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Dynamics of a fluid equation with Neumann boundary conditions

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Abstract: We study the dynamics of a Neumann boundary value problem arising in fluid dynamics. We prove the nonexistence, existence and uniqueness of positive solutions under suitable conditions. At the same time, under stricter conditions, we also obtain the dynamic properties of the Neumann boundary value problem, such as the stability and instability of positive solutions. The methods of proof mainly involve the upper and lower solutions method, eigenvalue theory and some analysis techniques.

Key words: Dynamic analysis, Neumann conditions, fluid dynamics

1. Introduction

In this paper, we study the dynamics of positive solutions of the Neumann boundary value problem

$$\begin{cases} -u'' + a(x)u = b(x)u^p, \ x \in (0,1), \\ u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $p > 0$, $p \neq 1$, a and b are continuous positive functions on $[0,1]$. By the upper and lower solutions method, eigenvalue theory and some analysis techniques, we will establish the existence and dynamic properties for (1.1).

Many papers, such as [9, 18, 19] and the references therein, have been devoted to the study of the existence and multiplicity of positive solutions for Neumann boundary value problem using degree theory, fixed point theorem and upper and lower solutions methods.

We study problem (1.1) for three motivations. Firstly, (1.1) has proved to be important in applications. For example, fluid dynamics [1, 10] can be described by

$$\begin{cases} -u'' + q^2 u = (1 + \sin \pi x)u^2, \ x \in (0,1), \ q > 0, \\ u'(0) = u'(1) = 0. \end{cases} \tag{1.2}$$

Subsequently, Bensedik and Bouchekif [2] and Torres [13] discussed the existence and symmetry of positive solutions of such problems. Secondly, positivity of Green's function plays an important role in finding solutions to problem (1.2) in the literature [2, 13]. However, in this paper, we do not need to depend on the properties of

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Green's function, and make full use of the characteristics of problem (1.1) to study. Finally, previous results on Neumann boundary value problem are concerned with existence, but few on dynamical properties [12, 19], such as stability and instability. In this paper, inspired by the stability study of periodic boundary value problems [3–5, 15–17], we will continue to discuss this topic and study the existence and dynamic analysis of positive solutions for a fluid dynamics equation with Neumann boundary value problem (1.1).

Throughout the paper, it is assumed that u is a continuous function and denote

$$u_M = \max_{x\in[0,1]} u(x), \quad u_m = \min_{x\in[0,1]} u(x),$$

and $\Delta_u = \frac{u_M}{u_m}$ when $u_m > 0$. The remaining part of the paper is organized as follows. In section 2, some preliminary results are given and in section 3, the existence results are proved. Finally, section 4 studies the dynamic properties of problem (1.1).

2. Preliminaries

Let $\lambda_1(s) < \lambda_2(s) \le \lambda_3(s) \le \cdots$ all be eigenvalues of the linear equation

$$\begin{cases} u'' + [s(x)+\lambda]u = 0, \; x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{2.1}$$

It is well known that the first eigenvalue $\lambda_1(s)$ is simple and the corresponding eigenfunction does not change sign. When $s(x) \equiv 0$, the first eigenvalue is equal to 0 and the second eigenvalue is equal to π^2. Further properties of eigenvalues concerning (2.1) can be found in [7].

Consider the following Neumann boundary value problem

$$\begin{cases} u'' + g(x,u) = f(x), \; x \in (0,1), \\ u'(0) = u'(1) = 0. \end{cases} \tag{2.2}$$

Definition 2.1 *[19] A function $\alpha \in C^2[0,1]$ is called a lower solution of (2.2) if α satisfies*

$$\begin{cases} \alpha'' + g(x,\alpha) \ge f(x), \; x \in (0,1), \\ \alpha'(0) \ge 0 \ge \alpha'(1). \end{cases} \tag{2.3}$$

A function $\beta \in C^2[0,1]$ is called an upper solution of (2.2) if β satisfies

$$\begin{cases} \beta'' + g(x,\beta) \le f(x), \; x \in (0,1), \\ \beta'(0) \le 0 \le \beta'(1). \end{cases} \tag{2.4}$$

Lemma 2.2 *[12] Suppose that u is the solution of (2.2), then u is stable if the first eigenvalue $\lambda_1(g'_u(x,u))$ of the equation*

$$\begin{cases} \varphi'' + [g'_u(x,u)+\lambda]\varphi = 0, \; x \in (0,1), \\ \varphi'(0) = \varphi'(1) = 0, \end{cases} \tag{2.5}$$

is strictly positive. The solution u is unstable if the first eigenvalue $\lambda_1(g'_u(x,u))$ is strictly negative.

The following two lemmas can be deduced directly from the differential mean value theorem combined with Lemmas 2.3 and 2.4 in [19].

Lemma 2.3 *Let* $\alpha, \beta \in C^2[0,1]$ *be the lower and upper solutions of (2.2) for all* $x \in [0,1]$ *with* $\alpha(x) \leq \beta(x)$. *Suppose that there exists sufficiently large* $L > 0$ *such that* $g'_u(x,u) > -L$, *for all* $x \in [0,1]$ *with* $\alpha(x) \leq u \leq \beta(x)$. *Then the problem (2.2) has a solution* $u \in [\alpha_m, \beta_M]$.

Lemma 2.4 *Let* $\alpha, \beta \in C^2[0,1]$ *be the lower and upper solutions of (2.2) for all* $x \in [0,1]$ *with* $\beta(x) \leq \alpha(x)$. *Suppose that*

$$g'_u(x,u) < \frac{\pi^2}{4}, \tag{2.6}$$

for all $x \in [0,1]$ *with* $\beta(x) \leq u \leq \alpha(x)$. *Then the problem (2.2) has a solution* $u \in [\beta_m, \alpha_M]$.

Lemma 2.5 *[19] Let* $[w,v] = \{u \in C^2[0,1] | w \leq u \leq v\}$. *Assume that for all* $x \in [0,1]$ *and* $u \in [w,v]$, *(2.6) holds. Then*

(1) (2.2) has a unique solution on $[w,v]$ *if a solution exists and* $g'_u(x,u) < 0$ *on* $[w,v]$.

(2) (2.2) has a unique solution on $[w,v]$ *if a solution exists and* $g'_u(x,u) > 0$ *on* $[w,v]$.

3. Existence analysis for problem (1.1)

3.1. Nonexistence

Integrating the equation in the problem (1.1) on $[0,1]$, and taking into account the boundary conditions, we get a necessary condition for the existence of positive solutions, namely

$$\int_0^1 a(x)u(x)dx = \int_0^1 b(x)u^p(x)dx.$$

So it is easy to obtain the following nonexistence result.

Theorem 3.1 *If one of the following conditions holds*

(1) $a_m \geq 0$ *and* $b_M < 0$,

(2) $a_m > 0$ *and* $b_M \leq 0$,

(3) $a_M \leq 0$ *and* $b_m > 0$,

(4) $a_M < 0$ *and* $b_m \geq 0$,

then problem (1.1) does not have positive solutions.

3.2. Existence

The method of lower and upper solutions is a quite useful and flexible tool for investigating not only the existence, localization and multiplicity of periodic solutions, but also qualitative properties, such as stability and instability. The monograph [6] presents a nice and complete historical review of the subject. We also refer to [8, 11, 14] for details and further references related to the method of lower and upper solutions.

Theorem 3.2 *Let* $0 < p < 1$, *constants* $c_1, c_2, d_1, d_2 \in (0, +\infty)$, *functions* $\omega, \sigma \in C^2[0,1]$ *be such that the equalities*

$$\ddot{\omega} = c_1 b(x) - c_2 a(x),\ x \in (0,1),\ \omega'(0) = \omega'(1) = 0, \tag{3.1}$$

and

$$\ddot{\sigma} = d_1 b(x) - d_2 a(x),\ x \in [0,1],\ \sigma'(0) = \sigma'(1) = 0 \tag{3.2}$$

are fulfilled and let there exist $u_0 \in (0, +\infty)$ *such that*

$$u_0(\omega(x) - m_\omega) + \sigma(x) - m_\sigma \le c_2 u_0 + d_2 - (c_1 u_0 + d_1)^{1/p} \ for\ x \in [0,1]. \tag{3.3}$$

Then problem (1.1) has at least one positive solution.

Proof In order to prove Theorem 3.2 from Lemma 2.3, only a pair of positive well-ordered lower and upper solutions need to be constructed.

Step 1. Construction of a lower solution.

Put

$$\alpha(x) = c_2 u_0 + d_2 - [u_0(\omega(x) - m_\omega) + \sigma(x) - m_\sigma] \ for\ x \in [0,1].$$

Then, obviously $\alpha \in C^2[0,1]$ and in view of (3.1) and (3.2) we have

$$\alpha'' = (c_2 u_0 + d_2) a(x) - (c_1 u_0 + d_1) b(x) \ for\ x \in [0,1]. \tag{3.4}$$

Moreover, according to (3.3),

$$(c_1 u_0 + d_1)^{1/p} \le \alpha(x) \le c_2 u_0 + d_2 \ for\ x \in [0,1]. \tag{3.5}$$

Now (3.4) and (3.5) imply

$$\alpha'' + b(x)\alpha^p - a(x)\alpha \ge 0 \ for\ \ x \in [0,1].$$

Consequently, α is a lower function to (1.1).

Step 2. Construction of an upper solution.

Note that the inequality $0 < p < 1$ implies

$$\lim_{u \to +\infty} [(c_1 u + d_1)^{1/p} - (c_2 u + d_2)] = +\infty.$$

Therefore, we can choose $u_1 > u_0$ such that

$$u_1(\omega(x) - m_\omega) + \sigma(x) - m_\sigma \le (c_1 u_1 + d_1)^{1/p} - (c_2 u_1 + d_2) \ for\ x \in [0,1] \tag{3.6}$$

and put

$$\beta(x) = (c_1 u_1 + d_1)^{\frac{1}{p}} - [u_1(\omega(x) - m_\omega) + \sigma(x) - m_\sigma] \ for\ x \in [0,1]. \tag{3.7}$$

Then, $\beta \in C^2[0,1]$ and in view of (3.1) and (3.2) we have

$$\beta'' = (c_2 u_1 + d_2) a(x) - (c_1 u_1 + d_1) b(x) \ for\ \ x \in [0,1]. \tag{3.8}$$

Moreover, according to (3.6) and (3.7),

$$c_2u_1 + d_2 \le \beta(x) \le (c_1u_1 + d_1)^{1/p}. \tag{3.9}$$

Now (3.8) and (3.9) imply

$$\beta'' + b(x)\beta^p - a(x)\beta \le 0 \ for \ \ x \in [0,1].$$

Consequently, β is an upper function to (1.1).

Moreover, (3.5) and (3.9) imply

$$0 < \alpha(x) \le \beta(x) \ for \ x \in [0,1].$$

Thus α and β are a pair of well-ordered lower and upper solutions.

□

Theorem 3.3 *Let $p > 1$, constants $c_1, c_2, d_1, d_2 \in (0, +\infty)$, functions $\omega, \sigma \in C^2[0,1]$ be such that the equalities (3.1) and (3.2) are fulfilled and let there exist $u_0 \in (0, +\infty)$ such that*

$$u_0(\omega(x) - m_\omega) + \sigma(x) - m_\sigma \le (c_1u_0 + d_1)^{1/p} - (c_2u_0 + d_2) \ for \ x \in [0,1]. \tag{3.10}$$

Then problem (1.1) has at least one positive solution provided that

$$pb_M\alpha_M^{p-1} - a_m < \frac{\pi^2}{4}, \tag{3.11}$$

where α is defined by (3.16).

Proof The existence is proved using Lemma 2.4. To do so, let us construct a pair of unwell-ordered lower and upper solutions.

Let

$$\beta(x) = (c_1u_0 + d_1)^{\frac{1}{p}} - [u_0(\omega(x) - m_\omega) + \sigma(x) - m_\sigma] \ for \ x \in [0,1]. \tag{3.12}$$

Then, $\beta \in C^2[0,1]$ and in view of (3.1) and (3.2) we have

$$\beta'' = (c_2u_0 + d_2)a(x) - (c_1u_0 + d_1)b(x) \ for \ \ x \in [0,1]. \tag{3.13}$$

Moreover, according to (3.10) and (3.12),

$$c_2u_0 + d_2 \le \beta(x) \le (c_1u_0 + d_1)^{1/p}. \tag{3.14}$$

Now (3.13) and (3.14) imply

$$\beta'' + b(x)\beta^p - a(x)\beta \le 0 \ for \ \ x \in [0,1].$$

Consequently, β is an upper function to (1.1).

Note that the inequality $p > 1$ implies

$$\lim_{u \to +\infty}[c_2u + d_2 - (c_1u + d_1)^{1/p}] = +\infty.$$

Therefore, we can choose $u_1 > u_0$ such that

$$u_1(\omega(x) - m_\omega) + \sigma(x) - m_\sigma \leq c_2 u_1 + d_2 - (c_1 u_1 + d_1)^{1/p} \ for \ x \in [0,1] \tag{3.15}$$

and put

$$\alpha(x) = c_2 u_1 + d_2 - [u_1(\omega(x) - m_\omega) + \sigma(x) - m_\sigma] \ for \ x \in [0,1]. \tag{3.16}$$

Then, obviously $\alpha \in C^2[0,1]$ and in view of (3.1) and (3.2) we have

$$\alpha'' = (c_2 u_1 + d_2)a(x) - (c_1 u_1 + d_1)b(x) \ for \ x \in [0,1]. \tag{3.17}$$

Moreover, according to (3.15),

$$(c_1 u_1 + d_1)^{1/p} \leq \alpha(x) \leq c_2 u_1 + d_2 \ for \ x \in [0,1]. \tag{3.18}$$

Now (3.17) and (3.18) imply

$$\alpha'' + b(x)\alpha^p - a(x)\alpha \geq 0 \ for \ \ x \in [0,1].$$

Consequently, α is a lower function to (1.1).

Moreover, (3.14) and (3.18) imply

$$0 < \beta(x) \leq \alpha(x) \ for \ x \in [0,1].$$

Thus α and β are a pair of unwell-ordered lower and upper solutions.

Let us fix

$$g(x,u) = b(x)u^p - a(x)u.$$

Then by $p > 1$, for $u \in [\beta_m, \alpha_M]$, we have

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\leq pb_M \alpha_M^{p-1} - a_m. \end{aligned}$$

The above inequality and (3.11) imply that condition (2.6) of Lemma 2.4 is satisfied, so the result is guaranteed.

□

Theorem 3.4 *Assume that* $0 < p < 1, a_m > 0$*. Then problem* (1.1) *has at least one solution such that*

$$\left(\frac{b_m}{a_M}\right)^{\frac{1}{1-p}} \leq u(x) \leq \left(\frac{b_M}{a_m}\right)^{\frac{1}{1-p}}.$$

Proof Note that

$$\beta \equiv \left(\frac{b_M}{a_m}\right)^{\frac{1}{1-p}}$$

is a constant upper function and

$$\alpha \equiv \left(\frac{b_m}{a_M}\right)^{\frac{1}{1-p}}$$

is a constant lower function on the well order $\alpha < \beta$.

Note that

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\geq pb(x)\beta^{p-1} - a(x) \\ &\geq pb_m \frac{a_m}{b_M} - a_M. \end{aligned}$$

Take $L = a_M - pb_m \frac{a_m}{b_M}$. Then $L > 0$ because $p < 1$. By Lemma 2.3, problem (1.1) has at least one solution.

□

Theorem 3.5 *Assume that* $p > 1, b_m > 0$ *and the following inequality holds*

$$4pb_M \cdot \frac{a_M}{b_m} - 4a_m < \pi^2. \tag{3.19}$$

Then problem (1.1) *has at least one solution such that*

$$\left(\frac{a_m}{b_M}\right)^{\frac{1}{p-1}} \leq u(x) \leq \left(\frac{a_M}{b_m}\right)^{\frac{1}{p-1}}.$$

Proof Note that

$$\beta \equiv \left(\frac{a_m}{b_M}\right)^{\frac{1}{p-1}}$$

is a constant upper function and

$$\alpha \equiv \left(\frac{a_M}{b_m}\right)^{\frac{1}{p-1}}$$

is a constant lower function on the reserved order $\alpha > \beta$.

Note that

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\leq pb(x)\alpha^{p-1} - a(x) \\ &\leq pb_M \cdot \frac{a_M}{b_m} - a_m. \end{aligned}$$

By Lemma 2.4, a sufficient condition for the existence of solution of (1.1) is

$$pb_M \cdot \frac{a_M}{b_m} - a_m < \frac{\pi^2}{4},$$

which is equivalent to condition (3.19).

□

4. Dynamic analysis for problem (1.1)

In this section, based on the existence of positive solutions obtained in the previous section, the dynamic behaviors of equation (1.1) are further studied, including stability and instability.

4.1. Stability

Theorem 4.1 *Under the conditions of Theorem 3.2, if* $a_m - pb_M \cdot (c_1u_0 + d_1)^{\frac{p-1}{p}} > 0$, *then the solution* u *provided by Theorem 3.2 is unique and stable.*

Proof From $a_m - pb_M \cdot (c_1u_0 + d_1)^{\frac{p-1}{p}} > 0$, for $u \in [\alpha_m, \beta_M]$, we have

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\leq pb(x)\alpha_m^{p-1} - a(x) \\ &\leq pb_M \cdot (c_1u_0 + d_1)^{\frac{p-1}{p}} - a_m \\ &< 0, \end{aligned}$$

it follows from the second conclusion of Lemma 2.5 that (1.1) has a unique solution u. By the comparison theorem of eigenvalue [7] we have $\lambda_1(g'_u(x,u)) > 0$, so the unique solution u is stable.

□

Theorem 4.2 *Under the conditions of Theorem 3.4, if moreover* $b_m > 0$ *and* $p\Delta < 1$, *then the solution* u *provided by Theorem 3.4 is unique and stable.*

Proof From $0 < p\Delta < 1$, for $u \in [\alpha, \beta]$, we have

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\leq pb(x)\alpha^{p-1} - a(x) \\ &\leq pb_M \cdot \frac{a_M}{b_m} - a_m \\ &< 0. \end{aligned}$$

The rest of the proof is similar to that in the proof of Theorem 4.1, so we omit the details. □

4.2. Instability

Theorem 4.3 *Under the conditions of Theorem 3.3, if* $pb_m \cdot (c_2u_0 + d_2)^{p-1} - a_M > 0$, *then the solution* u *provided by Theorem 3.3 is unique and unstable.*

Proof From $pb_m \cdot (c_2u_0 + d_2)^{p-1} - a_M > 0$, for $u \in [\beta_m, \alpha_M]$, we have

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\geq pb(x)\beta_m^{p-1} - a(x) \\ &\geq pb_m \cdot (c_2u_0 + d_2)^{p-1} - a_M \\ &> 0. \end{aligned}$$

Therefore, it follows from the third conclusion of Lemma 2.5 that (1.1) has a unique solution u. By the comparison theorem of eigenvalue we have $\lambda_1(g'_u(x,u)) < 0$, so the unique solution u is unstable.

□

Theorem 4.4 *Under the conditions of Theorem 3.5, if moreover* $a_m > 0$ *and* $p > \Delta \geq 1$*, then the solution* u *provided by Theorem 3.5 is unique and unstable.*

Proof From $p > \Delta$, for $u \in [\beta_m, \alpha_M]$, we have

$$\begin{aligned} g'_u(x,u) &= pb(x)u^{p-1} - a(x) \\ &\geq pb(x)\beta_m^{p-1} - a(x) \\ &\geq pb_m \cdot \frac{a_m}{b_M} - a_M \\ &> 0. \end{aligned}$$

We follow the same strategy as in the proof of Theorem 4.3.

□

Example 4.5 *Let the coefficients in (1.1) be*

$$a(x) \equiv q^2, \text{ and } b(x) = 1 + \sin \pi x, \quad 0 \leq x \leq 1, \ q > 0.$$

(1) (1.1) has at least one positive solution if $0 < p < 1$*.*

(2) (1.1) has a unique stable solution if $0 < p < \frac{1}{2}$*.*

Proof First we prove (1). It is easy to see that $a_m > 0$. The result follows from Theorem 3.4. Next we prove (2). Notice that $\triangle = 2$, so the uniqueness and stability are obtained from Theorem 4.2.

□

Remark 4.6 *According to Theorem 3.5, it is easy to obtain the existence of positive solutions of the problem 1.2, but its further dynamic properties are still an open problem for us, because the condition of Theorem 4.4 is not satisfied.*

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