}. By the definition of Γ_n^2 we know that the degrees of such u's are 1 and the degrees of such v's are 2, that is, the closed neighborhood of these vertices are $N[u] = \{\alpha010, \alpha000\}$ and $N[v] = \{\beta0101, \beta0100, \beta0001\}$. Let D be a minimal dominating set for Γ_n^2. Then to dominate each u and v, D must include at least one vertex from $N[u]$ and one vertex from $N[v]$. Since $N[u] \cap N[v] = \emptyset$, we have $|D| \geq f_{n-1} + f_{n-2} = f_n$ which completes the proof. $\square$

Remark 4.5 *In Tables 4 and 2 we observe that* $\gamma(\Gamma_n^6) = \gamma(\Gamma_n^7) = \gamma(\Gamma_n^8) = f_{n-1}$ *for* $6 \leq n \leq 12$ *and* $\gamma_t(\Gamma_n^3) = \gamma_t(\Gamma_n^4) = f_n$ *for* $2 \leq n \leq 12$. *However the technique we used in the proof of Proposition 4.4 is not enough to prove these observations.*

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