Lattice ordered semigroups and Γ-hypersemigroups

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Abstract: As we have already seen in Turkish Journal of Mathematics (2019) 43: 2592–2601 many results on hypersemigroups do not need any proof as they can be obtained from lattice ordered semigroups. The present paper goes a step further, to show that many results on Γ-hypersemigroups as well can be obtained from lattice ordered semigroups. It can be instructive to prove them directly, but even in that case the proofs go along the lines of lattice ordered semigroups (or *poe*-semigroups). In the investigation, we faced the problem to correct the definition of Γ-hypersemigroups given in the existing bibliography.

Key words: Lattice ordered semigroup, Γ-hypersemigroup, regular, intra-regular, ideal element, ideal

1. Introduction

Kovács was the first who observed that the regular rings (introduced by J. von Neumann) can be characterized by the property $A \cap B = AB$ for every right ideal A and every left ideal B, where AB is the set of all finite sums of the form $\sum a_i b_i$; $a_i \in A$, $b_i \in B$ [14]. Iséki studied regularity for semigroups characterizing the von Neumann regular semigroups as semigroups satisfying the property $A \cap B = AB$ for every right ideal A and every left ideal B [5]. The author of the present paper introduced the concept of regularity in case of ordered semigroups as follows: An ordered semigroup is said to be regular if for every $a \in S$ there exists $x \in S$ such that $a \leq axa$. It is shown in [9] that an ordered semigroup S is regular if and only if for every right ideal A and every left ideal B of S we have $A \cap B = (AB]$, equivalently $A \cap B \subseteq (AB]$, where $(AB]$ is the subset of S defined by $(AB] = \{t \in S \mid t \leq ab \text{ for some } a \in A, b \in B\}$.

The concept of a Γ-semigroup has been introduced by M.K. Sen in the International Symposium New Delhi 1981, as an extension of the concept of a Γ-ring introduced by Nobusawa [15], as follows: Given two nonempty sets S and Γ, S is called a Γ-semigroup [17] if the following assertions are satisfied:

(1) $a\alpha b \in S$ and $\alpha a \beta \in \Gamma$ and

(2) $(a\alpha b)\beta c = a(\alpha b\beta)c = a\alpha(b\beta c)$

for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$.

In 1986 Sen and Saha gave a second definition of a Γ-semigroup as follows:

Definition 1.1 *Let* $S = \{a, b, c, ...\}$ *and* $\Gamma = \{\alpha, \beta, \gamma, ...\}$ *be two nonempty sets. Then* S *is called a* Γ*-semigroup [18] if*

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    (1) $a\alpha b \in S$ *and*

    (2) $(a\alpha b)\beta c = a\alpha(b\beta c)$

for all $a, b, c \in S$ and all $\alpha, \beta \in \Gamma$.

One can find this definition of Γ-semigroups in [21] where the notion of a radical in a Γ-semigroup and the notion of ΓS-act over a Γ-semigroup have been introduced, in [19] and [20] where the notions of regular and orthodox Γ-semigroups have been introduced and studied and in [16] where the maximum idempotent-separating congruence on an inverse Γ-semigroup has been studied. But still, we cannot say that Γ is a set of binary operations on M to work on it. Perhaps that was the reason that later some authors defined the Γ-semigroup as follows: If S is a semigroup and Γ is a nonempty set, then S is called a Γ-semigroup if there is a mapping

$$S \times \Gamma \times S \to S \mid (a, \gamma, b) \to a\gamma b$$

such that $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in S$ and all $\gamma, \mu \in \Gamma$. Which means that a semigroup S is a Γ-semigroup if there is a mapping of $S \times \Gamma \times S$ into S $(a, \gamma, b) \to a\gamma b$ such that

$$f\Big(f(a, \gamma, b), \mu, c\Big) = f\Big(a, \gamma, f(b, \mu, c)\Big)$$

for all $a, b, c \in M$ and all $\gamma, \mu \in \Gamma$. As we see, the definition given by Sen and Saha (Definition 1.1) is much more useful for the investigation if we add the missing third condition in it. Adding the uniqueness condition in Definition 1.1, in an expression of the form $A_1 \Gamma A_2 \Gamma A_3 \ldots \Gamma A_n$ (or $a_1 \gamma_1 a_2 \gamma_2 a_3 \ldots \gamma_n a_n$), we can put parentheses in any place beginning with some a_i and ending in some a_j. So the definition of a Γ-semigroup has been given in Scientiae Mathematicae Japonicae [10] as follows:

Definition 1.2 *[10] For two nonempty sets M and Γ, denote by $M\Gamma M$ the set defined by*

$$M\Gamma M := \{a\gamma b \mid a, b \in M, \gamma \in \Gamma\}.$$

Then M is called a Γ-semigroup if the following assertions are satisfied:

    (1) $M\Gamma M \subseteq M$

    (2) *if $a, b, c, d \in M$ and $\gamma, \mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$*

    (3) $(a\gamma b)\mu c = a\gamma(b\mu c)$ *for all $a, b, c \in M$ and all $\gamma, \mu \in \Gamma$.*

    We can omit the definition of the $M\Gamma M$ and write condition (1) as follows as well: For every $a, b \in M$ and every $\gamma \in \Gamma$, we have $a\gamma b \in M$.

    In other words, a Γ-semigroup is a nonempty set M with a set Γ of binary operations on M, satisfying the associativity condition $(a\gamma b)\mu c = a\gamma(b\mu c)$ for all $a, b, c \in M$ and all $\gamma, \mu \in \Gamma$.

    If we have only the conditions (1) and (2), then this is the definition of a Γ-groupoid.

    An hypergroupoid is a nonempty set S with an hyperoperation "$\circ$" on S (: to each $a, b \in S$ assigns a unique nonempty subset $a \circ b$ of S) and an operation "$*$" on $\mathcal{P}^*(S)$ such that $A * B = \bigcup\limits_{(a,b) \in A \times B} (a \circ b)$ for every $A, B \in \mathcal{P}^*(S)$; $\mathcal{P}^*(S)$ being the set of (all) nonempty subsets of S. An hypergroupoid S is called hypersemigroup if $\{x\} * (y \circ z) = (x \circ y) * \{z\}$ for every $x, y, z \in S$. For any $x, y \in S$, we have $\{x\} * \{y\} = x \circ y$ [11].

We have seen in [13] that many results on hypersemigroups do not need any proof since they are obtained from lattice ordered semigroups. We can prove them just to show how a direct proof (in the language of semigroups) works. In the present paper we go a step further to show that many results on Γ-hypersemigroups as well can be obtained from lattice ordered semigroups and do not need any proof unless we want to show how one can prove them independently.

After some preliminaries from lattice ordered semigroups, the definition of a Γ-hypersemigroup is given and the associative relation is shown in it (Proposition 3.17) to be able to put parentheses in expressions of the form $A_1\Gamma A_2\Gamma A_3.....\Gamma A_n$ (otherwise no investigation on the subject is possible).

The concept of regular Γ-hypersemigroup has been introduced and, extending the theorem by Iséki [5], it is shown that a Γ-hypersemigroup M is regular if and only if for every right ideal A and every left ideal B of M we have $A \cap B = A\Gamma B$, equivalently $A \cap B \subseteq A\Gamma B$. It is proved that the same result can be also obtained as corollary of the corresponding result on lattice ordered semigroup.

Under the same methodology, every result on lattice ordered semigroups in Turkish Journal of Mathematics [12, 13], every result in section 2 of the present paper, and many results on lattice ordered semigroups, $\vee e$ or *poe*-semigroups, hold automatically for Γ-hypersemigroups as well, do not need any proof -as they follow from more general statements about ordered semigroups; one can prove them just to show how a direct proof works, but even in that case this direct proof goes along the lines of le, $\vee e$ or *poe*-semigroups.

When is convenient and no confusion is possible, we identify the singleton $\{a\}$ by the element a and write, for example, $a\Gamma M\Gamma a$ instead of $\{a\}\Gamma M\Gamma\{a\}$, $a\Gamma M$ instead of $\{a\}\Gamma M$.

2. On lattice ordered semigroups

An ordered groupoid (shortly *po-groupoid*) is a groupoid S with an order "$\leq$" on it such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for every $c \in S$. If the multiplication on S is associative, then S is called an *ordered semigroup* (*po-semigroup*). A *lattice ordered semigroup* (shortly *l-semigroup*) is a semigroup S which is at the same time a lattice such that $a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc$ for all $a,b,c \in S$ [1–4]. An *le-semigroup* is an *l*-semigroup having a greatest element with respect to the order of S, usually denoted by e (: $e \geq a$ for all $a \in S$). The *poe*-groupoids and the *poe*-semigroups are also *po*-groupoids and *po*-semigroups respectively having a greatest element with respect to the order. An element a of a *poe*-groupoid S is called a *right ideal element* if $ae \leq a$; it is called a *left ideal element* if $ea \leq a$ [6, 7]. By an *ideal element* of S we mean an element that is both a right and a left ideal element of S. An element b of a *poe*-semigroup S is called a *bi-ideal element* of S [6, 8] if $beb \leq b$. For an element a of an *le*-semigroup S, denote by $r(a)$, $l(a)$ the right and the left ideal element of S, respectively, generated by a and we have $r(a) = a \vee ae$ and $l(a) = a \vee ea$ (see also [13]). Denote by $r(l(a))$ the ideal element of S generated by a; and we have $r(l(a)) = a \vee ea \vee ae \vee eae$. Indeed, the element $a \vee ea \vee ae \vee eae$ is an ideal element of S containing a; and if t is an ideal element of S such that $t \geq a$, then $a \vee ea \vee ae \vee eae \leq t \vee et \vee te \vee ete = t$. It might be noted here that, for any $a \in S$, $r(l(a)) = l(r(a))$. For further information see also [7, 13].

Definition 2.1 *[6, 7] A poe-semigroup S is said to be regular if, for every $a \in S$, we have $a \leq aea$.*

The following theorem holds:

Theorem 2.2 *(see also [6, 8]) Let S be an le-semigroup. The following are equivalent:*

(1) S is regular.

(2) $a \wedge b = ab$ for every right ideal element a and every left ideal element b of S.

(3) $a \wedge b \leq ab$ for every right ideal element a and every left ideal element b of S.

Proof $(1) \implies (2)$. Since S is regular, we have

$$a \wedge b \leq (a \wedge b)e(a \wedge b) \leq (ae)b \leq ab \leq ae \wedge eb \leq a \wedge b,$$

thus we get $a \wedge b = ab$.

The implication $(2) \Rightarrow (3)$ is obvious.

$(3) \implies (1)$. Let $a \in S$. By hypothesis, we have

$$
\begin{aligned}
a \leq r(a) \wedge l(a) & \leq r(a)l(a) = (a \vee ae)(a \vee ea) = a^2 \vee aea \vee ae^2a \\
& = a^2 \vee aea,
\end{aligned}
$$

then $a^2 \leq a^3 \vee aea^2 \leq aea$ from which $a \leq aea$ and so S is regular. $\square$

Definition 2.3 *[7] A poe-semigroup S is said to be intra-regular if, for every $a \in S$, we have $a \leq ea^2e$.*

Theorem 2.4 *(see also [7]) An le-semigroup S is intra-regular if and only if*

for every right ideal element a and every left ideal element b of S, we have $a \wedge b \leq ba$.

Proof $\implies$. Let a be a right ideal element and b a left ideal element of S. Since S is intra-regular, we have

$$b \wedge a \leq e(b \wedge a)^2 e = e(b \wedge a)(b \wedge a)e.$$

Since $e(b \wedge a) \leq eb$ and $(b \wedge a)e \leq ae$, we have $e(b \wedge a)(b \wedge a)e \leq (eb)(ae) \leq ba$. Thus we have $b \wedge a \leq ba$.

$\impliedby$. Let $a \in S$. By hypothesis, we have

$$
\begin{aligned}
a & \leq r(a) \wedge l(a) = l(a)r(a) = (a \vee ea)(a \vee ae) \\
& = a^2 \vee ea^2 \vee a^2e \vee ea^2e.
\end{aligned}
$$

Then we have

$$a^2 \leq (a^2 \vee ea^2 \vee a^2e \vee ea^2e)(a^2 \vee ea^2 \vee a^2e \vee ea^2e) \leq ea^2e \vee ea^2,$$

$a^2e \leq ea^2e^2 \vee ea^2e = ea^2e$, then $a \leq ea^2 \vee ea^2e$, and then $ea^2 \leq e(ea^2 \vee ea^2e)a = e^2a^3 \vee e^2a^2ea \leq ea^2e$. Thus we get $a \leq ea^2e$ and so S is intra-regular. $\square$

Definition 2.5 *[6, 7] An le-semigroup S is called right (resp. left) regular if, for every $a \in S$, we have*

$$a \leq a^2e \ \ (resp. \ a \leq ea^2).$$

Definition 2.6 *[6] A poe-groupoid S is called right (resp. left) duo if the right (resp. left) ideal elements of S are left (resp. right) ideal elements of S as well (that is, ideal elements of S).*

Theorem 2.7 *An le-semigroup S is right regular and right duo if and only if for any right ideal elements a, b of S, we have*

$$a \wedge b = ab.$$

Proof $\implies$. Let a, b be right ideal elements of S. Then $ab \le ae \le a$; since S is right duo, b is a left ideal element of S as well, so $ab \le eb \le b$. Thus we have $ab \le a \wedge b$. Since S is right regular, we have $a \wedge b \le (a \wedge b)(a \wedge b)e$. Since $a \wedge b \le a, b$, we have $(a \wedge b)e \le ae \wedge be \le a \wedge b$. Then $a \wedge b \le (a \wedge b)(a \wedge b) \le ab$. Thus we have $a \wedge b = ab$.

$\impliedby$. Let a be a right ideal element of S. Since e is a right ideal element of S, by hypothesis, we have $a = e \wedge a = ea$, so a is a left ideal element of S and S is right duo. Let now $a \in S$. By hypothesis, we have

$$
\begin{aligned}
a \le r(a) \wedge r(a) &= r(a)r(a) = (a \vee ae)(a \vee ae) \\
&= a^2 \vee aea \vee a^2 e \vee aeae
\end{aligned}
$$

Since ae is a right ideal element of S and S is right duo, ae is a left ideal element of S as well; that is $e(ae) \le ae$ and so $a(eae) \le a^2 e$. Hence we obtain $a \le a^2 \vee aea \vee a^2 e$, then $a^2 \le a^3 \vee a^2 ea \vee a^3 e \le aea \vee a^2 e$. Thus we get $a \le aea \vee a^2 e$, and then $a(ea) \le aeaea \vee a^2 e^2 a \le aeaea \vee a^2 e$. Since $a(eae) \le a^2 e$, we have $aeaea \le a^2 ea \le a^2 e$, and thus $aea \le a^2 e$. Since $a \le aea \vee a^2 e$ and $aea \le a^2 e$, we have $a \le a^2 e$ and S is right regular. $\square$

In a similar way we prove the following theorem.

Theorem 2.8 *An le-semigroup S is left regular and left duo if and only if for any left ideal elements a, b of S, we have*

$$a \wedge b = ba.$$

Definition 2.9 *[7] Let S be a po-groupoid. An element t of S is called semiprime if, for any $a \in S$ such that $a^2 \le t$, we have $a \le t$.*

Theorem 2.10 *(see also [7]) An le-semigroup S is intra-regular if and only if the ideal elements of S are semiprime.*

Proof $\implies$. Let t be an ideal element of S and $a \in S$ such that $a^2 \le t$. Since S is intra-regular, we have $a \le ea^2 e \le ete \le t$, so $a \le t$ and t is semiprime.

$\impliedby$. Let $a \in S$. The element $r(l(a^2)$ is an ideal element of S such that $a^2 \le r(l(a^2))$. Since $r(l(a^2))$ is semiprime, we have $a \le r(l(a^2)) = a^2 \vee ea^2 \vee a^2 e \vee ea^2 e$. Then

$$a^2 \le (a^2 \vee ea^2 \vee a^2 e \vee ea^2 e)(a^2 \vee ea^2 \vee a^2 e \vee ea^2 e) \le ea^2 \vee ea^2 e.$$

Then $a^2 e \le ea^2 e \vee ea^2 e^2 = ea^2 e$. Hence we obtain $a \le ea^2 \vee ea^2 e$ and then $a^2 \le ea^2 a \vee ea^2 ea \le ea^2 e$, from which $ea^2 \le e^2 a^2 e \le ea^2 e$. We have $a \le ea^2 \vee ea^2 e$ and $ea^2 \le ea^2 e$, thus we have $a \le ea^2 e$ and S is intra-regular. $\square$

Proposition 2.11 *If a poe-semigroup S is right (or left) regular, then it is intra-regular.*

Proof Let S be right regular and $a \in S$. Then $a \le a^2 e \le a(a^2 e)e \le ea^2 e$, then $a \le ea^2 e$ and S is intra-regular. If S is left regular, then $a \le ea^2 \le e(ea^2)a \le ea^2 e$ and again S is intra-regular. $\square$

Proposition 2.12 *An le-semigroup S is right regular if and only if, for any $a \in S$, we have*

$$r(a) = r(a^2), \text{ equivalently } r(a) \leq r(a^2).$$

Proof $\implies$. Let $a \in S$. Since S is right regular, we have

$$r(a) = a \vee ae \leq a^2 e \vee a^2 e^2 = a^2 e \leq a^2 \vee a^2 e = r(a^2).$$

On the other hand, $r(a^2) = a^2 \vee a^2 e \leq ae \leq a \vee ae = r(a)$. Thus we have $r(a) = r(a^2)$.
$\impliedby$. Let $a \in S$. By hypothesis, we have

$$a \leq r(a) \leq r(a^2) = a^2 \vee a^2 e \leq ae,$$

then $a^2 \leq a^2 e$. Since $a \leq a^2 \vee a^2 e$ and $a^2 \leq a^2 e$, we have $a \leq a^2 e$ and S is right regular. $\qquad\square$

In a similar way we prove the following proposition.

Proposition 2.13 *An le-semigroup S is left regular if and only if, for any $a \in S$, we have*

$$l(a) = l(a^2), \text{ equivalently } l(a) \leq l(a^2).$$

Theorem 2.14 *An le-semigroup S is right regular if and only if the right ideal elements of S are semiprime.*

Proof $\implies$. Let t be a right ideal element of S and $a \in S$ such that $a^2 \leq t$. Since S is right regular, we have $a \leq a^2 e \leq te \leq t$ and so $a \leq t$.
$\impliedby$. Let $a \in S$. Since $r(a^2)$ is a right ideal element of S, by hypothesis, it is semiprime. Since $a^2 \leq r(a^2)$, we have $a \leq r(a^2) = a^2 \vee a^2 e$. Then $a^2 \leq a^3 \vee a^3 e \leq a^2 e$. Since $a \leq a^2 \vee a^2 e$ and $a^2 \leq a^2 e$, we have $a \leq a^2 e$ and so S is right regular. $\qquad\square$

In a similar way the following theorem holds.

Theorem 2.15 *An le-semigroup S is left regular if and only if the left ideal elements of S are semiprime.*

Definition 2.16 *A poe-groupoid S is called right (resp. left) simple if the element "e" is the only right (resp. left) ideal element of S. That is, if a is a right (resp. left) ideal element of S, then $a = e$.*

Proposition 2.17 *A poe-semigroup S is right (resp. left) simple if and only if, for every $a \in S$, we have $ae = e$ (resp. $ea = e$).*

Proof Let S be right simple and $a \in S$. The element ae is a right ideal element of S (as $(ae)e = ae^2 \leq ae$). Since S is right simple, we have $ae = e$.
Assuming $ae = e$ for every $a \in S$, let t be a right ideal element of S. Then we have $e = te \leq t$ and then $t = e$ and so S is right simple. $\qquad\square$

A right (left) ideal element or bi-ideal element a of S is called *proper* if $a \neq e$.

Theorem 2.18 *A poe-semigroup S is right (resp. left) simple if and only if does not contain proper right (resp. left) ideal elements.*

Proof $\Longrightarrow$. Let S be a right simple and a be a right ideal element of S. Since S is right simple, by Proposition 2.17, we have $ae = e$. Since a is a right ideal element of S, we have $ae \leq a$. Thus we have $e \leq a$ and so $a = e$.
$\Longleftarrow$. Assuming S does not contain proper right ideal elements, let $a \in S$. Since ae is a right ideal element of S, by hypothesis, we have $ae = e$. By Proposition 2.17, S is right simple. $\square$

Proposition 2.19 *If a poe-semigroup S is both right and left simple, then it is regular.*

Proof Let $a \in S$. Since S is right and left simple, by Proposition 2.17, we have $ae = ea = e$. Then we have $a \leq e = ae = a(ea)$, thus $a \leq aea$, and S is regular. $\square$

Theorem 2.20 *A poe-semigroup S is both right and left simple if and only if does not contain proper bi-ideal elements.*

Proof $\Longrightarrow$. Let b be a bi-ideal element of S. Then $beb \leq b$. Since S is right and left simple, by Proposition 2.17, we have $be = eb = e$. Thus we have $b \geq (be)b = eb = e$ and so $b = e$.
$\Longleftarrow$. Let a be a right ideal element of S. Then a is a bi-ideal element of S; indeed $(ae)a \leq a^2 \leq ae \leq a$. Since S does not contain proper bi-ideal elements, we have $a = e$ and so S is right simple. Let now a be a left ideal element of S. Since a is a bi-ideal element of S, by hypothesis, we have $a = e$ and so S is left simple. $\square$

3. On Γ-hypersemigroups

Taking into account the definition of a Γ-groupoid given in [10], a Γ-hypergroupoid can be defined as follows:

Definition 3.1 *Let M and Γ be two nonempty sets. The set M is called a Γ-hypergroupoid if the following assertions are satisfied:*

    (i) if $a, b \in M$ and $\gamma \in \Gamma$, then $\emptyset \neq a\gamma b \subseteq M$ and

    (ii) if $a, b, c, d \in M$ and $\gamma, \mu \in \Gamma$ such that $a = c$, $\gamma = \mu$ and $b = d$, then $a\gamma b = c\mu d$.

    In other words, a Γ-hypergroupoid is a nonempty set M with a set Γ of binary hyperoperations on M.

Definition 3.2 *If M is a Γ-hypergroupoid, then for every $\gamma \in \Gamma$ we denote by $\overline{\gamma}$ the operation on $\mathcal{P}^*(M)$ (induced by the hyperoperation γ) defined by*

$$\overline{\gamma} : \mathcal{P}^*(M) \times \mathcal{P}^*(M) \to \mathcal{P}^*(M) \mid (A, B) \to A\overline{\gamma}B,$$

where

$$A\overline{\gamma}B := \bigcup_{a \in A, b \in B} a\gamma b.$$

It is easy to see that this operation is well defined.

Definition 3.3 *If M is a Γ-hypergroupoid, we denote by Γ the operation on $\mathcal{P}^*(M)$ defined by*

$$\Gamma : \mathcal{P}^*(M) \times \mathcal{P}^*(M) \to \mathcal{P}^*(M) \mid (A, B) \to A\Gamma B,$$

where

$$A\Gamma B := \bigcup_{\gamma \in \Gamma} A\overline{\gamma}B.$$

This is also well defined.

Remark 3.4 *For any nonempty subsets A, B of a Γ-hypergroupoid, we have*

$$A\Gamma B = \bigcup_{\gamma \in \Gamma} A\overline{\gamma}B = \bigcup_{\gamma \in \Gamma} \Big(\bigcup_{a \in A, b \in B} a\gamma b \Big) = \bigcup_{a \in A, b \in B, \gamma \in \Gamma} a\gamma b.$$

Lemma 3.5 *If M is an hypergroupoid then, for any $x, y \in M$ and any $\gamma \in \Gamma$, we have*

$$\{x\}\overline{\gamma}\{y\} = x\gamma y.$$

Proof Let $x, y \in M$ and $\gamma \in \Gamma$. By Definition 3.2, we have

$$\{x\}\overline{\gamma}\{y\} = \bigcup_{u \in \{x\}, v \in \{y\}} u\gamma v = x\gamma y.$$

□

Lemma 3.6 *Let M be a Γ-hypergroupoid, A,B,C,D nonempty subsets of M and $\gamma \in \Gamma$. If $A \subseteq B$ and $C \subseteq D$, then $A\overline{\gamma}C \subseteq B\overline{\gamma}D$.*

Proof Let $x \in A\overline{\gamma}C$. By Definition 3.2, $x \in a\gamma c$ for some $a \in A$, $c \in C$. Since $a \in B$ and $c \in D$, again by Definition 3.2, we get $a\gamma c \subseteq B\overline{\gamma}D$, so $x \in B\overline{\gamma}D$. Thus we have $A\overline{\gamma}C \subseteq B\overline{\gamma}D$. □

Lemma 3.7 *Let M be a Γ-hypergroupoid, $x \in M$ and A, B nonempty subsets of M. Then we have the following:*

    (1) $x \in A\Gamma B$ *if and only if* $x \in a\gamma b$ *for some* $a \in A, b \in B, \gamma \in \Gamma$

    (2) *if* $a \in A$, $b \in B$ *and* $\gamma \in \Gamma$, *then* $a\gamma b \subseteq A\Gamma B$.

Proof (1). It follows immediately from Remark 3.4. The property (2) is an immediate consequence of (1). □

Lemma 3.8 *If M is a Γ-hypergroupoid, A, B, C, D nonempty subsets of M, $A \subseteq B$ and $C \subseteq D$, then*

$$A\Gamma C \subseteq B\Gamma D.$$

Proof Let $x \in A\Gamma C$. By Lemma 3.7(1), we have $x \in a\gamma c$ for some $a \in A$, $\gamma \in \Gamma$, $c \in C$. Since $a \in B$, $\gamma \in \Gamma$ and $c \in D$, by Lemma 3.7(2), we have $a\gamma c \subseteq B\Gamma D$, so $x \in B\Gamma D$. Thus we have $A\Gamma C \subseteq B\Gamma D$. □

Definition 3.9 *Let M be a Γ-hypergroupoid. A nonempty subset A of M is called a right (resp. left) ideal of M if $A\Gamma M \subseteq A$ (resp. $M\Gamma A \subseteq A$). If a subset of M is both a right and a left ideal of M, then it is called an ideal of M.*

Proposition 3.10 *Let M be a Γ-hypergroupoid. The following are equivalent:*

    (1) *A is a right ideal of M.*

    (2) *If $a \in A$, $\gamma \in \Gamma$ and $m \in M$, then $a\gamma m \subseteq A$.*

Proof $(1) \Longrightarrow (2)$. Let $a \in A$, $\gamma \in \Gamma$ and $m \in M$. By Lemma 3.7(2), we have $a\gamma m \subseteq A\Gamma M$. By hypothesis, $A\Gamma M \subseteq A$. Thus we have $a\gamma m \subseteq A$.

$(2) \Longrightarrow (1)$. Let $x \in A\Gamma M$. According to Lemma 3.7(1), we have $x \in a\gamma m$ for some $a \in A$, $\gamma \in \Gamma$, $m \in M$. By hypothesis, we have $a\gamma m \subseteq A$. Thus we have $x \in A$, so $A\Gamma M \subseteq A$ and A is a right ideal of M. $\qquad \square$

In a similar way we prove the following proposition.

Proposition 3.11 *If M is a Γ-hypergroupoid, then the following are equivalent:*

    (1) *B is a left ideal of M.*

    (2) *If $a \in B$, $\gamma \in \Gamma$ and $m \in M$, then $m\gamma a \subseteq B$.*

Proposition 3.12 *Let M be a Γ-hypergroupoid. If A is a right ideal of M and B is a left ideal of M, then*

$$A \cap B \neq \emptyset.$$

Proof Let A be a right ideal and B a left ideal of M. Take an element $a \in A$, an element $b \in B$ and an element $\gamma \in \Gamma$ $(A, B, \Gamma \neq \emptyset)$. Since $a \in A$, $b \in M$ and $\gamma \in \Gamma$, by Lemma 3.7(2), we have $a\gamma b \subseteq A\Gamma M$. Since A is a right ideal of M, we have $A\Gamma M \subseteq A$ and so $a\gamma b \subseteq A$. Since $a \in M$, $b \in B$ and $\gamma \in \Gamma$, again by Lemma 3.7(2), we have $a\gamma b \subseteq M\Gamma B$. Since B is a left ideal of M, we have $M\Gamma B \subseteq B$, thus we have $a\gamma b \subseteq B$. Since $a\gamma b \subseteq A \cap B$ and $a\gamma b \neq \emptyset$ (by Def. 3.1), we have $A \cap B \neq \emptyset$. $\qquad \square$

Proposition 3.13 *Let M be a Γ-hypergroupoid and $A_i, B \in \mathcal{P}^*(M)$, $i \in I$. Then we have the following:*

    (1) $(\bigcup\limits_{i \in I} A_i)\Gamma B = \bigcup\limits_{i \in I} (A_i \Gamma B)$

    (2) $B\Gamma(\bigcup\limits_{i \in I} A_i) = \bigcup\limits_{i \in I} (B\Gamma A_i)$.

Proof (1) Let $x \in (\bigcup\limits_{i \in I} A_i)\Gamma B$. By Lemma 3.7(1), $x \in a\gamma b$ for some $a \in \bigcup\limits_{i \in I} A_i$, $\gamma \in \Gamma$ $b \in B$. Since $a \in A_j$ for some $j \in I$, $\gamma \in \Gamma$ and $b \in B$, by Lemma 3.7(2), we have $a\gamma b \subseteq A_j\Gamma B \subseteq \bigcup\limits_{i \in I} (A_i\Gamma B)$. Then $x \in \bigcup\limits_{i \in I} (A_i\Gamma B)$.

Let now $x \in \bigcup\limits_{i \in I} (A_i\Gamma B)$. Then $x \in A_j\Gamma B$ for some $j \in I$. By Lemma 3.7(1), $x \in a\gamma b$ for some $a \in A_j$, $\gamma \in \Gamma$, $b \in B$. Since $a \in \bigcup\limits_{i \in I} A_i$, $\gamma \in \Gamma$ and $b \in B$, by Lemma 3.7(2), we have $a\gamma b \subseteq (\bigcup\limits_{i \in I} A_i)\Gamma B$. Then we have $x \in (\bigcup\limits_{i \in I} A_i)\Gamma B$.

The proof of property (2) is similar. $\qquad \square$

Definition 3.14 *A Γ-hypergroupoid M is called Γ-hypersemigroup if, for any $a, b, c \in M$ and any $\gamma, \mu \in \Gamma$, we have*

$$\{a\}\overline{\gamma}(b\mu c) = (a\gamma b)\overline{\mu}\{c\}.$$

Proposition 3.15 *Every semigroup $(S, \cdot)$ is a Γ-hypersemigroup.*

Proof Take an element γ not contained in S, consider $\Gamma = \{\gamma\}$, and the hyperoperation γ on S defined by

$$\gamma : S \times S \to \mathcal{P}^*(S) \mid (a, b) \to a\gamma b := \{ab\}.$$

Then (S, Γ) is a Γ-hypersemigroup. Indeed, if $a, b, c \in M$ then, by Definition 3.2, we have

$$\{a\}\overline{\gamma}(b\gamma c) = \bigcup_{u \in b\gamma c} a\gamma u = \bigcup_{u=bc} a\gamma u = a\gamma(bc) = \{a(bc)\},$$

$$(a\gamma b)\overline{\gamma}\{c\} = \bigcup_{u \in a\gamma b} u\gamma c = \bigcup_{u=ab} u\gamma c = (ab)\gamma c = \{(ab)c\} = \{a(bc)\},$$

and so $\{a\}\overline{\gamma}(b\gamma c) = (a\gamma b)\overline{\gamma}\{c\}$.

Second Proof Take an element $\gamma \notin S$, consider $\Gamma = \{\gamma\}$, and the hyperoperation γ on S defined by

$$\gamma : S \times S \to \mathcal{P}^*(S) \mid (a, b) \to a\gamma b := \{a, b, ab\}.$$

The operation γ is well defined and we have $\{a\}\overline{\gamma}(b\gamma c) = (a\gamma b)\overline{\gamma}\{c\}$. Indeed, by Definition 3.2,

$$
\begin{aligned}
\{a\}\overline{\gamma}(b\gamma c) &= \bigcup_{u \in b\gamma c} a\gamma u = \bigcup_{u \in \{b,c,bc\}} a\gamma u = a\gamma b \cup a\gamma c \cup a\gamma(bc) \\
&= \{a, b, c, ab, ac, bc, a(bc)\},
\end{aligned}
$$

$$
\begin{aligned}
(a\gamma b)\overline{\gamma}\{c\} &= \bigcup_{u \in a\gamma b} u\gamma c = \bigcup_{u \in \{a,b,ab\}} u\gamma c = a\gamma c \cup b\gamma c \cup (ab)\gamma c \\
&= \{a, b, c, ab, ac, bc, (ab)c\}.
\end{aligned}
$$

Since $a(bc) = (ab)c$, we have $\{a\}\overline{\gamma}(b\gamma c) = (a\gamma b)\overline{\gamma}\{c\}$. $\qquad\qquad\square$

Example 3.16 *We consider the semigroup defined by Table 1.*

Table 1. Multiplication table of the semigroup of the Example 3.16.

·	a	b	c
a	a	a	a
b	a	a	a
c	a	b	c

According to the first proof of Proposition 3.15, Table 2 defines a Γ-hypersemigroup.

Table 2. The hypersemigroup of the Example 3.16 that corresponds to the first proof of Proposition 3.15.

γ	a	b	c
a	{a}	{a}	{a}
b	{a}	{a}	{a}
c	{a}	{b}	{c}

According to the second proof of Proposition 3.15, Table 3 defines a Γ-hypersemigroup.

Table 3. The hypersemigroup of the Example 3.16 that corresponds to the second proof of Proposition 3.15.

γ	a	b	c
a	{a}	{a,b}	{a,c}
b	{a,b}	{a,b}	{a,b,c}
c	{a,c}	{b,c}	{c}

Proposition 3.17 *If M is a Γ-hypersemigroup then, for any nonempty subsets A, B, C of M we have*

$$(A\Gamma B)\Gamma C = A\Gamma(B\Gamma C).$$

Proof Let $x \in (A\Gamma B)\Gamma C$. By Lemma 3.7(1), we have $x \in u\mu c$ for some $u \in A\Gamma B$, $\mu \in \Gamma$, $c \in C$. Since $u \in A\Gamma B$, again by Lemma 3.7(1), we have $u \in a\gamma b$ for some $a \in A$, $\gamma \in \Gamma$, $b \in B$. Then we have

$$
\begin{aligned}
x \in u\mu c &= \{u\}\overline{\mu}\{c\} \text{ (by Lemma 3.5)} \\
&\subseteq (a\gamma b)\overline{\mu}\{c\} \text{ (by Lemma 3.6)} \\
&= \{a\}\overline{\gamma}(b\mu c) \text{ (by Definition 3.14)} \\
&\subseteq \{a\}\Gamma(b\mu c) \text{ (by Definition 3.3).}
\end{aligned}
$$

Since $b \in B$, $\mu \in \Gamma$ and $c \in C$, by Lemma 3.7(2), we have $b\mu c \subseteq B\Gamma C$. Then, by Lemma 3.8, $\{a\}\Gamma(b\mu c) \subseteq A\Gamma(B\Gamma C)$, thus we get $x \in A\Gamma(B\Gamma C)$ and so $(A\Gamma B)\Gamma C \subseteq A\Gamma(B\Gamma C)$.

Let now $x \in A\Gamma(B\Gamma C)$. By Lemma 3.7(1), we have $x \in a\gamma u$ for some $a \in A$, $\gamma \in \Gamma$, $u \in B\Gamma C$ and $u \in b\mu c$ for some $b \in B$, $\mu \in \Gamma$, $c \in C$. As in the previous case, we have

$$x \in a\gamma u = \{a\}\overline{\gamma}\{u\} \subseteq \{a\}\overline{\gamma}(b\mu c) = (a\gamma b)\overline{\mu}\{c\} \subseteq (a\gamma b)\Gamma\{c\}.$$

Since $a \in A$, $\gamma \in \Gamma$, $b \in B$, by Lemma 3.7(2), we have $a\gamma b \subseteq A\Gamma B$. Then, by Lemma 3.8, $(a\gamma b)\Gamma\{c\} \subseteq (A\Gamma B)\Gamma C$, then $x \in (A\Gamma B)\Gamma C$ and so $A\Gamma(B\Gamma C) \subseteq (A\Gamma B)\Gamma C$. □

From Proposition 3.17 we have the following corollary.

Corollary 3.18 *If M is a Γ-hypersemigroup, then the set $(\mathcal{P}^*(M), \Gamma)$ is a semigroup.*

Definition 3.19 *A Γ-hypersemigroup M is called regular if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that*

$$a \in (a\gamma x)\overline{\mu}\{a\}.$$

Proposition 3.20 *A Γ-hypersemigroup M is regular if and only if for any nonempty subset A of M, we have*

$$A \subseteq A\Gamma M\Gamma A.$$

Proof $\implies$. Let A be a nonempty set and $a \in A$. Since M is regular, there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that $a \in (a\gamma x)\overline{\mu}\{a\}$. Since $a \in A$, $\gamma \in \Gamma$, $x \in M$, by Lemma 3.7(2), we have $a\gamma x \subseteq A\Gamma M$. Then, by Lemma 3.6, $(a\gamma x)\overline{\mu}\{a\} \subseteq (A\Gamma M)\overline{\mu}\{a\}$. By Definition 3.3, $(A\Gamma M)\overline{\mu}\{a\} \subseteq (A\Gamma M)\Gamma\{a\}$. By Lemma 3.8 and Proposition 3.17, we have $(A\Gamma M)\Gamma\{a\} \subseteq (A\Gamma M)\Gamma A = A\Gamma M\Gamma A$. Thus we get $a \in A\Gamma M\Gamma A$ and so $A \subseteq A\Gamma M\Gamma A$.

$\Longleftarrow$. Let $a \in M$. By hypothesis, we have $\{a\} \subseteq \Big(\{a\}\Gamma M\Big)\Gamma\{a\}$. By Lemma 3.7(1), $a \in u\mu a$ for some $u \in \{a\}\Gamma M$, $\mu \in \Gamma$. Since $u \in \{a\}\Gamma M$, again by Lemma 3.7(1), $u \in a\gamma x$ for some $\gamma \in \Gamma$, $x \in M$. By Lemma 3.5, $u\mu a = \{u\}\overline{\mu}\{a\}$ and, by Lemma 3.6, $\{u\}\overline{\mu}\{a\} \subseteq (a\gamma x)\overline{\mu}\{a\}$. Thus we get $a \in (a\gamma x)\overline{\mu}\{a\}$, and M is regular. $\qquad\square$

Proposition 3.20 gives us a second, equivalent concept of regularity in Γ-hypersemigroups.

Denote by $R(A)$ (resp. $L(A)$) the right (resp. left) ideal of M generated by A, and we have the following proposition.

Proposition 3.21 *If M is a Γ-hypersemigroup then, for any nonempty subset A of M, we have*

(1) $R(A) = A \cup A\Gamma M$ *and*

(2) $L(A) = A \cup M\Gamma A$.

Proof (1) The set $A \cup A\Gamma M$ is a right ideal of M containing A. In fact, it is a nonempty subset of S and we have

$$
\begin{aligned}
(A \cup A\Gamma M)\Gamma M &= A\Gamma M \cup (A\Gamma M)\Gamma M \text{ (by Prop. 3.13(1))} \\
&= A\Gamma M \cup A\Gamma(M\Gamma M) \text{ (by Prop. 3.17)} \\
&= A\Gamma M \text{ (since } M\Gamma M \subseteq M) \\
&\subseteq A \cup A\Gamma M.
\end{aligned}
$$

If now T is a right ideal of M containing A, then we have $A \cup A\Gamma M \subseteq T \cup T\Gamma M = T$ and property (1) holds. The proof of property (2) is similar. $\qquad\square$

Theorem 3.22 *Let M be a Γ-hypersemigroup. The following are equivalent:*

(1) *M is regular.*

(2) *$A \cap B = A\Gamma B$ for any right ideal A and any left ideal B of M.*

(3) *$A \cap B \subseteq A\Gamma B$ for any right ideal A and any left ideal B of M.*

Proof (1) $\Longrightarrow$ (2). Let M be regular, A be a right ideal and B a left ideal of M. By Proposition 3.12, $A \cap B \neq \emptyset$. Since M is regular, by Proposition 3.20, Lemma 3.8 and Proposition 3.17, we have

$$
\begin{aligned}
A \cap B &\subseteq (A \cap B)\Gamma M\Gamma(A \cap B) \subseteq A\Gamma M\Gamma B = A\Gamma(M\Gamma B) \subseteq A\Gamma B \\
&\subseteq A\Gamma M \cap M\Gamma B \subseteq A \cap B.
\end{aligned}
$$

Thus we have $A \cap B = A\Gamma B$.

The implication (2) $\Longrightarrow$ (3) is obvious.

(3) $\Longrightarrow$ (1). Let A be a nonempty subset of M. By hypothesis, we have

$$
\begin{aligned}
A &\subseteq R(A) \cap L(A) = R(A)\Gamma L(A) = (A \cup A\Gamma M)\Gamma(A \cup M\Gamma A) \text{ (by Prop. 3.21)} \\
&= A\Gamma A \cup A\Gamma M\Gamma A \cup A\Gamma(M\Gamma M)\Gamma A \text{ (by Prop. 3.13 and Prop. 3.17).}
\end{aligned}
$$

Since $M\Gamma M \subseteq M$, by Lemma 3.8, we have $A\Gamma(M\Gamma M)\Gamma A \subseteq A\Gamma M\Gamma A$. Then we have $A \subseteq A\Gamma A \cup A\Gamma M\Gamma A$, and then

$$
\begin{aligned}
A\Gamma A \;\; &\subseteq \;\; (A\Gamma A \cup A\Gamma M\Gamma A)\Gamma A \text{ (by Lemma 3.8)}\\
&= \;\; A\Gamma A\Gamma A \cup A\Gamma (M\Gamma A)\Gamma A \text{ (by Prop. 3.13(1) and Prop. 3.17)}\\
&\subseteq \;\; A\Gamma M\Gamma A \text{ (since } A \subseteq M \text{ and } M\Gamma A \subseteq M).
\end{aligned}
$$

Then we have $A \subseteq A\Gamma M\Gamma A$ and, by Proposition 3.20, M is regular. $\qquad\qquad\qquad\qquad\square$

Remark 3.23 *We have $A \subseteq A\Gamma M\Gamma A$ for every $A \in \mathcal{P}^*(M)$ if and only if $a \in \{a\}\Gamma M\Gamma\{a\}$ for every $a \in M$. Indeed, if $a \in a\Gamma M\Gamma a$ for every $a \in M$ and $b \in A$, then $b \in (b\Gamma M)\Gamma b$, then $b \in x\gamma b$ for some $x \in b\Gamma M$, $\gamma \in \Gamma$ and $x \in b\mu y$ for some $\mu \in \Gamma$, $y \in M$, and so*

$$
b \in x\gamma b = \{x\}\overline{\gamma}\{b\} \subseteq (b\mu y)\overline{\gamma}\{b\} \subseteq (b\mu y)\Gamma\{b\} \subseteq (A\Gamma M)\Gamma A = A\Gamma M\Gamma A,
$$

thus we have $A \subseteq A\Gamma M\Gamma A$. The rest is obvious.

Example 3.24 *We consider the set $M = \{a, b, c\}$ and let $\Gamma = \{\gamma, \mu\}$ the hyperoperations on M defined by Tables 4 and 5.*

Table 4. The γ-hyperoperation of the Example 3.24.

γ	a	b	c
a	{a,b}	{a,b}	{c}
b	{a,b}	{a,b}	{c}
c	{c}	{c}	{c}

Table 5. The μ-hyperoperation of the Example 3.24.

μ	a	b	c
a	{a,b}	{a,b}	{c}
b	{a,b}	{b}	{c}
c	{c}	{c}	{c}

This is a Γ-hypersemigroup as $\{x\}\overline{\gamma}(y\mu z) = (x\gamma y)\overline{\mu}\{z\}$ for all $x, y, z \in M$ and all $\gamma, \mu \in \Gamma$. We had to check 108 cases to verify it by hand (computer programs are necessary). It is an example of a regular Γ-hypersemigroup as well. Indeed, for example,

$$
a \in (a\gamma a)\overline{\mu}\{a\} = \bigcup_{u \in a\gamma a} u\mu a = \bigcup_{u \in \{a,b\}} u\mu a = a\mu a \cup b\mu a = \{a, b\} \text{ (by Def. 3.2)},
$$

$b \in (b\gamma a)\overline{\mu}\{b\}$ as $(b\gamma a)\overline{\mu}\{b\} = \{a,b\}\overline{\mu}\{b\} = a\mu b \cup b\mu b = \{a, b\}$ (by Def. 3.2),

$c \in (c\gamma c)\overline{\mu}\{c\}$ as $(c\gamma c)\overline{\mu}\{c\} = \{c\}\overline{\mu}\{c\} = c\mu c = \{c\}$ (by Lemma 3.5).

We remark that the sets $\{c\}$ and M are the only right (resp. left) ideals of S; this being so, they are the only ideals of S. Theorem 3.22 can be applied. Indeed, $\{c\}\cap\{c\} = \{c\}\Gamma\{c\}(= \{c\})$ and $\{c\}\cap M = \{c\}\Gamma M(= \{c\})$ (the hyperoperations γ and μ are commutative).

Theorem 3.22 as Corollary to Theorem 2.2

Let (M, Γ) be a Γ-hypersemigroup. By Corollary 3.18, $(\mathcal{P}^*(M), \Gamma)$ is a semigroup. The inclusion relation "$\subseteq$" is clearly an order on $\mathcal{P}^*(M)$, M being the greatest element of $\mathcal{P}^*(M)$ and, for any $A, B \in \mathcal{P}^*(M)$, $A \cup B$ and $A \cap B$ being the supremum and the infimum of A and B, respectively (with respect to this order). Moreover, by Proposition 3.13, for any nonempty subsets A, B, C of M, we have $(A \cup B)\Gamma C = A\Gamma C \cup B\Gamma C$ and $A\Gamma(B \cup C) = A\Gamma B \cup A\Gamma C$. Thus $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is an le-semigroup.

Let now M be a regular Γ-hypersemigroup, A a right ideal and B a left ideal in M. Then $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is a regular le-semigroup, A is a right ideal element and B is a left ideal element of $(\mathcal{P}^*(M), \Gamma, \subseteq)$. By Theorem 2.2 $(1) \Rightarrow (2)$, we have $\inf\{A, B\} = A\Gamma B$, that is $A \cap B = A\Gamma B$.

Suppose M is a Γ-hypersemigroup such that $A \cap B \subseteq A\Gamma B$ for every right ideal A and every left ideal B of M. Then $A \cap B \subseteq A\Gamma B$ for every right ideal element A and every left ideal element B of $(\mathcal{P}^*(M), \Gamma, \subseteq)$. By Theorem 2.2 $(3) \Rightarrow (1)$, $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is a regular le-semigroup, that is $A \subseteq A\Gamma M\Gamma A$ for any $A \in \mathcal{P}^*(M)$ and so, by Proposition 3.20, M is regular.

Note From every result of section 2, a corresponding result on a Γ-hypersemigroup can be obtained. Thus we have the following results and the necessary definitions.

Definition 3.25 *A Γ-hypersemigroup M is called intra-regular if for every $a \in M$ there exist $x, y \in M$ and $\gamma, \mu, \rho \in \Gamma$ such that*

$$a \in (x\gamma a)\overline{\mu}(a\rho y).$$

The Example 3.24 is an example of an intra-regular Γ-hypersemigroup as, for example,

$$a \in (a\gamma a)\overline{\mu}(a\gamma a) = \{a, b\}, \ b \in (a\gamma b)\overline{\mu}(b\gamma b) = \{a, b\}, \ c \in (a\gamma c)\overline{\mu}(c\gamma a) = \{c\}.$$

Theorem 3.26 *(see Theorem 2.4) A Γ-hypersemigroup M is intra-regular if and only if, for every right ideal A and every left ideal B of M, we have $A \cap B \subseteq B\Gamma A$.*

Definition 3.27 *A Γ-hypersemigroup M is called right (resp. left) regular if for every $a \in M$ there exist $x \in M$ and $\gamma, \mu \in \Gamma$ such that*

$$a \in (a\gamma a)\overline{\mu}\{x\} \ (resp. \ a \in \{x\}\overline{\gamma}(a\mu a)).$$

The Example 3.24 is an example of right regular Γ-hypersemigroups as, for example,

$$a \in (a\gamma a)\overline{\mu}\{a\}, \ b \in (b\gamma b)\overline{\mu}\{b\}, \ c \in (c\gamma c)\overline{\mu}\{c\}.$$

The Example 3.24 is at the same time an example of left regular Γ-hypersemigroups as, for example,

$$a \in \{a\}\overline{\gamma}(a\gamma a), \ b \in \{a\}\overline{\mu}(b\gamma b), \ c \in \{c\}\overline{\gamma}(c\gamma c).$$

A Γ-hypersemigroup M is called *right* (resp. *left*) *duo* if every right (resp. left) ideal of M is at the same time a left (resp. right) ideal (that is, an ideal) of M.

Theorem 3.28 *(see Theorem 2.7 and Theorem 2.8) A Γ-hypersemigroup M is right (resp. left) regular and right (resp. left) duo if and only if for every right (resp. left) ideals A, B of S we have $A \cap B = A\Gamma B$ (resp. $A \cap B = B\Gamma A$).*

A nonempty subset T of a Γ-hypergroupoid M is called *semiprime* if for any nonempty subset A of M such that $A\Gamma A \subseteq T$ we have $A \subseteq T$.

Theorem 3.29 *(see Theorem 2.10) A Γ-hypersemigroup M is intra-regular if and only if the ideals of M are semiprime.*

Proposition 3.30 *(see Proposition 2.11) If a Γ-hypersemigroup is right (or left regular), then it is intra-regular.*

We have already seen that the Example 3.24 is an example of a right (resp. left) regular Γ-hypersemigroup. This being so, by Proposition 3.30, it is an example of intra-regular Γ-hypersemigroup as well (its independent proof has been given above).

Proposition 3.31 *(see Proposition 2.12 and Proposition 2.13) A Γ-hypersemigroup M is right regular if and only if, for every nonempty subset A of M, we have $R(A) = R(A\Gamma A)$, equivalently $R(A) \subseteq R(A\Gamma A)$.*

It is left regular if and only if, for every nonempty subset A of M, we have $L(A) = L(A\Gamma A)$, equivalently $L(A) \subseteq L(A\Gamma A)$.

Theorem 3.32 *(see Theorem 2.14 and Theorem 2.15) A Γ-hypersemigroup M is right (resp. left) regular if and only if the right (resp. left) ideals of M are semiprime.*

A Γ-hypergroupoid M is called *right* (resp. *left*) *simple* if M is the only right (resp. left) ideal of M.

Proposition 3.33 *(see Proposition 2.17) A Γ-hypersemigroup M is right (resp. left simple) if and only if, for every nonempty subset A of S, we have $A\Gamma M = M$ (resp. $M\Gamma A = M$).*

Theorem 3.34 *(see Theorem 2.18) A Γ-hypersemigroup M is right (resp. left) simple if and only if does not contain proper right (resp. left) ideals.*

Proposition 3.35 *(see Proposition 2.19) If a Γ-hypersemigroup is both right and left simple, then it is regular.*

A nonempty subset B of a Γ-hypersemigroup M is called a *bi-ideal* of M if $B\Gamma B\Gamma B \subseteq B$.

Theorem 3.36 *(see Theorem 2.20) A Γ-hypersemigroup is both right and left simple if and only if does not contain proper bi-ideals.*

We do not need to prove the results given above. According to section 2, we know that they hold. They are obtained as corollaries of the corresponding results of section 2 in the way indicated to get Theorem 3.22 as corollary to Theorem 2.2.

Let us only give the proof of Theorem 3.36 to compare its proof with that one on *poe*-semigroups given in Theorem 2.20 and see how similar is.

Proof of Theorem 3.36 $\Longrightarrow$. Let B be a bi-ideal of M. Then $B\Gamma M\Gamma B \subseteq B$. By Proposition 3.33, we have $B\Gamma M = M\Gamma B = M$. Thus we have $B \supseteq (B\Gamma M)\Gamma B = M\Gamma B = M$ and so $B = M$.
$\Longleftarrow$. Let A be a right (or left) ideal of M. Then A is a bi-ideal of M and so $A = M$.

Theorem 3.36 as Corollary to Theorem 2.20
$\Longrightarrow$. Let M be a Γ-hypersemigroup both right and left simple. Then $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is a *poe*-semigroup both

right and left simple. By Theorem 2.20, $\mathcal{P}^*(M)$ does not contain proper bi-ideal elements. So M does not contain proper bi-ideals.

$\Longleftarrow$. Let M be a Γ-hypersemigroup not containing proper bi-ideals. Then $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is a poe-semigroup not containing proper bi-ideal elements. By Theorem 2.20, $\mathcal{P}^*(M)$ is right and left simple. So M is right and left simple as well. $\qquad \square$

4. Conclusion

From every result of an le-semigroup based on right (left) ideal elements, ideal elements, bi-ideal elements or quasi-ideal elements [13], a corresponding result of a Γ- hypersemigroup based on right (left) ideals, ideals, bi-ideals or quasi-ideals can be obtained. From many results on regular or intra-regular le-semigroups (and not only), analogous results for Γ-hypersemigroups can be obtained. This is because, if M is a Γ-hypersemigroup, then the set $\mathcal{P}^*(M)$ of nonempty subsets of M endowed with the operation "Γ" and the inclusion relation "$\subseteq$" is an le-semigroup. Moreover

(1) A is a right (resp. left) ideal of M if and only if it is a right (resp. left) ideal element of $(\mathcal{P}^*(M), \Gamma, \subseteq)$.

(2) A is a bi-ideal of M if and only if it is a bi-ideal element of $(\mathcal{P}^*(M), \Gamma, \subseteq)$.

(3) The Γ-hypersemigroup M is regular if and only if the le-semigroup $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is regular.

(4) The Γ-hypersemigroup M is intra-regular if and only if the le-semigroup $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is intra-regular.

(5) The Γ-hypersemigroup M is right (resp. left) regular if and only if the le-semigroup $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is right (resp. left) regular.

(6) The Γ-hypersemigroup M is right (resp. left) simple if and only if the poe-semigroup $(\mathcal{P}^*(M), \Gamma, \subseteq)$ is right (resp. left) simple.

We can say the same if we replace the word le-semigroup with $\vee e$-semigroup or poe-semigroup.

So, many results on Γ-hypersemigroups are direct consequences of more general theorems about lattice ordered semigroups. It can be instructive to prove them directly, just to show how an independent proof works, but this independent proof goes along the lines of the le, $\vee e$ or poe-semigroups.

This is not exactly the case for ordered Γ-hypersemigroups, but even in that case, the main idea also comes from the le, $\vee e$ or poe-semigroups and the proofs go along the lines of the le, $\vee e$ or poe-semigroups. We will see it in a forthcoming paper in which we will examine the results given (without proof), in the last part of the paper, for ordered Γ-hypersemigroups.

The results on ordered semigroups that hold for Γ-hypersemigroups (most of them hold), by easy modification for ordered Γ-hypersemigroups also hold. $\qquad \square$

As there are people interested in examples of Γ-hypersemigroups, we might say the following:

Creation of examples for Γ-hypersemigroups is not an easy task. For a given example a computer program to generate it and another one to check its validation is needed.

Even for an ordered semigroup of order 4 or 5, for example, it is impossible to write an example given by a table of multiplication and an order by hand. For somebody who is no expert on the subject, it is difficult

even to check the examples by hand if he does not know the Light's associativity test and its extended form for ordered semigroups.

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