

## Existence results of positive solutions for Kirchhoff type biharmonic equation via bifurcation methods\*

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**Abstract:** This paper is concerned with the existence of positive solutions for the fourth order Kirchhoff type problem

$$\begin{cases} \Delta^2 u - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda f(u(x)), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases}$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) is a bounded domain with smooth boundary  $\partial\Omega$ ,  $a > 0, b \geq 0$  are constants,  $\lambda \in \mathbb{R}$  is a parameter. For the case  $f(u) \equiv u$ , we use an argument based on the linear eigenvalue problems of fourth order elliptic equations to show that there exists a unique positive solution for all  $\lambda > \Lambda_{1,a}$ , here  $\Lambda_{1,a}$  is the first eigenvalue of the above problem with  $b = 0$ ; For the case  $f$  is sublinear, we prove that there exists a positive solution for all  $\lambda > 0$  and no positive solution for  $\lambda < 0$  by using bifurcation method.

**Key words:** Kirchhoff type biharmonic equation, global bifurcation, positive solution

### 1. Introduction

In this paper, we consider the following nonlinear fourth order Kirchhoff type problem

$$\begin{cases} \Delta^2 u - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda f(u(x)), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega \subset \mathbb{R}^N$  ( $N \geq 1$ ) is a bounded domain with smooth boundary  $\partial\Omega$ ,  $a > 0, b \geq 0$  are constants,  $\lambda \in \mathbb{R}$  is a parameter,  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous.

Problem (1.1) is closely related to the extensible beam model and Berger plate model which was proposed in [32] and [6], respectively.

In [32], to describe the deflection of an extensible beam of length  $L$  with hinged ends, Woinowsky-Krieger studied the equation

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho A} \frac{\partial^4 u}{\partial x^4} - \left( \frac{H}{\rho} + \frac{E}{2\rho L} \int_0^L \left| \frac{\partial u}{\partial x} \right|^2 dx \right) \frac{\partial^2 u}{\partial x^2} = 0, \quad (1.2)$$

where  $u = u(x, t)$  is the lateral displacement at the space coordinate  $x$  and the time  $t$ ; the letters  $H, E, \rho, I$  and  $A$  denote, respectively, the tension in the rest position, the Young elasticity modulus, the density, the cross-sectional moment of inertia and the cross-sectional area. This model was proposed to modify the theory

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of the dynamic Euler-Bernoulli beam, assuming a nonlinear dependence of the axial strain on the deformation of the gradient. Owing to its importance in engineering, physics and material mechanics, since such model was proposed, this class of problems has been studied. These studies are focused on the properties of its solutions, as can be seen in [4,5,10,27] and references therein.

In 1955, Berger [6] studied the equation

$$\frac{\partial^2 u}{\partial t^2} + \Delta^2 u + (Q + \int_{\Omega} |\nabla u|^2 dx) \Delta u = f(u, u_t, x) \quad (1.3)$$

which is called the Berger plate model [8], as a simplification of the von Karman plate equation which describes large deflection of plate, where the parameter  $Q$  describes in-plane forces applied to the plate and the function  $f$  represents transverse loads which may depend on the displacement  $u$  and the velocity  $u_t$ .

Problem (1.1) is a generalization of the stationary problem associated with problem (1.2) in dimension one or problem (1.3) in dimension two.

In one-dimensional case ( $N = 1$ ), by using variational methods and some fixed point theorems in cones, the existence and multiplicity results for (1.1) with other boundary conditions are considered in [22–25]. In multidimensional case, also by using the variational methods, [12,13,31] studied the existence and multiplicity of nontrivial solutions for (1.1); and for similar problems in the whole space  $\mathbb{R}^N$ , see [3,7,11,26,29,33–35] and the reference therein.

To the best of our knowledge, when parameter  $\lambda$  varies in  $\mathbb{R}$ , the bifurcation phenomena and global behavior of positive solutions for fourth order nonlocal problem like (1.1) have not been discussed. It is worth pointing out that, although global bifurcation theory has been applied to deal with fourth order local problems[similar to (1.1) with  $b = 0$ ] in [16,18,21,28], but when  $b \neq 0$ , the nonlocal term under the integral sign in equation will causes some mathematical difficulties which make the study of the problem particularly interesting, as shown by [1,9,14,19,20,30] in which second order Kirchhoff type problems have been considered.

Motivated by the above works described, this paper will study global bifurcation phenomena and the existence of positive solutions for problem (1.1).

Concretely, we are concerned with problem (1.1) under the two cases:  $f(u) \equiv u$  or  $f$  is sublinear. For  $f(u) \equiv u$ , (1.1) can be seen as a nonlinear eigenvalue problem, we use an argument based on the linear eigenvalue problems of fourth order equations to get the existence and uniqueness of positive solution for all  $\lambda > \Lambda_{1,a}$ , where  $\Lambda_{1,a}$  is the first eigenvalue of (1.1) with  $b = 0$ ; For the case  $f$  is sublinear, we study global bifurcation phenomena of (1.1) and prove that there exists a positive solution for all  $\lambda > 0$  and no positive solution for  $\lambda < 0$ .

The rest of paper is arranged as follows: In Section 2, as preliminaries, we first construct the operator equation corresponding to (1.1). In Section 3, we deal with the case  $f(u) \equiv u$  based on the linear eigenvalue problem of fourth order equations and their properties. Finally, for the case  $f$  is sublinear, we discuss the global bifurcation phenomena and existence of positive solutions for (1.1) in Section 4.

**2. Preliminaries**

Let  $\Omega \subset \mathbb{R}^N$  be a bounded domain with smooth boundary  $\partial\Omega$ . Denote  $\mu_k$  ( $k = 1, 2, \dots$ ) the eigenvalues and  $\varphi_k$  the corresponding eigenfunctions of the eigenvalue problem

$$\begin{cases} -\Delta u = \mu u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

it is well known that  $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \dots \leq \mu_k \rightarrow \infty$  and  $\varphi_1 > 0$  for  $x \in \Omega$ .

Consider the space  $H = H^2(\Omega) \cap H_0^1(\Omega)$ . The inequalities below are well known:

$$\int_{\Omega} |\Delta u|^2 dx \geq \mu_1 \int_{\Omega} |\nabla u|^2 dx, \quad \int_{\Omega} |\Delta u|^2 dx \geq \mu_1^2 \int_{\Omega} |u|^2 dx, \quad u \in H. \tag{2.2}$$

Then, it is easy to show that  $H$  is a Hilbert space with the norm

$$\|u\| = \left( \int_{\Omega} |\Delta u|^2 dx \right)^{\frac{1}{2}}. \tag{2.3}$$

We say that  $u \in H$  is a weak solution of problem (1.1) if

$$\int_{\Omega} \Delta u \Delta v dx + (a + b \int_{\Omega} |\nabla u|^2 dx) \int_{\Omega} \nabla u \cdot \nabla v dx = \lambda \int_{\Omega} f(u) v dx, \tag{2.4}$$

for any  $v \in H$ .

Let  $P := \{u \in H : u \geq 0, \text{ a.e. in } \Omega\}$  and  $U := P \cup (-P)$ .

**Proposition 2.1** For each  $g \in H$ , there exists a weak solution  $u \in H$  to the problem

$$\begin{cases} \Delta^2 u - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = g(x), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.5}$$

and if  $g \in U$ , then  $u$  is unique. Moreover, the operator  $T : U \rightarrow U$  defined by

$$T(g) := u$$

is compact.

**Proof.** When  $g \equiv 0$ , it is obviously that (2.5) has only a unique solution  $u = 0$ . Next, we prove the existence and uniqueness of solutions for (2.5) with  $g \neq 0$ .

For constant  $R \geq 0$ , consider the linear problem

$$\begin{cases} \Delta^2 u - (a + bR) \Delta u = g(x), & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega \end{cases} \tag{2.6}$$

which is equivalent to the system

$$\begin{cases} -\Delta u = w, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \\ -\Delta w + (a + bR)w = g(x), & \text{in } \Omega, \\ w = 0, & \text{on } \partial\Omega. \end{cases}$$

Since  $-(a + bR) < \mu_1$ , then by the elliptic a priori estimates, (see e.g., Gilbarg and Trudinger [15], Gupta [17]), (2.6) has a unique solution  $u_R \in H \cap H^4(\Omega)$  and

$$\|u_R\|_{H^4(\Omega)} \leq C_1 \|g\|_{L^2(\Omega)} \tag{2.7}$$

for some constant  $C_1 > 0$ . Moreover, if  $u_R \in H$  is a weak solution then  $\Delta u_R = 0$  on  $\partial\Omega$  in the trace sense and, in particular,  $\Delta u_R \in H$  and there exist some constant  $C_2 > 0$  such that

$$\|\Delta u_R\|_H \leq C_2 \|g\|_{L^2(\Omega)}. \tag{2.8}$$

Multiplying the equation in (2.6) by  $u_R$  and integrating, we have

$$\int_{\Omega} |\nabla u_R|^2 dx = \frac{\int_{\Omega} g(x)u_R(x)dx - \int_{\Omega} |\Delta u_R|^2 dx}{a + bR}. \tag{2.9}$$

Then to get a solution of (2.5), we only need to find  $R$  such that

$$R = y(R) := \frac{\int_{\Omega} g(x)u_R(x)dx - \int_{\Omega} |\Delta u_R|^2 dx}{a + bR} = \int_{\Omega} |\nabla u_R|^2 dx, \tag{2.10}$$

that is, find a fixed point of  $R = y(R)$ . Obviously,  $y(0) > 0$ . On the other hand, by (2.7) and (2.8) we have

$$|y(R)| = \frac{|\int_{\Omega} g(x)u_R(x)dx - \int_{\Omega} |\Delta u_R|^2 dx|}{a + bR} \leq C. \tag{2.11}$$

This concludes the existence of fixed point for  $R = y(R)$ .

Now, we show that if  $g \in U$ , the solution of (2.5) is unique. Without loss of generality, we assume on the contrary that for some  $g \in P$ , there exist two solutions  $u \neq \tilde{u}$ . By the maximum principle for the Laplacian, we have

$$u(x) \geq 0, \quad \Delta u(x) \leq 0, \quad \text{a.e. } x \in \Omega; \quad \tilde{u}(x) \geq 0, \quad \Delta \tilde{u}(x) \leq 0, \quad \text{a.e. } x \in \Omega. \tag{2.12}$$

Since  $u$  and  $\tilde{u}$  satisfy the equation in (2.5), then

$$\Delta^2 u - \Delta^2 \tilde{u} - [a + b \int_{\Omega} |\nabla u|^2 dx](\Delta u - \Delta \tilde{u}) - b[\int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |\nabla \tilde{u}|^2 dx]\Delta \tilde{u} = 0. \tag{2.13}$$

If  $\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |\nabla \tilde{u}|^2 dx$ , (2.13) implies that  $u - \tilde{u}$  is the trivial solution of (2.5) with  $g \equiv 0$ , then  $u = \tilde{u}$ .

If we assume that  $\int_{\Omega} |\nabla u|^2 dx > \int_{\Omega} |\nabla \tilde{u}|^2 dx$ , then by (2.13) and (2.12) we have

$$\Delta^2 u - \Delta^2 \tilde{u} - [a + b \int_{\Omega} |\nabla u|^2 dx](\Delta u - \Delta \tilde{u}) \leq 0,$$

by the maximum principle again, we conclude that

$$\Delta(u - \tilde{u}) \geq 0, \quad u - \tilde{u} \leq 0. \tag{2.14}$$

On the other hand, from the assumption  $\int_{\Omega} |\nabla u|^2 dx > \int_{\Omega} |\nabla \tilde{u}|^2 dx$  we have

$$\begin{aligned} 0 &< \int_{\Omega} |\nabla u|^2 dx - \int_{\Omega} |\nabla \tilde{u}|^2 dx \\ &= - \int_{\Omega} u \Delta u dx + \int_{\Omega} \tilde{u} \Delta \tilde{u} dx \\ &= \int_{\Omega} [\tilde{u}(\Delta \tilde{u} - \Delta u) + \Delta u(\tilde{u} - u)] dx, \end{aligned} \tag{2.15}$$

which contradicts with (2.14). The uniqueness of solutions for (2.5) is proved.

At the end, let  $T : U \rightarrow H \cap H^4(\Omega)$  be the operator defined by  $Tg = u$ , where  $u$  is the solution of (2.5). Then by (2.7), (2.8) and the maximum principle, we can easily get that  $T : U \rightarrow U$  is compact.  $\square$

### 3. Nonlinear eigenvalue problem

In this section, we study (1.1) with  $f(u) \equiv u$ , that is the nonlinear eigenvalue problem

$$\begin{cases} \Delta^2 u - (a + b \int_{\Omega} |\nabla u|^2 dx) \Delta u = \lambda u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

The solutions of (3.1) are closely related to the following linear eigenvalue problem:

$$\begin{cases} \Delta^2 u - A \Delta u = \Lambda u, & \text{in } \Omega, \\ u = \Delta u = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.2}$$

Given a positive constant  $A$ , denoting the eigenvalues of problem (3.2) by  $\Lambda_{k,A}$  ( $k = 1, 2, \dots$ ), then we have the following results:

**Lemma 3.1**

- (i) If  $A_1, A_2$  are positive constants such that  $A_1 < A_2$ , then  $\Lambda_{1,A_1} < \Lambda_{1,A_2}$ .
- (ii) Let  $B, C$  be two fixed positive constants. Consider the map

$$\Lambda_1(\tau) := \Lambda_{1,B+\tau C}, \quad \tau \geq 0,$$

then  $\Lambda_1(\cdot)$  is a continuous and strictly increasing function and

$$\Lambda_1(0) = \Lambda_{1,B}, \quad \lim_{\tau \rightarrow +\infty} \Lambda_1(\tau) = +\infty. \tag{3.3}$$

**Proof.** We can easily see that the eigenvalues of problem (3.2) are  $\Lambda_{k,A} = \mu_k(\mu_k + A)$  ( $k = 1, 2, \dots$ ), and the corresponding eigenfunctions are still  $\varphi_k$ . In particular, the principal eigenvalue of problem (3.2) is  $\Lambda_{1,A} = \mu_1(\mu_1 + A)$  and the  $\Lambda_{1,A}$ -eigenfunction  $\varphi_1 > 0$  in  $\Omega$ . Then (i) and (ii) are immediate consequences.  $\square$

By using Lemma 3.1, we prove the following results on the nonlinear eigenvalue problem (3.1):

**Theorem 3.1** Problem (3.1) has a positive solution  $u_\lambda$  if and only if  $\lambda \in (\Lambda_{1,a}, +\infty)$ , moreover, the solution  $u_\lambda$  is unique and satisfying

$$\lim_{\lambda \rightarrow \Lambda_{1,a}} \|u_\lambda\| = 0, \quad \lim_{\lambda \rightarrow +\infty} \|u_\lambda\| = +\infty. \tag{3.4}$$

**Proof.** Assume that  $u$  is a positive solution of (3.1), then  $\int_{\Omega} |\nabla u|^2 dx > 0$ , consequently by Lemma 3.1 (i) we have

$$\lambda = \Lambda_{1,a+b \int_{\Omega} |\nabla u|^2 dx} > \Lambda_{1,a}.$$

To any  $\lambda \in (\Lambda_{1,a}, +\infty)$ , by Lemma 3.1 (ii), there exists a unique  $t_0(\lambda) > 0$  such that

$$\Lambda_{1,a+bt_0} = \lambda,$$

moreover,

$$\lim_{\lambda \rightarrow \Lambda_{1,a}} t_0(\lambda) = 0, \quad \lim_{\lambda \rightarrow +\infty} t_0(\lambda) = +\infty. \tag{3.5}$$

For the fixed  $t_0$ , take appropriate principal eigenfunction  $\varphi_1(x)$  of (3.2) associated to  $\Lambda_{1,a+bt_0}$  such that

$$\int_{\Omega} |\nabla \varphi_1|^2 dx = t_0, \tag{3.6}$$

then it is easy to see that  $u_{\lambda} = \varphi_1 > 0$  is a positive solution of (3.1). And in fact,  $\varphi_1(x) = c_{\lambda} \varphi_1(x)$ , where  $c_{\lambda}$  is a positive constant depending on  $\lambda$ .

To prove the uniqueness, we assume that there exist two positive solutions  $u \neq v$ , since

$$\lambda = \Lambda_{1,a+b \int_{\Omega} |\nabla u|^2 dx} = \Lambda_{1,a+b \int_{\Omega} |\nabla v|^2 dx},$$

then Lemma 3.1 (ii) guarantees that  $\int_{\Omega} |\nabla u|^2 dx = \int_{\Omega} |\nabla v|^2 dx$  and  $u$  is proportional to  $v$ , which implies that  $u = v$ .

Finally, we prove (3.4). Since the unique positive solution of (3.1) is  $u_{\lambda} = \varphi_1(x) = c_{\lambda} \varphi_1(x)$ , then by (3.6) and (3.5), we have

$$\lim_{\lambda \rightarrow \Lambda_{1,a}} \int_{\Omega} |\nabla u_{\lambda}|^2 dx = \lim_{\lambda \rightarrow \Lambda_{1,a}} c_{\lambda} \int_{\Omega} |\nabla \varphi_1|^2 dx \rightarrow 0, \tag{3.7}$$

and similarly

$$\lim_{\lambda \rightarrow +\infty} \int_{\Omega} |\nabla u_{\lambda}|^2 dx = \lim_{\lambda \rightarrow +\infty} c_{\lambda} \int_{\Omega} |\nabla \varphi_1|^2 dx \rightarrow +\infty, \tag{3.8}$$

that is

$$\lim_{\lambda \rightarrow \Lambda_{1,a}} c_{\lambda} \rightarrow 0, \quad \lim_{\lambda \rightarrow +\infty} c_{\lambda} \rightarrow +\infty, \tag{3.9}$$

then (3.4) is an immediate consequence. □

#### 4. The sublinear case

In this section, we consider (1.1) when the nonlinear term  $f$  is sublinear which means that  $f$  satisfying:

(H1)  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous,  $f(s) > 0$  for all  $s > 0$ ,  $f(0) = 0$  and  $f_0 := \lim_{s \rightarrow 0^+} \frac{f(s)}{s} = +\infty$ ;

(H2)  $f_{\infty} := \lim_{s \rightarrow +\infty} \frac{f(s)}{s} = 0$ .

We will study global bifurcation phenomena of (1.1) and prove that there exists a positive solution for all  $\lambda > 0$  and no positive solution for  $\lambda < 0$ .

We first state some notations. Let  $B_\rho := \{u \in H : \|u\| < \rho\}$ . For any  $u \in H$ , denote  $u^+ = \max\{u, 0\}$ . Define the operator  $F : \mathbb{R} \times H \mapsto H$  by

$$F(\lambda, u)(x) := T(\lambda f(u^+(x))), \tag{4.1}$$

where  $T$  is the operator defined in Proposition 2.1. Then, it is easy to see that  $u \in H$  is a weak nonnegative solution of (1.1) if and only if

$$u = F(\lambda, u). \tag{4.2}$$

In order to prove the main result of this section, we need the following lemmas.

**Lemma 4.1** For any fixed  $\lambda < 0$ , there exists a number  $\rho > 0$  such that

$$\text{deg}(I - F(\lambda, \cdot), B_\rho(0), 0) = 1.$$

**Proof.** First, we claim that there exists  $\delta > 0$  such that

$$u \neq tF(\lambda, u) = tT(\lambda f(u^+)) \quad \text{for all } u \in \overline{B_\delta}, u \neq 0 \text{ and } t \in [0, 1].$$

Suppose on the contrary that there exist sequence  $\{u_n\}$  in  $H \setminus 0$  with  $\|u_n\| \rightarrow 0$  and  $\{t_n\}$  in  $[0, 1]$  such that

$$u_n = t_n F(\lambda, u_n) = t_n T(\lambda f(u_n^+)),$$

that is

$$\begin{cases} \Delta^2 u_n - (a + b \int_\Omega |\nabla u_n|^2 dx) \Delta u_n = t_n \lambda f(u_n^+(x)) \leq 0, & \text{in } \Omega, \\ u_n = \Delta u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.3}$$

By (4.3) and the maximum principle we can easily get that  $u_n(x) \leq 0$  for  $x \in \Omega$ , which implies  $f(u_n^+) = 0$  according to (H1). Then (4.3) has only a unique solution  $u_n = 0$ , a contradiction with  $u_n \in H \setminus 0$ .

Take  $\rho \in (0, \delta]$ , according to the homotopy invariance of topological degree, we have

$$\text{deg}(I - F(\lambda, \cdot), B_\rho(0), 0) = \text{deg}(I, B_\rho(0), 0) = 1.$$

□

**Lemma 4.2** For any fixed  $\lambda > 0$ , there exists a number  $\rho > 0$  such that

$$\text{deg}(I - F(\lambda, \cdot), B_\rho(0), 0) = 0.$$

**Proof.** First, take a  $\psi \in H, \psi > 0$  in  $\Omega$ , we claim that there exists  $\delta > 0$  such that

$$u \neq T(\lambda f(u^+) + t\psi) \quad \text{for all } u \in \overline{B_\delta}, u \neq 0 \text{ and } t \in [0, 1].$$

Suppose on the contrary that there exist sequence  $\{u_n\}$  in  $H \setminus 0$  with  $\|u_n\| \rightarrow 0$  and  $\{t_n\}$  in  $[0, 1]$  such that

$$u_n = T(\lambda f(u_n^+) + t_n \psi),$$

that is

$$\begin{cases} \Delta^2 u_n - (a + b \int_\Omega |\nabla u_n|^2 dx) \Delta u_n = \lambda f(u_n^+) + t_n \psi, & \text{in } \Omega, \\ u_n = \Delta u_n = 0, & \text{on } \partial\Omega. \end{cases} \tag{4.4}$$

Since  $t_n\psi > 0$ , by the maximum principle we have that  $u_n(x) > 0$  for a.e.  $x \in \Omega$ .

On the other hand,  $\|u_n\| \rightarrow 0$  implies that

$$\int_{\Omega} |\nabla u_n|^2 dx \leq C$$

for some positive constant  $C$ . Hence, according to Lemma 3.1 we have that

$$\Lambda_{1,a+b \int_{\Omega} |\nabla u_n|^2 dx} \leq \Lambda_{1,a+bC} := \Lambda.$$

Fix this value of  $\Lambda$ , since  $u_n(x) \rightarrow 0$  for a.e.  $x \in \Omega$ , then according to (H1), for  $n$  large we have that  $\lambda f(u_n^+) > \Lambda u_n$ . Combining this with  $\Delta u_n(x) \leq 0$  for a.e.  $x \in \Omega$  we can get

$$\Delta^2 u_n - (a + bC)\Delta u_n \geq \Delta^2 u_n - (a + b \int_{\Omega} |\nabla u_n|^2 dx)\Delta u_n = \lambda f(u_n^+) + t_n\psi > \Lambda u_n,$$

which implies that  $\Lambda_{1,a+bC} > \Lambda$ , a contradiction.

Take  $\rho \in (0, \delta]$ , according to the homotopy invariance of topological degree, we have

$$\text{deg}(I - F(\lambda, \cdot), B_{\rho}(0), 0) = \text{deg}(I - T(\lambda f(\cdot) + \psi), B_{\rho}(0), 0) = 0.$$

□

Now, we are ready to consider the bifurcation of positive solutions of (1.1) from the line of trivial solutions  $\{(\lambda, 0) \in \mathbb{R} \times H : \lambda \in \mathbb{R}\}$ .

**Theorem 4.1** Assume that (H1) and (H2) hold. Then from  $(0, 0)$  there emanate an unbounded continuum  $\mathcal{C}_0$  of positive solutions of (1.1) in  $\mathbb{R} \times H$ .

**Proof of Theorem 4.1.** By an argument similar to that of [2, Proposition 3.5], using Lemma 4.1 and 4.2, we can show that  $(0, 0)$  is a bifurcation point from the line of trivial solutions  $\{(\lambda, 0) \in \mathbb{R} \times H : \lambda \in \mathbb{R}\}$  for the equation (4.2), and there exists a connected component  $\mathcal{C}_0$  of positive solutions of (4.2) containing  $(0, 0)$ , either

- (i)  $\mathcal{C}_0$  is unbounded in  $\mathbb{R} \times H$ , or
- (ii)  $\mathcal{C}_0 \cap [\mathbb{R} \setminus 0 \times \{0\}] \neq \emptyset$ .

To prove the unboundedness of  $\mathcal{C}_0$ , we only need to show that the case (ii) cannot occur, that is:  $\mathcal{C}_0$  cannot meet  $(\lambda, 0)$  for any  $\lambda \neq 0$ . It is easy to see that for  $\lambda < 0$  problem (1.1) does not possess a positive solution. For the case  $\lambda > 0$ , we assume on the contrary that there exist some  $\lambda_0 > 0$  and a sequence of parameters  $\{\lambda_n\}$  and corresponding positive solutions  $\{u_n\}$  of (1.1) such that  $\lambda_n \rightarrow \lambda_0$  and  $\|u_n\| \rightarrow 0$ . Since  $u_n(x) \rightarrow 0$  for a.e.  $x \in \Omega$ , then by (H1), for fixed  $\varepsilon \in (0, \lambda_0)$  there exists  $n_0 \in \mathbb{N}$  such that when  $n > n_0$  we have

$$\Delta^2 u_n - (a + b \int_{\Omega} |\nabla u_n|^2 dx)\Delta u_n = \lambda_n f(u_n) \geq (\lambda_0 - \varepsilon)f(u_n) > \Lambda u_n, \tag{4.5}$$

where  $\Lambda$  is defined as in Lemma 4.2. Now, we can get a contradiction in a similar way that in the proof of Lemma 4.2. □

The main result of this section is following:

**Theorem 4.2** Assume that (H1) and (H2) hold, then (1.1) has a positive solution if and only if  $\lambda > 0$ .

**Proof.** By Theorem 4.1, there exists an unbounded continuum  $\mathcal{C}_0 \in \mathbb{R} \times H$  of positive solutions of (1.1). We will show that  $\|u\|$  is bounded for any fixed  $\lambda > 0$ , that is,  $\mathcal{C}_0$  cannot blow up at finite  $\lambda \in (0, +\infty)$ . Assume on



the contrary that there exist  $\lambda_0 > 0$  and a sequence of parameters  $\{\lambda_n\}$  and corresponding positive solutions  $\{u_n\}$  of (1.1) such that  $\lambda_n \rightarrow \lambda_0$ ,  $\|u_n\| \rightarrow \infty$ . Since

$$\Delta^2 u_n - (a + b \int_{\Omega} |\nabla u_n|^2 dx) \Delta u_n = \lambda_n f(u_n), \tag{4.6}$$

divide (4.6) by  $\|u_n\|$  and set  $v_n = \frac{u_n}{\|u_n\|}$ , then we get

$$\Delta^2 v_n - (a + b \int_{\Omega} |\nabla u_n|^2 dx) \Delta v_n = \lambda_n \frac{f(u_n)}{\|u_n\|} = \lambda_n \frac{f(u_n)}{u_n} v_n. \tag{4.7}$$

Multiplying (4.7) by  $v_n$  and integrating, we obtain

$$\int_{\Omega} |\nabla v_n(x)|^2 dx = \frac{\int_{\Omega} \lambda_n \frac{f(u_n(x))}{u_n} v_n^2(x) dx - \int_{\Omega} |\Delta v_n(x)|^2 dx}{a + b \int_{\Omega} |\nabla u_n(x)|^2 dx}. \tag{4.8}$$

Since  $\int_{\Omega} |\Delta v_n|^2 dx = \|v_n\|^2 \equiv 1$ , and by (2.2) and (H2) we have  $\int_{\Omega} \lambda_n \frac{f(u_n(x))}{u_n} v_n^2(x) dx \rightarrow 0$  as  $n \rightarrow \infty$ , this contradict with (4.8).

The above conclusion means that  $\|u\|$  is bounded for any fixed  $\lambda > 0$ . Combining this with the unboundedness of  $\mathcal{C}_0$ , we conclude that  $\sup\{\lambda \mid (\lambda, u) \in \mathcal{C}_0\} = \infty$ , then for any  $\lambda > 0$  there exists a positive solution for (1.1). □

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