

Application of spectral mapping method to Dirac operator

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Abstract: In the present study, theorems related to the uniqueness of the solution of inverse problems for Dirac equations system are proved by applying spectral mapping method. With the help of this method, the inverse problem is reduced to the so-called main equation, which corresponds to the problem of existence and uniqueness of the solution of the system of linear equations in the Banach space.

Key words: Dirac operator, Weyl function, inverse problem

1. Introduction

Let E be the 2×2 identity matrix and let

$$\sigma_1 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

are the well-known Pauli matrices, which have the following properties: $\sigma_k^2 = E$, $\sigma_k^* = \sigma_k$ (self-adjointness), $\sigma_k \sigma_j = -\sigma_j \sigma_k$, $k \neq j$ (anticommutativity), $j, k = 1, 2, 3$. Let $p, q, r \in L[0, \pi]$, i.e. $p(x), q(x), r(x)$ is a real valued, summable on $[0, \pi]$ functions. We consider the canonical Dirac system

$$iY := \left\{ B \frac{d}{dx} + \Omega(x) \right\} Y = \lambda Y, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \lambda \in \mathbb{C}$$

where $B = -i\sigma_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and the matrix is

$$\Omega(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix}$$

usually called a potential. By $L(p, q, \alpha, \beta) = L(\Omega, \alpha, \beta)$ we will denote the boundary value problems

$$iY = \lambda Y, \quad x \in (0, \pi), \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad \lambda \in \mathbb{C} \tag{1.1}$$

$$U(y) := y_1(0) \sin \alpha + y_2(0) \cos \alpha = 0 \tag{1.2}$$

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$$V(y) := y_1(\pi) \sin \beta + y_2(\pi) \cos \beta = 0 \quad (1.3)$$

By the same $L(\Omega, \alpha, \beta)$ we denote the self-adjoint operator, generated by differential expression l in Hilbert space of two component vector-functions $L^2([0, \pi], \mathbb{C}^2)$ with the domain

$$D_L = \left\{ Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} : y_1, y_2 \in AC[0, \pi], lY \in L^2[0, \pi], \right. \\ \left. y_1(0) \sin \alpha + y_2(0) \cos \alpha = 0, y_1(\pi) \sin \beta + y_2(\pi) \cos \beta = 0 \right\}$$

Under $AC[0, \pi]$ we understand the set of absolutely continuous function, defined on $[0, \pi]$.

In this paper we study the inverse problem for the Dirac operator with continuous potential. The Dirac equation, which is the relativistic generalization of the Schrödinger equation, describes the relativistic motion of elementary spin- $\frac{1}{2}$ particles, such as electrons and quarks, in quantum mechanics. A good way to understand the quantum system is to study the scattering and boundary value problems theory since the system cannot be seen directly. In the scattering process, particles, which are originally far away from the range of the potential, are sent in a direction towards the scattering centre and scattered by the effect of the potential.

The inverse scattering problem can also be approached by the inverse boundary value problem. As is well known (see [6, 11]), for the Dirac equation with compact supported coefficients, the inverse scattering problem at fixed energy is equivalent to the inverse boundary value problem on a region containing the supports of the coefficients. In the [7] study, uniqueness theorems were obtained for the solution of the scattering problem in the semi-axis for the system of Dirac equations of $2n$ order.

The uniqueness of inverse problems for the Dirac operator is studied by several authors [4, 10, 15, 16] for particular potentials, such as the electric potential and the electro-magnetic potential. In particular, in the paper [13] the uniqueness of both the inverse boundary value problem and inverse scattering problem for Dirac equation with a magnetic potential and an electrical potential are proved. Also, a relation between the Dirichlet to Dirichlet map for the inverse boundary value problem and the scattering amplitude for the inverse scattering problem is given.

In recent years of the problems for the Dirac operators of attention because of the applications of these problems in physics, mechanics and engineering. Hereby, this type of problems were widely investigated by many authors, also the investigations were continued and developed in many directions. For example, some aspects of the direct spectral problems were studied in [6, 8, 11, 12, 19] and the references therein, moreover, the inverse spectral problems according to spectral data are examined in the works [6, 8, 10–12, 14, 19] where further references can be found. In the paper [3] real numbers A_1, A_2 are obtained which are such that for $\lambda > A_1$ or $\lambda < A_2$ there are no nontrivial solutions of the Dirac equation which are in $L^2(a, \infty)$ for some $a > 0$. Precise values for A_1 and A_2 are obtained in certain cases. In the [5, 18] studies for the Sturm-Liouville problem, the contour integral method was applied in the solution of inverse problems and some uniqueness theorems were proved with the help of this method. In study [1] regularized trace, which has an important place in the solution of the inverse problem according to two spectrums in the finite interval for Dirac systems, was calculated. Inverse problems for the singular Dirac systems in the finite interval are examined in [2, 4, 17] studies and various uniqueness theorems are obtained for their solution.

In the present paper, as different from other studies solve the inverse problem of recovering Dirac operator from the given spectral data by using ideas of the contour integral method.

2. Preliminaries

Denote by $\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}$, $\psi(x, \lambda) = \begin{pmatrix} \psi_1(x, \lambda) \\ \psi_2(x, \lambda) \end{pmatrix}$ the solutions of the system (1.1), satisfying the initial conditions

$$\varphi_1(0, \lambda) = \cos \alpha, \quad \varphi_2(0, \lambda) = -\sin \alpha$$

$$\psi_1(\pi, \lambda) = \cos \beta, \quad \psi_2(\pi, \lambda) = -\sin \beta$$

Definition 2.1 *If the boundary value problem L has a nontrivial solution $Y(x, \lambda) \neq 0$ for certain λ_1 , then λ_1 is called an eigenvalue and $Y(x, \lambda_1)$ is called an eigenfunction of L .*

Definition 2.2 *Let the λ_n , $n \in \mathbb{N}$ numbers be the eigenvalues of the problem L . In this case, the normalizing numbers of the problem L are defined as follows:*

$$\alpha_n := \int_0^\pi \{ \varphi_1^2(x, \lambda_n) + \varphi_2^2(x, \lambda_n) \} dx.$$

Definition 2.3 *The characteristic function of the problem (1.1)-(1.3) is defined as follows:*

$$\begin{aligned} \Delta(\lambda) &= \begin{vmatrix} \varphi_1(x, \lambda) & \varphi_2(x, \lambda) \\ \psi_1(x, \lambda) & \psi_2(x, \lambda) \end{vmatrix} \\ &= \varphi_1(x, \lambda)\psi_2(x, \lambda) - \varphi_2(x, \lambda)\psi_1(x, \lambda). \end{aligned}$$

The function $\Delta(\lambda)$ does not depend on x . Therefore the following equations can be written for the $\Delta(\lambda)$ function:

$$\begin{aligned} \Delta(\lambda) &= \varphi_1(0, \lambda)\psi_2(0, \lambda) - \varphi_2(0, \lambda)\psi_1(0, \lambda) \\ &= \cos \alpha \psi_2(0, \lambda) + \sin \alpha \psi_1(0, \lambda) \\ &= \varphi_1(\pi, \lambda)\psi_2(\pi, \lambda) - \varphi_2(\pi, \lambda)\psi_1(\pi, \lambda) \\ &= -\sin \beta \varphi_1(\pi, \lambda) - \cos \beta \varphi_2(\pi, \lambda). \end{aligned}$$

In addition, the zeros of the $\Delta(\lambda)$ characteristic function coincide with the eigenvalues of the problem L .

Lemma 2.4 *As $|\lambda| \rightarrow \infty$ uniformly in x (for $0 \leq x \leq \pi$) the estimates*

$$\begin{cases} \varphi_1(x, \lambda) = \cos \{ \xi(x, \lambda) - \alpha \} + O(\lambda^{-1}) \\ \varphi_2(x, \lambda) = \sin \{ \xi(x, \lambda) - \alpha \} + O(\lambda^{-1}) \end{cases} \quad (2.1)$$

are valid.

Proof It was demonstrated in study [10] that the following equations are valid for functions $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$

$$\varphi_1(x, \lambda) = \cos \{ \xi(x, \lambda) - \alpha \} + \int_0^x \{ K_{11}(x, s) \cos(\lambda s - \alpha) + K_{12}(x, s) \sin(\lambda s - \alpha) \} ds \quad (2.2)$$

$$\varphi_2(x, \lambda) = \sin \{ \xi(x, \lambda) - \alpha \} + \int_0^x \{ K_{21}(x, s) \cos (\lambda s - \alpha) - K_{22}(x, s) \sin (\lambda s - \alpha) \} ds \tag{2.3}$$

where $\xi(x, \lambda) = \lambda x - \frac{1}{2} \int_0^x [p(\tau) - r(\tau)] d\tau$. It is clear that the equations (2.1) will be obtained, if partial integration is applied to the integrals once in the expressions of functions $\varphi_1(x, \lambda)$ and $\varphi_2(x, \lambda)$. □

Lemma 2.5 *The eigenvalues of the boundary-value problem L are simple.*

Proof Since $\varphi(x, \lambda)$ satisfies the boundary condition (1.2), to determine the eigenvalues of the problem under consideration, the functions $\varphi_1(x, \lambda)$, $\varphi_2(x, \lambda)$ should be substituted in the boundary condition (1.3), and its roots should be found. Put

$$D(\lambda) = \varphi_1(\pi, \lambda) \sin \beta + \varphi_2(\pi, \lambda) \cos \beta.$$

Then

$$\frac{dD(\lambda)}{d\lambda} = \frac{\partial \varphi_1}{\partial \lambda} \sin \beta + \frac{\partial \varphi_2}{\partial \lambda} \cos \beta.$$

Let λ_0 be a double eigenvalue and $\varphi^0(x, \lambda_0)$ one of the corresponding vector-valued eigenfunctions. Then the conditions $D(\lambda_0) = 0$, $\frac{dD(\lambda_0)}{d\lambda} = 0$ should be fulfilled simultaneously, i.e.

$$\varphi_1^0(\pi, \lambda_0) \sin \beta + \varphi_2^0(\pi, \lambda_0) \cos \beta = 0$$

$$\frac{\partial}{\partial \lambda} \varphi_1^0(\pi, \lambda_0) \sin \beta + \frac{\partial}{\partial \lambda} \varphi_2^0(\pi, \lambda_0) \cos \beta = 0.$$

Since $\sin \beta$ and $\cos \beta$ cannot vanish simultaneously, it follows from the last two equalities that

$$\varphi_2^0(\pi, \lambda) \frac{\partial \varphi_1^0(\pi, \lambda_0)}{\partial \lambda} - \varphi_1^0(\pi, \lambda) \frac{\partial \varphi_2^0(\pi, \lambda_0)}{\partial \lambda} = 0. \tag{2.4}$$

Now, differentiating the system (1.1) with respect to λ

$$\begin{aligned} \left(\frac{\partial y_2}{\partial \lambda} \right)'_{x,} + \{ p(x) - \lambda \} \frac{\partial y_1}{\partial \lambda} + q(x) \frac{\partial y_2}{\partial \lambda} &= y_1 \\ - \left(\frac{\partial y_1}{\partial \lambda} \right)'_{x,} + \{ r(x) - \lambda \} \frac{\partial y_2}{\partial \lambda} + q(x) \frac{\partial y_1}{\partial \lambda} &= y_2 \end{aligned}$$

Multiplying the equations (1.1) and recent equations obtained above by $-\frac{\partial y_1}{\partial \lambda}$, $-\frac{\partial y_2}{\partial \lambda}$, y_1 and y_2 , respectively, adding them together and integrating with respect to x from 0 to π , we obtain

$$\left\{ y_1(x, \lambda) \frac{\partial y_2}{\partial \lambda} - y_2(x, \lambda) \frac{\partial y_1}{\partial \lambda} \right\} \Big|_0^\pi = \int_0^\pi \{ y_1^2(x, \lambda) + y_2^2(x, \lambda) \} dx.$$

Putting $\lambda = \lambda_0$, taking into account that

$$\left. \frac{\partial \varphi_1^0(x, \lambda_0)}{\partial \lambda} \right|_{x=0} = \left. \frac{\partial \varphi_2^0(x, \lambda_0)}{\partial \lambda} \right|_{x=0} = 0$$

by (2.2), (2.3) and using the equality (2.4), we obtain the relation

$$\begin{aligned} & \int_0^\pi \left[\{\varphi_1^0(x, \lambda_0)\}^2 + \{\varphi_2^0(x, \lambda_0)\}^2 \right] dx \\ &= \varphi_1^0(\pi, \lambda) \frac{\partial \varphi_2^0(\pi, \lambda_0)}{\partial \lambda} - \varphi_2^0(\pi, \lambda) \frac{\partial \varphi_1^0(\pi, \lambda_0)}{\partial \lambda} = 0. \end{aligned}$$

Hence, $\varphi_1^0(x, \lambda_0) = \varphi_2^0(x, \lambda_0) \equiv 0$ or $\varphi^0(x, \lambda_0) \equiv 0$, which is impossible. The lemma is proved. □

Lemma 2.6 *The eigenvalues and normalizing constants of the problem L satisfy the asymptotic equalities:*

$$\lambda_n = n - \frac{\beta - \alpha}{\pi} + d_n, \quad d_n \in l_2, \quad 0 \leq \alpha < \beta \leq \pi \tag{2.5}$$

$$\alpha_n = \pi + \delta_n, \quad \delta_n = \frac{k_n}{n} + \frac{\gamma_n}{n^2} \in l_2 \tag{2.6}$$

Proof If the characteristic function of the L_0 problem is shown with $\Delta_0(\lambda)$ when $\Omega(x) = 0$, then it is clear that $\Delta_0(\lambda)$ provides the following equation:

$$\begin{aligned} \Delta_0(\lambda) &= -\sin \beta \cos(\xi(\pi, \lambda) - \alpha) - \cos \beta \sin(\xi(\pi, \lambda) - \alpha) \\ &= -\sin \beta \cos(\lambda\pi - \alpha) - \cos \beta \sin(\lambda\pi - \alpha) \\ &= -\sin(\lambda\pi + \beta - \alpha) \end{aligned}$$

On the other hand, if the equations (2.2) and (2.3) are used in the equation $\Delta(\lambda) = -\sin \beta \varphi_1(\pi, \lambda) - \cos \beta \varphi_2(\pi, \lambda)$

$$\begin{aligned} \Delta(\lambda) &= -\sin \beta \cos(\xi(\pi, \lambda) - \alpha) - \cos \beta \sin(\xi(\pi, \lambda) - \alpha) \\ &\quad - \sin \beta \int_0^\pi \{K_{11}(\pi, s) \cos(\lambda s - \alpha) + K_{12}(\pi, s) \sin(\lambda s - \alpha)\} ds \\ &\quad - \cos \beta \int_0^\pi \{K_{21}(\pi, s) \cos(\lambda s - \alpha) - K_{22}(\pi, s) \sin(\lambda s - \alpha)\} ds \\ &= \Delta_0(\lambda) - \sin \beta \int_0^\pi \{K_{11}(\pi, s) \cos(\lambda s - \alpha) + K_{12}(\pi, s) \sin(\lambda s - \alpha)\} ds \\ &\quad - \cos \beta \int_0^\pi \{K_{21}(\pi, s) \cos(\lambda s - \alpha) - K_{22}(\pi, s) \sin(\lambda s - \alpha)\} ds \end{aligned}$$

is obtained. Define $G_\delta = \{\lambda : |\lambda - \lambda_n^0| \geq \delta, n = 0, 1, 2, \dots\}$, $\delta > 0$ and $\Gamma_n = \{\lambda : |\lambda| = |\lambda_n^0| + \delta, \delta > 0, n = 0, 1, 2, \dots\}$. In this case, there is $C_\delta > 0$, which provides the inequality of $|\Delta_0(\lambda)| \geq C_\delta \exp |\operatorname{Im} \lambda| \pi$. Also in the sufficiently large values of λ , $|\Delta(\lambda) - \Delta_0(\lambda)| < \frac{C_\delta}{2} \exp |\operatorname{Im} \lambda| \pi$ and $|\Delta_0(\lambda)| \geq C_\delta \exp |\operatorname{Im} \lambda| \pi > \frac{C_\delta}{2} \exp |\operatorname{Im} \lambda| \pi > |\Delta(\lambda) - \Delta_0(\lambda)|$ inequalitys are provided. For sufficiently large n in Γ_n , the zeros of the functions $\Delta_0(\lambda)$ and $[\Delta(\lambda) - \Delta_0(\lambda)] + \Delta_0(\lambda) = \Delta(\lambda)$ are the same number. Therefore, for the sufficiently large values of n according to the Rouché theorem, the $\Delta(\lambda)$ function has only one zero in each of the circles $|\lambda - \lambda_n^0| < \delta$. Since δ is a sufficiently small positive number, $\lambda_n = \lambda_n^0 + d_n$ is obtained for d_n that satisfies $\lim_{n \rightarrow \infty} d_n = 0$. Since λ_n numbers are roots of the characteristic function $\Delta(\lambda)$

$$\begin{aligned} \Delta(\lambda_n) &= \Delta(\lambda_n^0 + d_n) \\ &= \Delta_0(\lambda_n^0 + d_n) - \sin \beta \int_0^\pi \{K_{11}(\pi, s) \cos((\lambda_n^0 + d_n)s - \alpha) + K_{12}(\pi, s) \sin((\lambda_n^0 + d_n)s - \alpha)\} ds \\ &\quad - \cos \beta \int_0^\pi \{K_{21}(\pi, s) \cos((\lambda_n^0 + d_n)s - \alpha) - K_{22}(\pi, s) \sin((\lambda_n^0 + d_n)s - \alpha)\} ds = 0 \end{aligned}$$

is obtained. On the other hand, since $\Delta_0(\lambda_n^0) = 0$, $\lambda_n^0 = n - \frac{\beta - \alpha}{\pi}$ and

$$\begin{aligned} \Delta_0(\lambda_n) &= \Delta_0(\lambda_n^0 + d_n) = \Delta_0(\lambda_n^0) + \dot{\Delta}_0(\lambda_n^0) d_n + \ddot{\Delta}_0(\lambda_n^0) \frac{d_n^2}{2} + \dots \\ &= \dot{\Delta}_0(\lambda_n^0) d_n + o(d_n) = (\dot{\Delta}_0(\lambda_n^0) + o(1)) d_n \end{aligned}$$

is written. In addition, since $\Delta_0(\lambda)$ is a "sine" type function [9], there is a number $\epsilon_\delta > 0$ for each $n \in \mathbb{N}$ such that $|\dot{\Delta}_0(\lambda_n^0)| \geq \epsilon_\delta$ is valid. The expression $\Delta_0(\lambda_n^0 + d_n)$ is replaced by $\Delta(\lambda_n)$ and if necessary arrangements are made

$$d_n = \frac{[K_{11}(\pi, \pi) + K_{21}(\pi, \pi)] \sin(\lambda_n^0 \pi - \alpha) - [K_{12}(\pi, \pi) - K_{22}(\pi, \pi)] \cos(\lambda_n^0 \pi - \alpha)}{\lambda_n^0 \left\{ \dot{\Delta}_0(\lambda_n^0) - \int_0^\pi s (K_{11}(\pi, s) - K_{21}(\pi, s)) \sin(\lambda_n^0 s - \alpha) ds + \int_0^\pi s (K_{12}(\pi, s) - K_{22}(\pi, s)) \cos(\lambda_n^0 s - \alpha) ds \right\}} + \frac{\epsilon_n}{\lambda_n^0} \in l_2$$

$$\epsilon_n = - \int_0^\pi \{ (K_{11s}(\pi, s) + K_{21s}(\pi, s)) \sin(\lambda_n^0 s - \alpha) - (K_{12s}(\pi, s) - K_{22s}(\pi, s)) \cos(\lambda_n^0 s - \alpha) \} ds \in l_2.$$

obtained

Now, in the expression $\alpha_n = \int_0^\pi \{ \varphi_1^2(x, \lambda_n) + \varphi_2^2(x, \lambda_n) \} dx$, (2.2), (2.3) and $\lambda_n = \lambda_n^0 + d_n$ equations are

used and if necessary calculations are made $\alpha_n = \pi + \delta_n = \pi + \frac{k_n}{n} + \frac{\gamma_n}{n^2}$ obtained, where

$$k_n = 2 \left\{ \int_0^\pi \cos(\xi(x, \lambda_n^0) - \alpha) [K_{11}(x, \pi) \sin(\lambda_n^0 \pi - \alpha) - K_{22}(x, \pi) \cos(\lambda_n^0 \pi - \alpha)] dx \right. \\ \left. + \int_0^\pi \sin(\xi(x, \lambda_n^0) - \alpha) [K_{21}(x, \pi) \cos(\lambda_n^0 \pi - \alpha) - K_{22}(x, \pi) \sin(\lambda_n^0 \pi - \alpha)] dx \right. \\ \left. + \delta_{n1} \int_0^\pi \cos(\xi(x, \lambda_n^0) - \alpha) dx + \delta_{n2} \int_0^\pi \sin(\xi(x, \lambda_n^0) - \alpha) dx \right\} \in l_2, \delta_{n1}, \delta_{n2} \in l_2.$$

$$\gamma_n = \int_0^\pi [K_{11}(x, \pi) \sin(\lambda_n^0 \pi - \alpha) - K_{22}(x, \pi) \cos(\lambda_n^0 \pi - \alpha)]^2 dx \\ + \int_0^\pi [K_{21}(x, \pi) \cos(\lambda_n^0 \pi - \alpha) - K_{22}(x, \pi) \sin(\lambda_n^0 \pi - \alpha)]^2 dx \\ + 2\delta_{n1} \int_0^\pi [K_{11}(x, \pi) \sin(\lambda_n^0 \pi - \alpha) - K_{22}(x, \pi) \cos(\lambda_n^0 \pi - \alpha)] dx \\ + 2\delta_{n2} \int_0^\pi [K_{21}(x, \pi) \cos(\lambda_n^0 \pi - \alpha) - K_{22}(x, \pi) \sin(\lambda_n^0 \pi - \alpha)] dx + \pi(\delta_{n1}^2 + \delta_{n2}^2) \in l_2, \delta_{n1}, \delta_{n2} \in l_2.$$

□

Assume that $\Phi(x, \lambda)$ is a solution of (1.1) which satisfies the conditions $U(\Phi) = 1$ and $V(\Phi) = 0$.

Let $C(x, \lambda) = \begin{pmatrix} C_1(x, \lambda) \\ C_2(x, \lambda) \end{pmatrix}$, $S(x, \lambda) = \begin{pmatrix} S_1(x, \lambda) \\ S_2(x, \lambda) \end{pmatrix}$ and $\Psi(x, \lambda) = \begin{pmatrix} \psi_1(x, \lambda) \\ \psi(x, \lambda) \end{pmatrix}$ denote solutions of equation (1.1) that satisfy the initial conditions $C(0, \lambda) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $S(0, \lambda) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\Psi(\pi, \lambda) = \begin{pmatrix} \cos \beta \\ -\sin \beta \end{pmatrix}$ respectively.

We set $M(\lambda) := \Phi_1(0, \lambda)$. The functions $\Phi(x, \lambda)$ and $M(\lambda)$ are called the Weyl solution and Weyl function of the boundary value problem L respectively.

It is obvious that

$$\Phi(x, \lambda) = -\frac{\Psi(x, \lambda)}{\Delta(\lambda)} = C(x, \lambda) + M(\lambda)\varphi(x, \lambda), \tag{2.7}$$

$$\langle \Phi(x, \lambda), \varphi(x, \lambda) \rangle \equiv 1. \tag{2.8}$$

We define the matrix $P(x, \lambda) = [P_{ij}(x, \lambda)]_{j,k=1,2}$ by the formula

$$P(x, \lambda) \begin{pmatrix} \tilde{\varphi}_1(x, \lambda) & \tilde{\Phi}_1(x, \lambda) \\ \tilde{\varphi}_2(x, \lambda) & \tilde{\Phi}_2(x, \lambda) \end{pmatrix} = \begin{pmatrix} \varphi_1(x, \lambda) & \Phi_1(x, \lambda) \\ \varphi_2(x, \lambda) & \Phi_2(x, \lambda) \end{pmatrix}. \tag{2.9}$$

Using (2.8) and (2.9), we get

$$\begin{cases} P_{11}(x, \lambda) &= \varphi_1(x, \lambda)\tilde{\Phi}_2(x, \lambda) - \Phi_1(x, \lambda)\tilde{\varphi}_2(x, \lambda) \\ P_{12}(x, \lambda) &= \Phi_1(x, \lambda)\tilde{\varphi}_1(x, \lambda) - \varphi_1(x, \lambda)\tilde{\Phi}_1(x, \lambda) \\ P_{21}(x, \lambda) &= \varphi_2(x, \lambda)\tilde{\Phi}_2(x, \lambda) - \Phi_2(x, \lambda)\tilde{\varphi}_2(x, \lambda) \\ P_{22}(x, \lambda) &= \Phi_2(x, \lambda)\tilde{\varphi}_1(x, \lambda) - \varphi_2(x, \lambda)\tilde{\Phi}_1(x, \lambda) \end{cases} \tag{2.10}$$

$$\begin{cases} P_{11}(x, \lambda) \equiv 1, P_{12}(x, \lambda) \equiv 0, P_{21}(x, \lambda) \equiv 0, P_{22}(x, \lambda) \equiv 1 \\ |P_{11}(x, \lambda) - 1| \leq \frac{c_\delta}{|\rho|}, \quad |P_{12}(x, \lambda)| \leq \frac{c_\delta}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^* \\ |P_{22}(x, \lambda) - 1| \leq \frac{c_\delta}{|\rho|}, \quad |P_{21}(x, \lambda)| \leq \frac{c_\delta}{|\rho|}, \quad \rho \in G_\delta, \quad |\rho| \geq \rho^* \end{cases} \quad (2.11)$$

and

$$\begin{cases} \varphi_1(x, \lambda) = P_{11}(x, \lambda)\tilde{\varphi}_1(x, \lambda) + P_{12}(x, \lambda)\tilde{\varphi}_2(x, \lambda) \\ \varphi_2(x, \lambda) = P_{21}(x, \lambda)\tilde{\varphi}_1(x, \lambda) + P_{22}(x, \lambda)\tilde{\varphi}_2(x, \lambda) \\ \Phi_1(x, \lambda) = P_{11}(x, \lambda)\tilde{\Phi}_1(x, \lambda) + P_{12}(x, \lambda)\tilde{\Phi}_2(x, \lambda) \\ \Phi'(x, \lambda) = P_{21}(x, \lambda)\tilde{\Phi}_1(x, \lambda) + P_{22}(x, \lambda)\tilde{\Phi}_2(x, \lambda). \end{cases} \quad (2.12)$$

3. The main equation of the inverse problem

In this section, we solve the inverse problem of recovering L from the given spectral data by using ideas of the contour integral method.

We denote that if $y(x, \lambda)$ and $z(x, \mu)$ are solutions of the equations $\ell y = \lambda y$ and $\ell z = \mu z$ separately, then

$$\frac{d}{dx} \langle y, z \rangle = (\mu - \lambda)y^t z, \quad \langle y, z \rangle := y_1(x, \lambda)z_2(x, \mu) - y_2(x, \lambda)z_1(x, \mu). \quad (3.1)$$

Note

$$D(x, \lambda, \mu) := \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\mu - \lambda} = \int_0^x \varphi^t(t, \lambda)\varphi(t, \mu)dt. \quad (3.2)$$

Let us select a model boundary value problem \tilde{L} such that $\beta - \alpha = \tilde{\beta} - \tilde{\alpha}$, $d_n = \tilde{d}_n$. Let $\{\tilde{\lambda}_n, \tilde{\alpha}_n\}_{n \geq 0}$ be the spectral data of \tilde{L} .

Furthermore, let

$$\xi_n := |\lambda_n - \tilde{\lambda}_n| + |\alpha_n - \tilde{\alpha}_n|$$

It follows from (2.5) and (2.6) and analogous formulas for \tilde{d}_n and $\tilde{\alpha}_n$ that

$$\Omega := \left(\sum_{n=0}^{\infty} [(n+1)\xi_n]^2 \right)^{\frac{1}{2}} < \infty, \quad \sum_n \xi_n < \infty. \quad (3.3)$$

Denote that

$$\begin{aligned} \lambda_{n0} &= \lambda_n, \quad \lambda_{n1} = \tilde{\lambda}_n, \quad \alpha_{n0} = \alpha_n, \quad \alpha_{n1} = \tilde{\alpha}_n, \quad \varphi_{ni}(x) = \varphi(x, \lambda_{ni}) \\ \tilde{\varphi}_{ni}(x) &= \tilde{\varphi}(x, \lambda_{ni}), \quad P_{ni, j}(x) = \frac{1}{\alpha_{kj}} D(x, \lambda_{ni}, \lambda_{kj}) \\ \tilde{P}_{ni, kj}(x) &= \frac{1}{\alpha_{kj}} \tilde{D}(x, \lambda_{ni}, \lambda_{kj}), \quad i, j \in \{0, 1\}, \quad n, k \geq 0. \end{aligned} \quad (3.4)$$

Then, according to (3.2),

$$\begin{aligned}
 P_{ni,kj}(x) &= \frac{\langle \varphi_{ni}(x), \varphi_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{kj} - \lambda_{ni})} = \frac{1}{\alpha_{kj}} \int_0^x \varphi_{ni}^t(t), \varphi_{kj}(t) dt \\
 \tilde{P}_{ni,kj}(x) &= \frac{\langle \tilde{\varphi}_{ni}(x), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{kj} - \lambda_{ni})} = \frac{1}{\alpha_{kj}} \int_0^x \tilde{\varphi}_{ni}^t(t), \tilde{\varphi}_{kj}(t) dt.
 \end{aligned}
 \tag{3.5}$$

It is clear that

$$P'_{ni,kj}(x) = \frac{1}{\alpha_{kj}} \varphi_{ni}^t(x) \varphi_{kj}(x), \quad \tilde{P}'_{ni,kj}(x) = \frac{1}{\alpha_{kj}} \tilde{\varphi}_{ni}^t(x) \tilde{\varphi}_{kj}(x)
 \tag{3.6}$$

Lemma 3.1 *Let $f(\rho)$ be an analytic function for*

$|\rho - \rho^0| < a$ such that $f(\rho^0) = 0$ and $|f(\rho)| \leq A$. Then $|f(\rho)| \leq \frac{A}{a} |\rho - \rho^0|$ for $|\rho - \rho^0| < a$.

Proof The proof of lemma is given in [4] □

Lemma 3.2 *The following estimates are valid for $x \in [0, \pi]$, $n, k \geq 0$, $i, j, v = 0, 1$;*

$$\begin{aligned}
 |\varphi_{1ni}(x)| &\leq c, \quad |\varphi_{2ni}(x)| \leq c \\
 |\varphi_{1n0}(x) - \varphi_{1n1}(x)| &\leq c\xi_n, \quad |\varphi_{2n0}(x) - \varphi_{2n1}(x)| \leq c\xi_n
 \end{aligned}
 \tag{3.7}$$

$$\begin{aligned}
 |P_{ni,kj}(x)| &\leq \frac{c}{|n - k| + 1} \\
 |P_{ni,k0}(x) - P_{ni,k1}(x)| &\leq \frac{c\xi_k}{|n - k| + 1}, \quad |P_{n0,kj}(x) - P_{n1,kj}(x)| \leq \frac{c\xi_n}{|n - k| + 1}
 \end{aligned}
 \tag{3.8}$$

$$|P_{n0,k0}(x) - P_{n1,k0}(x) - P_{n0,k1}(x) + P_{n1,k1}(x)| \leq \frac{c\xi_n \xi_k}{|n - k| + 1}.$$

The similar estimates are also current for $\tilde{\varphi}_{ni}(x)$, $\tilde{P}_{ni,kj}(x)$.

Proof It follows from (2.1), (2.5), and (2.6) that $|\varphi_{ni}(x)| \leq c$.

Applying Schwarz's lemma in the ρ -plane to the function $f(\rho) := \varphi(x, \lambda) - \varphi(x, \lambda_{n1})$ with fixed n , x and a , we obtain

$$\begin{aligned}
 |\varphi_1(x, \lambda) - \varphi_1(x, \lambda_{n1})| &\leq c|\rho - \rho_{n1}|, \quad |\rho - \rho_{n1}| \leq a \\
 |\varphi_2(x, \lambda) - \varphi_2(x, \lambda_{n1})| &\leq c|\rho - \rho_{n1}|, \quad |\rho - \rho_{n1}| \leq a
 \end{aligned}$$

As a result (3.7) is proved.

Let us show that

$$|D(x, \lambda, \lambda_{kj})| \leq \frac{ce^{|\tau|x}}{|\rho \mp k| + 1}, \quad \lambda = \rho^2, \quad \mp \operatorname{Re} \rho \geq 0, \quad \tau = \operatorname{Im} \rho, \quad k \geq 0.
 \tag{3.9}$$

For definiteness, let $\sigma := \operatorname{Re} \rho \geq 0$. Get a fixed $\delta_0 > 0$. For $|\rho - \rho_{kj}| \geq \delta_0$, we have by helping of (3.2), (2.1) and (2.5),

$$|D(x, \lambda, \lambda_{kj})| = \frac{|\langle \varphi(x, \lambda), \varphi(x, \lambda_{kj}) \rangle|}{|\lambda - \lambda_{kj}|} \leq ce^{|\tau|x} \frac{|\rho| + |\rho_{kj}|}{|\rho^2 - \rho_{kj}^2|}.$$

Since $\frac{|\rho| + |\rho_{kj}|}{|\rho + \rho_{kj}|} \leq \frac{\sqrt{\sigma^2 + \tau^2} + |\rho_{kj}|}{\sqrt{\sigma^2 + \tau^2 + \rho_{kj}^2}} \leq \sqrt{2}$, we obtain

$$|D(x, \lambda, \lambda_{kj})| \leq \frac{ce^{|\tau|x}}{|\rho - \rho_{kj}|}.$$

$$\frac{|\rho - k| + 1}{|\rho - \rho_{kj}|} \leq \frac{|\rho_{kj} - k| + 1}{|\rho - \rho_{kj}|} \leq c, \text{ and so } \frac{1}{|\rho - \rho_{kj}|} \leq \frac{c}{|\rho - k| + 1}.$$

Thus, this yields (3.9) for $|\rho - \rho_{kj}| \geq \delta_0$. For $|\rho - \rho_{kj}| \leq \delta_0$,

$$|D(x, \lambda, \lambda_{kj})| = \left| \int_0^x \varphi^t(t, \lambda) \varphi(t, \lambda_{kj}) dt \right| \leq ce^{|\tau|x},$$

i.e. (3.9) is also valid for $|\rho - \rho_{kj}| \leq \delta_0$. Likewise it can be shown that

$$|D(x, \lambda, \mu)| \leq \frac{ce^{|\tau|x}}{|\rho \mp \theta| + 1}, \mu = \theta^2, |\operatorname{Im} \theta| \leq c_0, \mp \operatorname{Re} \rho \operatorname{Re} \theta \geq 0$$

Using Schwarz's lemma, we get

$$|D(x, \lambda, \lambda_{k1}) - D(x, \lambda, \lambda_{k0})| \leq \frac{ce^{|\tau|x} \xi_k}{|\rho \mp k| + 1}, \mp \operatorname{Re} \rho \geq 0, k \geq 0. \tag{3.10}$$

Particularly, this yields

$$|D(x, \lambda_{ni}, \lambda_{k1}) - D(x, \lambda_{ni}, \lambda_{k0})| \leq \frac{c\xi_k}{|n - k| + 1}.$$

Symmetrically,

$$|D(x, \lambda_{n1}, \lambda_{kj}) - D(x, \lambda_{n0}, \lambda_{kj})| \leq \frac{c\xi_n}{|n - k| + 1}.$$

If we apply Schwarz's lemma to the function

$$Q_k(x, \lambda) := D(x, \lambda, \lambda_{k1}) - D(x, \lambda, \lambda_{k0})$$

for fixed k and x , we obtain

$$|D(x, \lambda_{n0}, \lambda_{k0}) - D(x, \lambda_{n1}, \lambda_{k0}) - D(x, \lambda_{n0}, \lambda_{k1}) + D(x, \lambda_{n1}, \lambda_{k1})| \leq \frac{c\xi_n \xi_k}{|n - k| + 1}.$$

These approximates together with (3.4), (3.6), (2.5) and (2.6) suggest (3.8). □

Lemma 3.3 *The following relations hold*

$$\begin{aligned} \tilde{\varphi}(x, \lambda) &= \varphi(x, \lambda) \\ &+ \sum_{k=0}^{\infty} \left(\frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda_{k0} - \lambda)} \varphi_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda_{k1} - \lambda)} \varphi_{k1}(x) \right) \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\mu - \lambda} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\mu - \lambda} \\ &+ \sum_{k=0}^{\infty} \left\{ \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle \langle \varphi_{k0}(x), \varphi(x, \mu) \rangle}{\alpha_{k0}(\lambda_{k0} - \lambda) \mu - \lambda_{k0}} \right. \\ &\left. - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle \langle \varphi_{k1}(x), \varphi(x, \mu) \rangle}{\alpha_{k1}(\lambda_{k1} - \lambda) \mu - \lambda_{k1}} \right\} = 0 \end{aligned} \tag{3.12}$$

Proof 1) Denote $\lambda' = \min_{ni} \lambda_{ni}$ and take a fixed $\delta > 0$. In the λ -plane we regard closed contours $\gamma_N = \gamma_N^+ \cup \gamma_N^- \cup \gamma' \cup \Gamma'_N$, as shown in Figure.

Where

$$\gamma_N^{\pm} = \left\{ \lambda : \pm \operatorname{Im} \lambda = \delta, \operatorname{Re} \lambda \geq \lambda', |\lambda| \leq \left(N + \frac{1}{2}\right)^2 \right\}$$

$$\gamma' = \left\{ \lambda : \lambda - \lambda' = \delta e^{i\alpha}, \alpha \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right) \right\}$$

$$\Gamma'_N = \Gamma_N \cap \{ \lambda : |\operatorname{Im} \lambda| \leq \delta, \operatorname{Re} \lambda > 0 \}, \Gamma_N = \left\{ \lambda : |\lambda| = \left(N + \frac{1}{2}\right)^2 \right\}$$

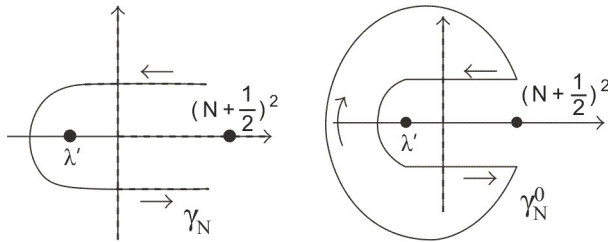


Figure 1. The curves γ_N and γ_N^0 .

Denote $\gamma_N^0 = \gamma_N^+ \cup \gamma_N^- \cup \gamma' \cup (\Gamma_N \setminus \Gamma'_N)$ (with clockwise circuit), as shown in Figure.

Let $P(x, \lambda) = [P_{jk}(x, \lambda)]_{j,k=1,2}$ be the matrix determined by (2.9). It follows from (2.7) and (2.10) that for each fixed x , the functions $P_{jk}(x, \lambda)$ are meromorphic in λ with simple poles $\{\lambda_n\}$ and $\{\tilde{\lambda}_n\}$. By Cauchy's integral formula,

$$P_1(x, k) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_N^0} \frac{P_{1k}(x, \xi) - \delta_{1k}}{\lambda - \xi} d\xi, k = 1, 2, \dots$$

where $\lambda \in \text{int}(\gamma_N^0)$ and δ_{jk} is the Kronecker delta. Therefore,

$$P_1(x, k) - \delta_{1k} = \frac{1}{2\pi i} \int_{\gamma_N} \frac{P_{1k}(x, \xi)}{\lambda - \xi} d\xi - \frac{1}{2\pi i} \int_{\Gamma_N} \frac{P_{1k}(x, \xi) - \delta_{1k}}{\lambda - \xi} d\xi,$$

where Γ_N is used with counterclockwise circuit. Substituting in to (2.12) we get

$$\varphi_1(x, \lambda) = \tilde{\varphi}_1(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \frac{\tilde{\varphi}_1(x, \lambda)P_{11}(x, \xi) + \tilde{\varphi}_2(x, \lambda)P_{12}(x, \xi)}{\lambda - \xi} d\xi + \varepsilon_N(x, \lambda)$$

where

$$\varepsilon_N(x, \lambda) = -\frac{1}{2\pi i} \int_{\Gamma_N} \frac{\tilde{\varphi}_1(x, \lambda) (P_{11}(x, \xi) - 1) + \tilde{\varphi}_2(x, \lambda)P_{12}(x, \xi)}{\lambda - \xi} d\xi.$$

By helping of. (2.11),

$$\lim_{N \rightarrow \infty} \varepsilon_N(x, \lambda) = 0 \tag{3.13}$$

uniformly with respect to $x \in [0, \pi]$ and λ on compact sets.

If we consider (2.10) into account we calculate

$$\begin{aligned} \varphi_1(x, \lambda) = & \tilde{\varphi}_1(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \left\{ \tilde{\varphi}_1(x, \lambda) \left[\varphi_1(x, \xi)\tilde{\Phi}_2(x, \xi) - \tilde{\Phi}_1(x, \xi)\tilde{\varphi}_2(x, \xi) \right] \right. \\ & \left. + \tilde{\varphi}_2(x, \lambda) \left[\tilde{\Phi}_1(x, \xi)\tilde{\varphi}_1(x, \xi) - \varphi_1(x, \xi)\tilde{\Phi}_1(x, \xi) \right] \right\} \frac{d\xi}{\lambda - \xi} + \varepsilon_N(x, \lambda). \end{aligned}$$

In view of (2.7), this supplies

$$\tilde{\varphi}_1(x, \lambda) = \varphi_1(x, \lambda) + \frac{1}{2\pi i} \int_{\gamma_N} \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle}{\xi - \lambda} \hat{M}(\xi) \varphi_1(x, \xi) d\xi + \varepsilon_N(x, \lambda), \tag{3.14}$$

where $\hat{M}(\lambda) = M(\lambda) - \tilde{M}(\lambda)$, since the terms with $C(x, \xi)$ vanish by Cauchy's theorem.

It follows from

$$\text{Re } sM(\lambda) = -\frac{\Delta^0(\lambda_n)}{\Delta(\lambda_n)} = -\frac{\beta_n}{\Delta(\lambda_n)} = \frac{1}{\alpha_n} \tag{3.15}$$

that

$$\text{Re } s \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle}{\xi - \lambda} \hat{M}(\xi) \varphi_1(x, \xi) = \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{kj}(x) \rangle}{\alpha_{kj}(\lambda_{kj} - \lambda)} \varphi_{1kj}(x).$$

Calculating the integral in (3.14) by the residue theorem and using (3.13) we get (3.11).

2) Since $\frac{1}{\mu - \lambda} \left(\frac{1}{\mu - \xi} - \frac{1}{\lambda - \xi} \right) = \frac{1}{(\lambda - \xi)(\xi - \mu)}$, we have by Cauchy's integral formula

$$\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\mu - \lambda} = \frac{1}{2\pi i} \int_{\gamma_N^0} \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi, k, j = 1, 2, \lambda, \mu \in \text{int}(\gamma_N^0).$$

Considering in the same way as above and using (2.11), we get

$$\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\mu - \lambda} = \frac{1}{2\pi i} \int_{\gamma_N} \frac{P_{jk}(x, \xi)}{(\lambda - \xi)(\xi - \mu)} d\xi + \varepsilon_{N_{jk}}(x, \lambda, \mu) \tag{3.16}$$

where $\lim_{n \rightarrow \infty} \varepsilon_{N_{jk}}(x, \lambda, \mu) = 0 \quad j, k = \overline{1, n}$. From (2.11) and (2.8), it follows that

$$\left. \begin{aligned} P_{11}(x, \lambda)\varphi_2(x, \lambda) - P_{21}(x, \lambda)\varphi_1(x, \lambda) &= \tilde{\varphi}_2(x, \lambda) \\ P_{22}(x, \lambda)\varphi_1(x, \lambda) - P_{12}(x, \lambda)\varphi_2(x, \lambda) &= \tilde{\varphi}_1(x, \lambda) \end{aligned} \right\} \tag{3.17}$$

$$\begin{aligned} P(x, \lambda) \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} &= \langle y(x), \tilde{\Phi}(x, \lambda) \rangle \begin{bmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{bmatrix} \\ &- \langle y(x), \tilde{\varphi}(x, \lambda) \rangle \begin{bmatrix} \Phi_1(x, \lambda) \\ \Phi_2(x, \lambda) \end{bmatrix} \end{aligned} \tag{3.18}$$

for any $y(x) \in C^1[0, 1]$.

If we consider (3.16) and (3.18) into account, we calculate

$$\begin{aligned} &\frac{P_{jk}(x, \lambda) - P_{jk}(x, \mu)}{\mu - \lambda} \begin{bmatrix} y(x) \\ y'(x) \end{bmatrix} \\ &= \frac{1}{2\pi i} \int_{\gamma_N} \left\{ \langle y(x), \tilde{\Phi}(x, \xi) \rangle \begin{bmatrix} \varphi_1(x, \xi) \\ \varphi_2(x, \xi) \end{bmatrix} \right. \\ &\quad \left. - \langle y(x), \tilde{\varphi}(x, \xi) \rangle \begin{bmatrix} \Phi_1(x, \xi) \\ \Phi_2(x, \xi) \end{bmatrix} \right\} \frac{d\xi}{(\lambda - \xi)(\xi - \mu)} \\ &+ \varepsilon_N^0(x, \lambda, \mu), \quad \lim_{N \rightarrow \infty} \varepsilon_N^0(x, \lambda, \mu) = 0. \end{aligned} \tag{3.19}$$

According to (2.9), $P(x, \lambda) \begin{bmatrix} \tilde{\varphi}_1(x, \lambda) \\ \tilde{\varphi}_2(x, \lambda) \end{bmatrix} = \begin{bmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{bmatrix}$. Thus,

$$\det \left(P(x, \lambda) \begin{bmatrix} \tilde{\varphi}_1(x, \lambda) \\ \tilde{\varphi}_2(x, \lambda) \end{bmatrix}, \begin{bmatrix} \varphi_1(x, \mu) \\ \varphi_2(x, \mu) \end{bmatrix} \right) = \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle .$$

Moreover, using (3.17) we obtain

$$\det \left(P(x, \mu) \begin{bmatrix} \tilde{\varphi}_1(x, \lambda) \\ \tilde{\varphi}_2(x, \lambda) \end{bmatrix}, \begin{bmatrix} \varphi_1(x, \mu) \\ \varphi_2(x, \mu) \end{bmatrix} \right) = \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle .$$

So,

$$\begin{aligned} &\det \left((P(x, \lambda) - P(x, \mu)) \begin{bmatrix} \tilde{\varphi}_1(x, \lambda) \\ \tilde{\varphi}_2(x, \lambda) \end{bmatrix}, \begin{bmatrix} \varphi_1(x, \mu) \\ \varphi_2(x, \mu) \end{bmatrix} \right) \\ &= \langle \varphi(x, \lambda), \varphi(x, \mu) \rangle - \langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle . \end{aligned}$$

As a result, (3.19) for $y(x) = \tilde{\varphi}(x, \lambda)$ supplies

$$\begin{aligned} & \frac{\langle \varphi(x, \lambda), \varphi(x, \mu) \rangle}{\mu - \lambda} - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \mu) \rangle}{\mu - \lambda} \\ &= \frac{1}{2\pi i} \int_{\gamma_N} \left\{ \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\Phi}(x, \xi) \rangle \langle \varphi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} \right. \\ & \quad \left. - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}(x, \xi) \rangle \langle \Phi(x, \xi), \varphi(x, \mu) \rangle}{(\lambda - \xi)(\xi - \mu)} \right\} d\xi \\ & + \varepsilon'_N(x, \lambda, \mu), \quad \lim_{N \rightarrow \infty} \varepsilon'_N(x, \lambda, \mu) = 0. \end{aligned}$$

By virtue of (2.7), (3.15) and residue theorem, we obtain (3.12). □

Similarly, it can be obtained the following relation

$$\begin{aligned} \tilde{\Phi}_1(x, \lambda) &= \Phi_1(x, \lambda) \\ &+ \sum_{k=0}^{\infty} \left(\frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda_{k0} - \lambda)} \varphi_{1k0}(x) - \frac{\langle \tilde{\Phi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda_{k1} - \lambda)} \varphi_{1k1}(x) \right). \end{aligned} \tag{3.20}$$

It follows from the definition of $\tilde{P}_{ni,kj}(x)$, $P_{ni,kj}(x)$ and (3.11), (3.12) that

$$\tilde{\varphi}_{ni}(x) = \varphi_{ni}(x) + \sum_{k=0}^{\infty} (\tilde{P}_{ni,k0}(x) \varphi_{k0}(x) - \tilde{P}_{ni,k1}(x) \varphi_{k1}(x)) \tag{3.21}$$

$$P_{ni,lj}(x) - \tilde{P}_{ni,lj}(x) + \sum_{k=0}^{\infty} (\tilde{P}_{ni,k0}(x) P_{k0,lj}(x) - \tilde{P}_{ni,k1}(x) P_{k1,lj}(x)) = 0. \tag{3.22}$$

Denote

$$\varepsilon_0(x) := \sum_{k=0}^{\infty} \left(\frac{1}{\alpha_{k0}} \tilde{\varphi}_{k0}(x) \varphi_{k0}^t(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{k1}(x) \varphi_{k1}^t(x) \right). \tag{3.23}$$

Lemma 3.4 *The serie in (3.23) converges absolutely and uniformly on $[0, \pi]$. The function $\varepsilon_0(x)$ is absolutely continuous.*

Proof We can write $\varepsilon_0(x)$ to the form

$$\varepsilon_0(x) = A_1(x) + A_2(x) \tag{3.24}$$

where

$$\begin{cases} A_1(x) = \sum_{k=0}^{\infty} \left(\frac{1}{\alpha_{k0}} - \frac{1}{\alpha_{k1}} \right) \tilde{\varphi}_{k0}(x) \varphi_{k0}^t(x) \\ A_2(x) = \sum_{k=0}^{\infty} \frac{1}{\alpha_{k1}} \{ (\tilde{\varphi}_{k0}(x) - \tilde{\varphi}_{k1}(x)) \varphi_{k0}^t(x) + \tilde{\varphi}_{k1}(x) (\varphi_{k0}^t(x) - \varphi_{k1}^t(x)) \}. \end{cases} \tag{3.25}$$

It follows from (2.5), (2.6), (3.3) and (3.7) that the series in (3.25) converge absolutely and uniformly on $[0, \pi]$ and

$$|A_j(x)| \leq c \sum_{k=0}^{\infty} \xi_k \leq c\Lambda, \quad j = 1, 2. \tag{3.26}$$

□

Lemma 3.5 *The following relation holds*

$$\Omega(x) = \tilde{\Omega}(x) + \Upsilon(x) \tag{3.27}$$

where $\Upsilon(x) = \begin{pmatrix} \Upsilon_0(x) & \Upsilon_1(x) \\ \Upsilon_1(x) & \Upsilon_2(x) \end{pmatrix},$

$$\begin{aligned} \Upsilon_0(x) &= 2 \sum_{k=0}^{\infty} \left\{ \frac{1}{\alpha_{k0}} \tilde{\varphi}_{2k0}(x) \varphi_{1k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{2k1}(x) \varphi_{1k1}(x) \right\} \\ \Upsilon_1(x) &= - \sum_{k=0}^{\infty} \left\{ \frac{1}{\alpha_{k0}} (\tilde{\varphi}_{1k0}(x) \varphi_{1k0}(x) - \tilde{\varphi}_{2k0}(x) \varphi_{2k0}(x)) \right. \\ &\quad \left. - \frac{1}{\alpha_{k1}} (\tilde{\varphi}_{1k1}(x) \varphi_{1k1}(x) - \tilde{\varphi}_{2k1}(x) \varphi_{2k1}(x)) \right\} \\ \Upsilon_2(x) &= -2 \sum_{k=0}^{\infty} \left\{ \frac{1}{\alpha_{k0}} \tilde{\varphi}_{1k0}(x) \varphi_{2k0}(x) - \frac{1}{\alpha_{k1}} \tilde{\varphi}_{1k1}(x) \varphi_{2k1}(x) \right\}. \end{aligned}$$

Proof If we differentiate (3.11) twice with respect to x and use (3.1) , (3.23), we obtain

$$\begin{aligned} &\tilde{\varphi}'(x, \lambda) - \varepsilon_0(x) \tilde{\varphi}(x, \lambda) = \varphi'(x, \lambda) \\ &+ \sum_{k=0}^{\infty} \left(\frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k0}(x) \rangle}{\alpha_{k0}(\lambda_{k0} - \lambda)} \varphi'_{k0}(x) - \frac{\langle \tilde{\varphi}(x, \lambda), \tilde{\varphi}_{k1}(x) \rangle}{\alpha_{k1}(\lambda_{k1} - \lambda)} \varphi'_{k1}(x) \right) \end{aligned} \tag{3.28}$$

If we replace $\varphi'(x, \lambda)$, $\tilde{\varphi}'(x, \lambda)$, $\varphi'_{k0}(x)$ and $\varphi'_{k1}(x)$ by using equation (1.1) and then replace $\varphi(x, \lambda)$ using equation (3.11) then

$$\begin{aligned} &(\tilde{p}(x) - p(x)) \tilde{\varphi}_1(x, \lambda) + (\tilde{q}(x) - q(x)) \tilde{\varphi}_2(x, \lambda) \\ &+ \Upsilon_0(x) \tilde{\varphi}_1(x, \lambda) + \Upsilon_1(x) \tilde{\varphi}_2(x, \lambda) \equiv 0, \forall \lambda \\ &(\tilde{q}(x) - q(x)) \tilde{\varphi}_1(x, \lambda) + (\tilde{r}(x) - r(x)) \tilde{\varphi}_2(x, \lambda) \\ &+ \Upsilon_1(x) \tilde{\varphi}_1(x, \lambda) + \Upsilon_2(x) \tilde{\varphi}_2(x, \lambda) \equiv 0, \forall \lambda \\ &\begin{pmatrix} \tilde{p}(x) & \tilde{q}(x) \\ \tilde{q}(x) & \tilde{r}(x) \end{pmatrix} - \begin{pmatrix} p(x) & q(x) \\ q(x) & r(x) \end{pmatrix} + \begin{pmatrix} \Upsilon_0(x) & \Upsilon_1(x) \\ \Upsilon_1(x) & \Upsilon_2(x) \end{pmatrix} = 0 \\ &\Omega(x) = \tilde{\Omega}(x) + \Upsilon(x). \end{aligned}$$

□

Remark 3.1. For each fixed $x \in [0, \pi]$ the equation (3.21) can be considered as a system of linear equations with respect to $\varphi_{ni}(x)$ $n \geq 0, i = 0, 1$. However, the series in (3.21) converges only "with brackets". Hence, it is not appropriate to use (3.21) as a main equation of the inverse problem. Below we will convert (3.21) to a linear equation in a corresponding Banach space of sequences.

Let T be a set of indices $u = (n, i), n \geq 0, i = 0, 1$. For each fixed $x \in [0, \pi]$, we define the vector

$$\Psi(x) = [\Psi_u(x)]_{u \in T} = \begin{bmatrix} \Psi_{1n0}(x) & \Psi_{2n0}(x) \\ \Psi_{1n1}(x) & \Psi_{2n1}(x) \end{bmatrix}_{n \geq 0}$$

by the formula

$$\begin{aligned} \begin{bmatrix} \Psi_{1n0}(x) & \Psi_{2n0}(x) \\ \Psi_{1n1}(x) & \Psi_{2n1}(x) \end{bmatrix} &= \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \varphi_{1n0}(x) & \varphi_{2n0}(x) \\ \varphi_{1n1}(x) & \varphi_{2n1}(x) \end{bmatrix} \\ &= \begin{bmatrix} \chi_n (\varphi_{1n0}(x) - \varphi_{1n1}(x)) & \chi_n (\varphi_{2n0}(x) - \varphi_{2n1}(x)) \\ \varphi_{1n1}(x) & \varphi_{2n1}(x) \end{bmatrix} \\ \chi_n &= \begin{cases} \xi_n^{-1} & , \xi_n \neq 0 \\ 0 & , \xi_n = 0 \end{cases} . \end{aligned}$$

We also define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v \in V} = \begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix}_{n,k \geq 0}, \quad u = (n, i), v = (k, j)$$

by the formula

$$\begin{aligned} \begin{bmatrix} H_{n0,k0}(x) & H_{n0,k1}(x) \\ H_{n1,k0}(x) & H_{n1,k1}(x) \end{bmatrix} &= \begin{bmatrix} \chi_n & -\chi_n \\ 0 & 1 \end{bmatrix} \begin{bmatrix} P_{n0,k0}(x) & P_{n0,k1}(x) \\ P_{n1,k0}(x) & P_{n1,k1}(x) \end{bmatrix} \begin{bmatrix} \xi_k & 1 \\ 0 & -1 \end{bmatrix} \\ &= \begin{bmatrix} \xi_k \chi_n (P_{n0,k0}(x) - P_{n1,k0}(x)) & \chi_n (P_{n0,k0}(x) - P_{n0,k1}(x) - P_{n1,k0}(x) + P_{n1,k1}(x)) \\ \xi_k P_{n1,k0}(x) & P_{n1,k0}(x) - P_{n1,k1}(x) \end{bmatrix} . \end{aligned}$$

Analogously we define $\tilde{\Psi}(x), \tilde{H}(x)$ by replacing in the former descriptions $\varphi_{ni}(x)$ by $\tilde{\varphi}_{ni}(x)$ and $P_{ni,kj}(x)$ by $\tilde{P}_{ni,kj}(x)$. It follows from (3.7) and (3.8) that

$$|\Psi_{ni}(x)| \leq c, \quad |H_{ni,kj}(x)| \leq \frac{c\xi_k}{|n-k|+1} \tag{3.29}$$

Similarly,

$$|\tilde{\Psi}_{ni}(x)| \leq c, \quad |\tilde{H}_{ni,kj}(x)| \leq \frac{c\xi_k}{|n-k|+1} \tag{3.30}$$

and also

$$\left| \tilde{H}_{ni,kj}(x) - \tilde{H}_{ni,kj}(x_0) \right| \leq c|x-x_0|\xi_k, \quad x, x_0 \in [0, \pi] \tag{3.31}$$

where c does not depend on x, x_0, n, i, j and k .

Let us consider the Banach space m of bounded sequences $\alpha = [\alpha_u]_{u \in T}$ with the norm $\|\alpha\|_m = \sup_{u \in T} |\alpha_u|$.

It follows from (3.29) and (3.30) that for each fixed $x \in [0, \pi]$, the operators $E + \tilde{H}(x)$ and $E - H(x)$ such that E is the identity operator mapping from m to m , are linear bounded operators and

$$\|H(x)\|, \|\tilde{H}(x)\| \leq c \sup_n \sum_k \frac{\xi_k}{|n-k|+1} < \infty. \tag{3.32}$$

Theorem 3.6 *In the Banach space m , the vector $\Psi(x) \in m$ satisfies the equation*

$$\tilde{\Psi}(x) = (E + \tilde{H}(x)) \Psi(x) \tag{3.33}$$

for each fixed $x \in [0, \pi]$. Furthermore, the operator $E + \tilde{H}(x)$ has a bounded inverse operator, i.e. equation (3.33) is uniquely solvable.

Proof We can write (3.21) in the form

$$\begin{aligned} \tilde{\varphi}_{n0}(x) - \tilde{\varphi}_{n1}(x) &= \varphi_{n0}(x) - \varphi_{n1}(x) \\ &+ \sum_{k=0}^{\infty} \left\{ \left(\tilde{P}_{n0,k0}(x) - \tilde{P}_{n1,k0}(x) \right) (\varphi_{k0}(x) - \varphi_{k1}(x)) \right. \\ &\left. + \left(\tilde{P}_{n0,k0}(x) - \tilde{P}_{n1,k0}(x) - \tilde{P}_{n0,k1}(x) + \tilde{P}_{n1,k1}(x) \right) \varphi_{k1}(x) \right\}, \\ \tilde{\varphi}_{n1}(x) &= \varphi_{n1}(x) + \sum_{k=0}^{\infty} \left\{ \tilde{P}_{n1,k0}(x) (\varphi_{k0}(x) - \varphi_{k1}(x)) + \left(\tilde{P}_{n1,k0}(x) - \tilde{P}_{n1,k1}(x) \right) \varphi_{k1}(x) \right\}. \end{aligned}$$

Considering our notations, we get

$$\tilde{\Psi}_{ni}(x) = \Psi_{ni}(x) + \sum_{k,j} \tilde{H}_{ni,kj}(x) \Psi_{kj}(x), \quad (n, i), (k, j) \in T \tag{3.34}$$

which is equivalent to (3.33). The series in (3.34) converges absolutely and uniformly for $x \in [0, \pi]$. Analogously, (3.22) takes the form

$$H_{ni,kj}(x) - \tilde{H}_{ni,kj}(x) + \sum_{l,s} \tilde{H}_{ni,ls}(x) H_{ls,kj}(x) = 0, \quad (n, i), (k, j), (l, s) \in T$$

or

$$(E + \tilde{H}(x))(E - H(x)) = E.$$

Replacing for L and \tilde{L} , similarly, we get

$$\Psi(x) = (E - H(x)) \tilde{\Psi}(x), \quad (E - H(x))(E + \tilde{H}(x)) = E.$$

Therefore, the operator $(E + \tilde{H}(x))^{-1}$ exists and it is a linear bounded operator. □

Equation (3.33) is called the main equation of the inverse problem. Solving (3.33), we find the vector $\Psi(x)$ and as a result, the functions $\varphi_{ni}(x)$. We can construct the function $\Omega(x)$ due to $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$ are the solutions of (1.1). So, we get the following algorithm for the solution of inverse problem.

Algorithm. Given the numbers $\{\lambda_n, \alpha_n\}_{n \geq 0}$.

1. Select \tilde{L} such that $\beta - \alpha = \tilde{\beta} - \tilde{\alpha}$, $d_n = \tilde{d}_n$ and construct $\tilde{\Psi}(x)$, $\tilde{H}(x)$.
2. Find $\Psi(x)$ by solving equation (3.33).
3. Calculate $\Omega(x)$ by (3.27).

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