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Vacuum isolating and blow-up analysis for edge hyperbolic system on edge Sobolev spaces

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Abstract: This paper deals with the study of the initial-boundary value problem of edge-hyperbolic system with damping term on the manifold with edge singularity. More precisely, it is analyzed the invariance and vacuum isolating of the solution sets to the edge-hyperbolic systems on edge Sobolev spaces. Then, by using a family of modified potential wells and concavity methods, it is obtained existence and nonexistence results of global solutions with exponential decay and is shown the blow-up in finite time of solutions on the manifold with edge singularities.

Key words: Semilinear hyperbolic equation, potential wells, cone Sobolev spaces, partial differential operator

1. Introduction

Boundary value problems for partial differential equations play a crucial role in many areas of mathematics and the applied sciences. For example, a principal task of quantum chemistry is the development of many-particle models in electronic structure theory which enable accurate predictions of molecular properties [13]. As boundary-initial value problem, it is important to know the existence and behaviour of the solutions of these models near coalescence points of particles. It is the purpose of our work to develop tools which help to deepen our understanding of the existence and reguality properties of the solutions which can be eventually used to improve such models and corresponding problems. The Coulomb singularities at coalescence points of particles are treated as embedded conical, edge and corner singularities in the configuration space of electrons [11]. Interesting phenomena are often connected with geometric singularities, for instance, in mechanics or cracks in a medium are described by hypersurfaces with a boundary. Configurations of that kind belong to the category of spaces (manifolds) with geometric singularities, here with edges. Singularities occur in physics too. To be more precise, they occur in the theories that physicists use. When one asks physics to calculate the self-energy of an electron, or the structure of space time at the center of a black hole, one encounter with mathematical bad behaviour, that is the singularities from the point view of mathematics. A spacetime singularity is a breakdown in spacetime, either in its geometry or in some other basic physical structure. When it is the fundamental geometry that breaks down, spacetime singularities are often viewed as an end, or edge points, of spacetime itself. Black holes are regions of spacetime from which nothing, not even light, can escape. A typical black hole is the result of the gravitational force becoming so strong that one would have to travel faster than light to escape its pull. Such black holes generically contain a spacetime singularity at their center; thus we cannot

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fully understand a black hole without also understanding the nature of singularities [12]. In recent years, from a mathematical point of view, the analysis on such (in general, stratified) spaces has become a mathematical structure theory with many deep relations with geometry, topology, and mathematical physics [10]. In [21], Melrose, Vasy and Wunsch investigated the geometric propagation and diffraction of singularities of solutions to the wave equation on manifolds with edge singularities. Let X be an $n-$dimensional manifold with boundary, where the boundary ∂X is endowed with a fibration $Z \to \partial X$ and $\partial X \to Y$ where Y, Z are without boundary. By an edge metric g on X, we mean a metric g on the interior of X which is a smooth 2-cotensor up to the boundary but which degenerates there in a way compatible with the fibration. A manifold with boundary equipped with such an edge metric also is called an edge manifold or a manifold with edge structure. If Z is point, then an edge metric on X is simply a metric in the usual sense, smooth up to the boundary, while if Y is a point, X is conic manifold [5]. A simple example of a more general edge metric is obtained by performing a real blowup on a submanifold B of a smooth, boundaryless manifold A. The blowup operation simply introduces polar coordinates near B, i.e. it replaces B by its spherical normal bundle, thus yielding a manifold X with boundary. The pullback of a smooth metric on A to X is then an edge metric [21]. Because of the two motives stated above, in this paper, we use the edge Sobolev inequality, Poincaré inequality and modified methods in [6] to prove on the global well-posedness of solutions to initial-boundary value problems for semilinear degenerate hyperbolic equations with damping term on manifolds with edge singularities. More precisely, we study the following initial-boundary value problem for semilinear edge hyperbolic equations

$$\begin{cases} u_{tt} - \Delta_{\mathbb{E}} u + V(z)u + \gamma u_t = f(z,u), & z \in int\mathbb{E}, t > 0, \\ u(z,0) = u_0(z), \qquad u_t(z,0) = u_1(z), & z \in int\mathbb{E} \\ u(z,t) = 0, & z \in \partial\mathbb{E}, t \geq 0, \end{cases} \tag{1.1}$$

where, γ is a nonnegative parameter and $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$, $N = 1 + n + q \geq 3$ is a dimension of $\mathbb{E}$ and coordinates $z := (r, x, y) = (r, x_1, ..., x_n, y_1, ..., y_q) \in \mathbb{E}$. Here the domain $\mathbb{E}$ is $[0, 1) \times X \times Y$, X is an $(n-1)$-dimensional closed compact manifold, $Y \subset \mathbb{R}^q$ is a bounded domain, which is regarded as the local model near the edge points on manifolds with edge singularities, and $\partial\mathbb{E} = \{0\} \times X \times Y$. Moreover, the operator $\Delta_{\mathbb{E}}$ in 1.1 is defined by $(r\partial_r)^2 + \partial_{x_1}^2 + ... + \partial_{x_n}^2 + (r\partial_{y_1})^2 + ... + (r\partial_{y_q})^2$, which is an elliptic operator with totally characteristic degeneracy on the boundary $r = 0$, we also call it Fuchsian type edge-Laplace operator, and the corresponding gradient operator by $\nabla_{\mathbb{E}} := (r\partial_r, \partial_{x_1}, ..., \partial_{x_n}, r\partial_{y_1}, ..., r\partial_{y_q})$.

Our study in hyperbolic system is in fact provoked by the study of [19] and we shall apply a potential method which was established by Sattinger [24]. So based on Martin and Schulze edge operators algebra [20], we study the existence and nonexistence global weak solutions for semilinear edge hyperbolic differential equations with respect to variable time with a positive potential function and a nonnegative weighted function. In [2], authors studied the problem 1.1 without damping term and the particular case of nonlinear term $f(x, u) = g(x)|u|^{p-1}u$. Furthermore,in the case of manifolds with conical singularities $\mathbb{B}$, the well-known operator $\Delta_{\mathbb{B}} + V(x)$ appears naturally in the nonlinear heat and wave equations, nonlinear and nonhomogeneous Schrödinger equations. For example, investigations have been done about the existence results, multiple solutions for nonhomogeneous degenerate Schrödinger equations in noncritical and critical cone Sobolev exponent on manifolds with conical singularities [3, 8, 15]. For finding such positive potential function any one can consider Poincaré's constant on manifold $\mathbb{B}$ [2]. Our problem can be seen then as a class of degenerate parabolic type equations in case that $V(z) = 0$ and $f(z, u) = f(u)$ then the problem 1.1 is reduced to problem 1.1 in [6] and in the classical sense our problem include the classical problem

$$\begin{cases} u_{tt} - \Delta u + \gamma u_t = f(u), & & x \in \Omega, t > 0, \\ u(x,0) = u_0(x), & \partial_t u(x,0) = u_1(x), & x \in \Omega \\ u(t,x) = 0, & & x \in \partial\Omega, t \geq 0, \end{cases} \tag{1.2}$$

where Ω is bounded domain of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and Δ is the standard Laplace operator and f is a suitable function [16, 18]. It is well-known that problem 1.2 has been studied by many authors, for example [18, 19]. Then, Runzhang [22] extended the results corresponding to the problem 1.2 in the [17] and [18] to the critical case and the authors in [23] studied the case with damping term and nonlinear term kind of problem 1.2. In this article, we shall find the existence and nonexistence theorems for the problem 1.1 in edge Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ which will be given in the next section. The similar results of type problem 1.1 studied on the manifold $\mathbb{B}$ with conical singularity in [1, 4]. We assume that V is a positive potential function which can be unbounded on the edge manifold $\mathbb{E}$ and is controlled by the following edge type Hardy's inequality [9]

$$\int_{\mathbb{E}} r^q V|u|^2 d\mu \leq C\|\nabla_{\mathbb{E}}\|^2_{L^{\frac{n+1}{2}}(\mathbb{E})} \qquad \forall u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}).$$

We suppose that $f : \mathbb{E} \times \mathbb{R}_+ \to \mathbb{R}$ is a Caratheodory function with the following assumptions:

A_1) $f(.\,, u) \in C^1(\mathbb{R})$ for all $u \in \mathbb{R}$ and also

$$u\Big(uf_2(., u) - f(., u)\Big) \geq 0 \qquad \forall u \in \mathbb{R},$$

where f_2 is the partial derivative with respect to second variable and the equality holds when $u = 0$.

A_2) There exists a positive constant c_0 such that

$$\mid f(., u) \mid \leq c_0 \mid u \mid^{p'},$$

where $1 < p' < \infty$ if $n = 1, 2$ and $2 < p' + 1 < 2^*$ if $n \geq 3$.

A_3) Let $z \in \mathbb{E}$, $2 < p + 1 \leq \theta \leq 2^*$ and for all $u \in \mathbb{R}$

A_{3-1}) $(p + 1)F(z, u) \leq uf(z, u)$ and

A_{3-2}) $\mid uf(z, u) \mid \leq \theta \mid F(z, u) \mid$ where, $F(., u) = \int_0^u f(., s)ds$.

The through of this paper we consider the following constants:

$$C_* = \inf\left\{\frac{\|\sqrt{V(z)}u(z)\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}} \quad ; \quad u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\right\},$$

$$C_{**} = \sup\left\{\frac{\|u(z)\|_{L_{q+1}^{\frac{n+1}{q+1}}(\mathbb{E})}}{\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}} \quad ; \quad u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})\right\}.$$

2. Edge Sobolev spaces

Consider X as a closed compact $C^\infty-$manifold of dimension n of the unit sphere in $\mathbb{R}^{n+1}$. We define an infinite cone in $\mathbb{R}^{n+1}$ as a quotient space $X^\Delta = \frac{\bar{\mathbb{R}}_+ \times X}{\{0\}\times X}$, with base X. The cylindrical coordinates $(r,\theta) \in X^\Delta - \{0\}$ in $\mathbb{R}^{n+1} - \{0\}$ are the standard coordinates. This gives us the description of $X^\Delta - \{0\}$ in the form $\mathbb{R}_+ \times X$. Then the stretched cone can be defined as $\bar{\mathbb{R}}_+ \times X = X^\wedge$. Now, consider $B = X^\Delta$ with a conical point, then by the similar way in [9, 10, 25], one can define the stretched manifold $\mathbb{B}$ with respect to B as a $C^\infty-$manifold with smooth boundary $\partial\mathbb{B} \cong X(0)$, where $X(0)$ is the cross section of singular point zero such that there is a diffeomorphism $B - \{0\} \cong \mathbb{B} - \partial\mathbb{B}$, the restriction of which to $U - \{0\} \cong V - \partial\mathbb{B}$ for an open neighborhood $U \subset B$ near the conic point zero and a collar neighborhood $V \subset \mathbb{B}$ with $V \cong [0,1) \times X(0)$. Therefore, we can take $\mathbb{B} = [0,1) \times X \subset \bar{\mathbb{R}}_+ \times X = X^\wedge$. In order to consider another type of a manifold with singularity of order one so-called wedge manifold, we consider a bounded domain Y in $\mathbb{R}^q$. Set $W = X^\Delta \times Y = B \times Y$. Then W is a corresponding wedge in $\mathbb{R}^{1+n+q}$. Therefore, the stretched wedge manifold $\mathbb{W}$ to W is $X^\wedge \times Y$ which is a manifold with smooth boundary $\{0\} \times X \times Y$. Set $(r,x) \in X^\wedge$. In order to define a finite wedge, it sufficient to consider the case $r \in [0,1)$. Thus, we define a finite wedge as

$$E = \frac{[0,1) \times X}{\{0\} \times X} \times Y \subset X^\Delta \times Y = W.$$

The stretched wedge manifold with respect to E is

$$\mathbb{E} = [0,1) \times X \times Y = \mathbb{B} \times Y \subset X^\wedge \times Y = W^\wedge,$$

with smooth boundary $\partial\mathbb{E} = \{0\} \times X \times Y$.

Definition 2.1 *For $(r,x,y) \in \mathbb{R}^N_+$ with $N = 1+n+q$, assume that $u(r,x,y) \in \mathcal{D}'(\mathbb{R}^N_+)$. We say that $u(r,x,y) \in L_p(\mathbb{R}^N_+; d\mu)$ if*

$$\|u\|_{L_p} = \left(\int_{\mathbb{R}^N_+} r^N |u(r,x,y)|^p d\mu\right)^{\frac{1}{p}} < +\infty,$$

where $d\mu = \frac{dr}{r} dx_1....dx_n \frac{dy_1}{r}...\frac{dy_q}{r}$ and for $1 \le p < \infty$.

Moreover, the weighted L_p spaces with wight $\gamma \in \mathbb{R}$ is denoted by $L_p^\gamma(\mathbb{R}^N_+; d\mu)$, which consists of function $u(r,,y)$ such that

$$\|u\|_{L_p^\gamma} = \left(\int_{\mathbb{R}^N_+} r^N |r^{-\gamma} u(r,x,y)|^p d\mu\right)^{\frac{1}{p}} < +\infty.$$

Now, we can define the weighted $p-$Sobolev spaces with natural scale for all $1 \le p < \infty$ on $\mathbb{R}_+^{N=1+n+q}$.

Definition 2.2 *For $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ and $N = 1+n+q$, the spaces*

$$\mathcal{H}_p^{m,\gamma}(\mathbb{R}^N_+) = \left\{ u \in \mathcal{D}'(\mathbb{R}^N_+) \quad | \quad r^{\frac{N}{p}-\gamma}(r\partial_r)^k \partial_x^\alpha (r\partial_y)^\beta u \in L_p(\mathbb{R}^N_+; d\mu) \right\}$$

for $k \in \mathbb{N}$, *multiindices* $\alpha \in \mathbb{N}^n$ *and* $\beta \in \mathbb{N}^q$ *with* $k + |\alpha| + |\beta| \leq m$. *In other words, if* $u(r,x,y) \in \mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ *then* $(r\partial_r)^k \partial_x^\alpha (r\partial_y)^\beta u \in L_p^\gamma(\mathbb{R}_+^N; d\mu)$. *Therefore,* $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ *is a Banach space with the following norm*

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)} = \sum_{k+|\alpha|+|\beta|\leq m} \left(\int_{\mathbb{R}_+^N} r^N |r^{-\gamma} (r\partial_r)^k \partial_x^\alpha (r\partial_y)^\beta u|^p d\mu \right)^{\frac{1}{p}}.$$

Moreover, the subspace $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{R}_+^N)$ *of* $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$ *denotes the closure of* $C_0^\infty(\mathbb{R}_+^N)$ *in* $\mathcal{H}_p^{m,\gamma}(\mathbb{R}_+^N)$. *Now, similarly to the definitions above, we can introduce the following weighted* $p-$ *Sobolev spaces on* $X^\wedge \times Y$, *where* $X^\wedge = \mathbb{R}_+ \times X$ *and* $X^\wedge \times Y$ *is an open stretched wedge.*

$$\mathcal{H}_p^{m,\gamma}(X^\wedge \times Y) := \left\{ u \in \mathcal{D}'(X^\wedge \times Y) \quad | \quad r^{\frac{N}{p}-\gamma} (r\partial_r)^k \partial_x^\alpha (r\partial_y)^\beta u \in L_p(X^\wedge \times Y; d\mu) \right\}$$

for $k \in \mathbb{N}$, *multiindices* $\alpha \in \mathbb{N}^n$ *and* $\beta \in \mathbb{N}^q$ *with* $k + |\alpha| + |\beta| \leq m$.

Then $\mathcal{H}_p^{m,\gamma}(X^\wedge \times Y)$ is a Banach space with the following norm

$$\|u\|_{\mathcal{H}_p^{m,\gamma}(X^\wedge \times Y)} = \sum_{k+|\alpha|+|\beta|\leq m} \left(\int_{X^\wedge \times Y} r^N |r^{-\gamma} (r\partial_r)^k \partial_x^\alpha (r\partial_y)^\beta u|^p d\mu \right)^{\frac{1}{p}}.$$

The subspace $\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge \times Y)$ of $\mathcal{H}_p^{m,\gamma}(X^\wedge \times Y)$ is defined as the closure of $C_0^\infty(X^\wedge \times Y)$.

Definition 2.3 *Let* $\mathbb{E}$ *be the stretched wedge to the finite wedge* E, *then* $\mathcal{H}_p^{m,\gamma}(\mathbb{E})$ *for* $m \in \mathbb{N}$, $\gamma \in \mathbb{R}$ *denotes the subset of all* $u \in W_{loc}^{m,p}(int\mathbb{E})$ *such that* $\omega u \in \mathcal{H}_p^{m,\gamma}(X^\wedge \times Y)$ *for any cut-off function* ω, *supported by a collar neighborhood of* $(0,1) \times \partial\mathbb{E}$. *Moreover, the subspace* $\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{E})$ *of* $\mathcal{H}_p^{m,\gamma}(\mathbb{E})$ *is defined as follows:*

$$\mathcal{H}_{p,0}^{m,\gamma}(\mathbb{E}) := [\omega]\mathcal{H}_{p,0}^{m,\gamma}(X^\wedge \times Y) + [1-\omega]W_0^{m,p}(int\mathbb{E})$$

where the classical Sobolev space $W_0^{m,p}(int\mathbb{E}$ *denotes the closure of* $C_0^\infty(int\mathbb{E})$ *in* $W^{m,p}(\tilde{\mathbb{E}})$ *for* $\tilde{\mathbb{E}}$ *that is a closed compact* C^∞ *manifold with boundary.*

If $u \in L_p^{\frac{n+1}{p}}(\mathbb{E})$ and $v \in L_{p'}^{\frac{n+1}{p'}}(\mathbb{E})$ with $p, p' \in (1,\infty)$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then one can obtain the following edge type Hölder inequality

$$\int_{\mathbb{E}} r^q |uv| d\mu \leq \left(\int_{\mathbb{E}} r^q |u|^p d\mu \right)^{\frac{1}{p}} \left(\int_{\mathbb{E}} r^q |v|^{p'} d\mu \right)^{\frac{1}{p'}}.$$

In the case $p = 2$, we have the corresponding edge type Schwartz inequality

$$\int_{\mathbb{E}} r^q |uv| d\mu \leq \left(\int_{\mathbb{E}} r^q |u|^2 d\mu \right)^{\frac{1}{2}} \left(r^q |v|^2 d\mu \right)^{\frac{1}{2}}.$$

In the sequel, for convenience we denote

$$(u,v)_2 = \int_{\mathbb{E}} r^q uv d\mu, \qquad \|u\|_{L_p^{\frac{n+1}{p}}(\mathbb{E})} = \left(\int_{\mathbb{E}} r^q |u|^p d\mu \right)^{\frac{1}{p}}.$$

-

3. Some auxiliary results

In this section we give some results about the potential wells for problem 1.1 and we obtain some properties of energy functional that we will use to prove the main results in Section 4.

Similar to the classical case, one can introduce the suitable functionals on the edge Sobolev space $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$:

$$J(u) = \frac{1}{2}\int\limits_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu + \frac{1}{2}\int\limits_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \int_{\mathbb{E}} r^q F(z,u)d\mu,$$

$$K(u) = \int\limits_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu + \int\limits_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \int_{\mathbb{E}} r^q u f(z,u)d\mu.$$

Then $J(u)$ and $K(u)$ are well-defined and belong to space $C^1(\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}),\mathbb{R})$. Now we define

$$\mathcal{N} = \left\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K(u) = 0, \quad \int\limits_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu \neq 0\right\},$$

$$d = \inf\left\{\sup_{\lambda\geq 0} J(\lambda u) \quad ; \quad u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), \quad \int\limits_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu \neq 0\right\}.$$

Thus, similar to the results in [4, 19] we obtain that $0 < d = \inf_{u\in\mathcal{N}} J(u)$. For $0 < \delta$ we define

$$K_\delta(u) = \delta\int\limits_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu + \int\limits_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \int\limits_{\mathbb{E}} r^q u f(z,u)d\mu,$$

$$\mathcal{N}_\delta = \left\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \quad ; \quad K_\delta(u) = 0, \quad \int\limits_{\mathbb{E}} r^q|\nabla_{\mathbb{E}}u|^2 d\mu \neq 0\right\},$$

$$d(\delta) = \inf_{u\in\mathcal{N}_\delta} J(u).$$

By preliminary results in [19] and as similar to [4], we provide some lemmas and propositions about the problem 1.1.

Lemma 3.1 *Suppose that the assumptions $(A_1)-(A_2)$ and (A_{3-1}) hold and consider $g(u) := \frac{f(z,u)}{u}$ for $u \neq 0$. Then*

- (i) $\lim_{u\to 0} g(u) = 0$;
- (ii) *g is increasing on $(0,\infty)$ and decreasing on $(-\infty,0)$;*
- (iii) *for any $z \in \mathbb{E}$, $uf(z,u) \geq 0$ for all $u \in \mathbb{R}$, where the equality holds only for $u = 0$;*
- (iv) *$f(z,u)$ is increasing with respect to second variable on $(-\infty,\infty)$;*
- (v) $0 \leq F(z,u) \leq \frac{c_0}{p'+1} \mid u \mid^{p'+1}$.

Lemma 3.2 *[24] Let $f(z,u)$ satisfy (A_1). Then $F(z,u) \geq c_1 \mid u \mid^{p+1}$ for $\mid u \mid\geq 1$ and positive constant c_1.*

Lemma 3.3 *Let* $f(z,u)$ *satisfy* (A_1), (A_{3-1}) *and define* $\varphi(\lambda) := \frac{1}{\lambda}\int_{\mathbb{E}} r^q u f(z,\lambda u)d\mu$. *Then*

(*i*) $\lim_{\lambda\to 0}\varphi(\lambda) = 0$, $\lim_{\lambda\to\infty}\varphi(\lambda) = \infty$;

(*ii*) $\varphi(\lambda)$ *is increasing on* $(0,\infty)$.

Proof (*i*) Since

$$\begin{aligned} \mid \varphi(\lambda) \mid &= \mid \frac{1}{\lambda}\int_{\mathbb{E}} r^q u f(z,\lambda u)d\mu \mid \le \frac{1}{\lambda}\int_{\mathbb{E}} r^q \mid u \mid\mid f(z,\lambda u) \mid d\mu \\ &\le \frac{1}{\lambda^2}\int_{\mathbb{E}} c_0 r^q \mid \lambda u \mid^{p'+1} d\mu = c_0\lambda^{p'-1}\|u\|^{p'+1}_{L_{p'+1}^{\frac{n}{p'+1}}(\mathbb{E})}. \end{aligned}$$

Hence, $\lim_{\lambda\to 0}\varphi(\lambda) = 0$.

By Lemma 3.2 and assumption (A_{3-1})

$$\begin{aligned} \varphi(\lambda) &= \frac{1}{\lambda^2}\int_{\mathbb{E}} \lambda r^q u f(z,\lambda u)d\mu \ge \frac{p+1}{\lambda^2}\int_{\mathbb{E}} r^q F(z,\lambda u)d\mu \\ &\ge \frac{(p+1)c_1}{\lambda^2}\int_{\mathbb{E}} r^q \mid \lambda u \mid^{p+1} d\mu = (p+1)c_1\lambda^{p-1}\int_{\mathbb{E}} r^q \mid u \mid^{p+1} d\mu \\ &= (p+1)c_1\lambda^{p-1}\|u\|^{p+1}_{L_{p+1}^{\frac{n}{p+1}}(\mathbb{E})}. \end{aligned}$$

Indeed, $\lim_{\lambda\to\infty}\varphi(\lambda) = \infty$.

(*ii*) $\varphi(\lambda)$ is increasing on $(0,\infty)$, since by assumption (A_1)

$$\begin{aligned} \varphi'(\lambda) &= \frac{1}{\lambda^2}\int_{\mathbb{E}} r^q(\lambda u^2 f_2(z,\lambda u) - \lambda u f(z,\lambda u))d\mu \\ &= \frac{1}{\lambda^3}\int_{\mathbb{E}} r^q \lambda u(\lambda u f_2(z,\lambda u) - f(z,\lambda u))d\mu > 0. \end{aligned}$$

□

Lemma 3.4 *Let* $f(z,u)$ *satisfy* (A_1), $u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$ *and* $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. *Then*

(*i*) $\lim_{\lambda\to 0} J(\lambda u) = 0$;

(*ii*) $\lim_{\lambda\to+\infty} J(\lambda u) = -\infty$;

(*iii*) *the functional* $J(\lambda u)$ *admits its maximal for* $0 < \lambda_* = \lambda(u) < \infty$;

(*iv*) $K(\lambda u) = \lambda\frac{\partial}{\partial\lambda}J(\lambda u)$;

(*v*) $K(\lambda u) > 0$ *for* $0 < \lambda < \lambda_*$, $K(\lambda u) < 0$ *for all* $\lambda_* < \lambda < \infty$ *and* $K(\lambda_* u) = 0$.

Proof Take $u \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$ and $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. Then

(i)

$$\begin{aligned}\lim_{\lambda\to 0} J(\lambda u) &= \frac{1}{2}\|\nabla_{\mathbb{E}}\lambda u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z) \mid \lambda u \mid^2 d\mu - \int_{\mathbb{E}} r^q F(z,\lambda u) d\mu \\ &= \lim_{\lambda\to 0}\frac{\lambda^2}{2}\left[\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu\right] - \int_{\mathbb{E}} r^q F(z,\lambda u) d\mu = 0.\end{aligned}$$

(ii)

$$\begin{aligned}\lim_{\lambda\to +\infty} J(\lambda u) &= \lim_{\lambda\infty}\frac{\lambda^2}{2}\left[\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu\right] - \int_{\mathbb{E}} r^q F(z,\lambda u) d\mu \\ &\geq \lim_{\lambda\to\infty}\left(\frac{\lambda^2}{2}(1+C_*^2)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{c_0\lambda^{p'+1}C_{**}^{p'+1}}{p'+1}\|\nabla_{\mathbb{E}} u\|^{p'+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\right) = -\infty.\end{aligned}$$

(iii) According to definition of functional J

$$J(\lambda u) = \frac{\lambda^2}{2}\left[\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu\right] - \int_{\mathbb{E}} r^q F(z,\lambda u) d\mu.$$

Thus

$$\frac{\partial J(\lambda u)}{\partial\lambda} = \lambda\left[\|\nabla_{\mathbb{E}}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu - \int_{\mathbb{E}} u r^q f(z,\lambda u) d\mu\right].$$

From $\frac{\partial J(\lambda u)}{\partial\lambda} = 0$ and by definition of $\varphi(\lambda)$ one can get

$$\varphi(\lambda) = \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu.$$

We take

$$\lambda_* := \varphi^{-1}\left(\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu\right),$$

and it follows λ_* is a maximal for $J(\lambda u)$ since for $\lambda = \lambda_*$, $\frac{\partial^2 J(\lambda u)}{\partial\lambda^2} = -\varphi'(\lambda) < 0$.

(iv) It follows from definition of the functionals J and K.

(v) It follows from definition of the functional K and λ_*. □

Proposition 3.5 *Let $f(x,u)$ satisfy assumption (A_2) and $0 < \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < l(\delta)$ where $l(\delta) = (\frac{\delta + C_*^2}{c_0 C_{**}^{q+1}})^{\frac{1}{q-1}}$. Then $K_\delta(u) > 0$. In particular, if*

$$0 < \|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < l(1)$$

then $K(u) > 0$.

Proof By definition of the functional K_δ

$$\begin{aligned} K_\delta(u) &= \delta\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu - \int_{\mathbb{E}} r^q u f(z,u) d\mu \\ &\geq (\delta + C_*^2)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \int_{\mathbb{E}} r^q u f(z,u) d\mu \\ &\geq (\delta + C_*^2)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \int_{\mathbb{E}} c_0 r^q \mid u \mid^{p'+1} d\mu \\ &\geq (\delta + C_*^2)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - c_0 C_{**}^{p'+1}\|\nabla_{\mathbb{E}}u\|^{p'+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &= \|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\left(\delta + C_*^2 - c_0 C_{**}^{p'+1}\|\nabla_{\mathbb{E}}u\|^{p'-1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\right). \end{aligned}$$

Therefore, $K_\delta(u) > 0$ from assumption. □

Proposition 3.6 *Let $f(z,u)$ satisfy assumption (A_2) and assume that $K_\delta(u) < 0$. Then $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > l(\delta)$. In particular, if $K(u) < 0$, then $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > l(1)$.*

Proof Since $K_\delta(u) < 0$ by assumption (A_2) we get that

$$\begin{aligned} \delta\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &< \int_{\mathbb{E}} r^q u f(z,u) d\mu - \int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu \\ &\leq \int_{\mathbb{E}} r^q \mid u \mid\mid f(z,u) \mid d\mu - C_*^2\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\leq c_0 \int_{\mathbb{E}} r^q \mid u \mid^{p'+1} d\mu - C_*^2\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\leq c_0 C_{**}^{p'+1}\|\nabla_{\mathbb{E}}u\|^{p'+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - C_*^2\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned}$$

Hence,

$$\delta\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + C_*^2\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < c_0 C_{**}^{p'+1}\|\nabla_{\mathbb{E}}u\|^{p'+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Indeed,

$$\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < \left(\frac{\delta + C_*^2}{c_0 C_{**}^{p'+1}}\right)^{\frac{1}{p'-1}} = l(\delta).$$

□

Corollary 3.7 *Let $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $K_\delta(u) = 0$ and $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. Then $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq l(\delta)$. In particular, if $K(u) = 0$ and $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq l(1)$.*

4. Invariance and vacuum isolating of the solutions

In this section, we express some properties of the solutions of the problem 1.1 such as an invariant set and vacuum isolating of solutions for the problem 1.1 under suitable conditions.

Proposition 4.1 *Let* $0 < \delta < \frac{p+1}{2}$ *and assumption* $(A_{3-1} - A_{3-2})$ *holds, then* $d(\delta) \geq a(\delta) l^2(\delta)$ *where* $a(\delta) = \left(\frac{(p+1) \ - \ 2\delta \ + \ C_*^2(p-1)}{2(p+1)}\right)$. *Moreover, we have*

$$d(\delta) = \inf_{u \in \mathcal{N}_\delta} J(u) = d\ \delta^2 a(\delta)[1 + c_*^2]^{-1}\ \frac{2(p+1)}{p-1}.$$

Proof Let $u \in \mathcal{N}_\delta$, so by proposition 3.6 we get that $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > l(\delta)$. Then by definition of J and K_δ we obtain that

$$\begin{aligned} J(u) &\geq \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q u f(z,u) d\mu \\ &= \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu \\ &- \frac{1}{p+1}\left(\delta\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - K_\delta(u) + \int_{\mathbb{E}} r^q V(z)|u(z)|^2 d\mu\right). \end{aligned}$$

Since $K_\delta(u) = 0$,

$$\begin{aligned} J(u) &\geq \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{p-1}{2(p+1)}\|V(z)^{\frac{1}{2}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\geq \left(\frac{1}{2} - \frac{\delta}{p+1}\right)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{p-1}{2(p+1)} C_*^2 \|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &= \left(\frac{1}{2} - \frac{\delta}{p+1} + \frac{(p-1)C_*^2}{2(p+1)}\right)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned}$$

Since $\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq l^2(\delta)$ then,

$$d(\delta) \geq a(\delta) l^2(\delta).$$

Now, we prove the second part of the assertion. Let $0 < \delta$ and $\bar{u} \in \mathcal{N}_\delta$ is minimizer of $d(\delta)$ that is $d(\delta) = J(\bar{u})$. we define $\lambda = \lambda(\delta)$ by

$$\|\nabla_{\mathbb{E}} \lambda \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|\lambda \bar{u}|^2 d\mu = \int_{\mathbb{E}} \lambda r^q \bar{u} f(z, \lambda \bar{u}) d\mu = \varphi(\lambda).$$

In fact, $\varphi : (0,\infty) \to (0,\infty)$ thus we can define $\lambda = \lambda(\alpha_0) = \varphi^{-1}(\alpha_0)$ where,

$$\alpha_0 := \|\nabla_{\mathbb{E}} \bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|\bar{u}|^2 d\mu.$$

Then for any $0<\delta$ up on definition of $\varphi(\lambda)$ there exists a unique λ which satisfies

$$\lambda=\varphi^{-1}\left(\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int\limits_{\mathbb{E}} r^qV(z)|u(z)|^2d\mu\right)=\varphi^{-1}(\varphi(\frac{1}{\delta}))=\frac{1}{\delta}.$$

Hence, for such λ, $\lambda\bar{u}\in\mathcal{N}$, so by definition of d we get that

$$\begin{aligned}
d &\leq J(\lambda\bar{u})=\frac{1}{2}\|\nabla_{\mathbb{E}}\lambda\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{2}\int\limits_{\mathbb{E}} r^qV(z)|\lambda\bar{u}|^2d\mu-\int\limits_{\mathbb{E}} r^qF(z,\lambda\bar{u})d\mu\\
&\leq \frac{\lambda 2}{2}\left[\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int_{\mathbb{E}} r^qV(z)\mid\bar{u}\mid^2 d\mu\right]-\frac{1}{\theta}\int_{\mathbb{E}}\lambda r^q\bar{u}f(z,\lambda\bar{u})d\mu\\
&= \frac{\lambda^2}{2}\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{\lambda^2}{2}\int\limits_{\mathbb{E}} r^qV(z)|\bar{u}|^2d\mu\\
&- \frac{1}{\theta}\left[\|\nabla_{\mathbb{E}}\lambda\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int\limits_{\mathbb{E}} r^qV(z)|\lambda\bar{u}|^2d\mu-K(\lambda\bar{u})\right]\\
&= \lambda^2\left[\frac{1}{2}\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{2}\int\limits_{\mathbb{E}} r^qV(z)|\bar{u}|^2d\mu-\frac{1}{\theta}\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}-\frac{1}{\theta}\int\limits_{\mathbb{E}} r^qV(z)|\bar{u}|^2d\mu\right]\\
&\leq \lambda^2(\frac{\theta-2}{2\theta})\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}-\lambda^2(\frac{2-\theta}{2\theta})C_*^2\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\\
&\leq \lambda^2(\frac{\theta-2}{2\theta})(1+C_*^2)\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

Therefore, by definition of $\lambda=\lambda(\delta)$ we have

$$d\leq\delta^2(\frac{\theta-2}{2\theta})(1+C_*^2)\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Moreover, $d(\delta)\geq a(\delta)\|\nabla_{\mathbb{E}}\bar{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$. Indeed,

$$d\leq J(\lambda\bar{u})\leq\delta^{-2}\left[\frac{\theta-2}{2\theta}\ (1+C_*^2)\right]\frac{d(\delta)}{a(\delta)}.$$

Hence,

$$d(\delta)\geq a(\delta)\ \delta^2\ [1+C_*^2]^{-1}\ (\frac{2\theta}{\theta-2})d.$$

Now, we let $0<\delta$ and $\tilde{u}\in\mathcal{N}$ is minimizer of d that is

$$d=J(\tilde{u})=\frac{1}{2}\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{2}\int\limits_{\mathbb{E}} r^qV(z)|\tilde{u}|^2d\mu-\int\limits_{\mathbb{E}} r^qF(z,\tilde{u})d\mu.$$

we define $\lambda=\lambda(\delta)$ by

$$\delta\|\nabla_{\mathbb{E}}\lambda\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int\limits_{\mathbb{E}} r^qV(z)|\lambda\tilde{u}|^2d\mu=\int\limits_{\mathbb{E}}\lambda r^q\tilde{u}f(z,\lambda\tilde{u})d\mu=\varphi(\lambda).$$

Then, using the mapping $\varphi:(0,\infty)\to(0,\infty)$ and for any $0<\delta$, there exists a unique λ which satisfies

$$\lambda=\lambda(\delta)=\varphi^{-1}\Big(\delta\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int_{\mathbb{E}}r^qV(z)\mid\tilde{u}\mid^2 d\mu\Big)=\varphi^{-1}(\varphi(\delta))=\delta.$$

Hence, for such λ, $\lambda\tilde{u}\in\mathcal{N}_\delta$ by definition of $d(\delta)$ we get that

$$\begin{aligned}
d &\geq \frac{1}{2}\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{2}\int_{\mathbb{E}}r^qV(z)|\tilde{u}|^2d\mu\\
&- \frac{1}{\theta}\Big(\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int_{\mathbb{E}}r^qV(z)|\tilde{u}|^2d\mu-K(\tilde{u})\Big)\\
&= (\frac{1}{2}-\frac{1}{\theta})\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\ (\frac{1}{2}-\frac{1}{\theta})\int_{\mathbb{E}}r^qV(z)|\tilde{u}|^2d\mu\\
&\geq \frac{\theta-2}{2\theta}\ [1+C_*^2]\ \|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
d(\delta) &\leq J(\lambda\tilde{u})=\frac{1}{2}\|\nabla_{\mathbb{E}}\lambda\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{1}{2}\int_{\mathbb{E}}r^qV(z)|\lambda\tilde{u}|^2d\mu-\int_{\mathbb{E}}r^qF(z,\lambda\tilde{u})d\mu\\
&\leq \frac{\lambda^2}{2}\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\frac{\lambda^2}{2}\int_{\mathbb{E}}r^qV(z)|\tilde{u}|^2d\mu\\
&- \frac{1}{\theta}\left(\delta\|\nabla_{\mathbb{E}}\lambda\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+\int_{\mathbb{E}}r^qV(z)|\lambda\tilde{u}|^2d\mu-K_\delta(\lambda\tilde{u})\ \right)\\
&= \lambda^2\left[(\frac{1}{2}-\frac{\delta}{\theta})\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}+(\frac{1}{2}-\frac{1}{\theta})\int_{\mathbb{E}}r^qV(z)|\tilde{u}|^2d\mu\ \right]\\
&= \lambda^2\left[(\frac{1}{2}-\frac{\delta}{\theta})\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}-\ \frac{(2-\theta)C_*^2}{2\theta}\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\ \right]\\
&\leq \lambda^2\left[\frac{1}{2}-\frac{\delta}{\theta}+\ \frac{(\theta-2)C_*^2}{2\theta}\ \right]\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

Assumption of (A_{3-2}) implies that

$$\frac{2(p+1)}{p-1}<\frac{2\theta}{\theta-2}$$

thus

$$d(\delta)\leq\lambda^2a(\delta)\|\nabla_{\mathbb{E}}\tilde{u}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.$$

Then, from definition of $\lambda(\delta)$ and above conclusions we get that

$$\begin{aligned}\delta^2(1+C_*^2)^{-1}\frac{2(p+1)}{p-1}a(\delta)d &< \delta^2(1+C_*^2)^{-1}\frac{2\theta}{\theta-2}a(\delta)d \\ &\leq d(\delta) \leq \delta^2(1+C_*^2)^{-1}\frac{2\theta}{\theta-2}a(\delta)d.\end{aligned}$$

Therefore,

$$d(\delta) = \inf J(u)_{u\in\mathcal{N}_\delta} = \delta^2\ a(\delta)\ d\ [1+C_*^2]^{-1}\ \frac{2\theta}{\theta-2}.$$

□

Remark 4.2 *According to $d(\delta)$ in proposition 4.1 we obtain that*
1) $\lim_{\delta\to 0} d(\delta) = 0,$
2) *we set* $C_1 := d[1+C_*^2]^{-1}\frac{2\theta}{\theta-2}$ *and* $C_2 := \frac{1}{2} + \frac{(p-1)C_*^2}{2(p+1)}$ *then*

$$d(\delta) = d[1+C_*^2]^{-1}\frac{2\theta}{\theta-2}\delta^2\left(\frac{1}{2} - \frac{\delta}{p+1} + \frac{C_*^2(p-1)}{p+1}\right) =$$

$$C_1C_2\delta^2 - \frac{C_1}{p+1}\delta^3 = C'\delta^2 - \frac{C_1}{p+1}\delta^3.$$

Then,

$$d'(\delta) = 2C'\delta - 3\frac{C_1}{p+1}\delta^2 = \delta\ [2C' - \frac{3C_1\delta}{p+1}] \Rightarrow$$

$d'(\delta) = 0 \Rightarrow \delta = \frac{2}{3}C_2(p+1)$. *Hence, if* $0 < \delta < \frac{2}{3}C_2(p+1)$ *then* $d(\delta)$ *is strictly increasing function and if* $\delta > \frac{2}{3}C_2(p+1)$ *then* $d(\delta)$ *is strictly decreasing function.*

Now, we introduce the following potential wells

$$W = \left\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \qquad ; \qquad K(u) > 0, \qquad J(u) < d\right\} \cup \{0\},$$

$$W_\delta = \left\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \qquad ; \qquad K_\delta(u) > 0, \qquad J(u) < d(\delta)\right\} \cup \{0\},$$

for $0 < \delta$, and corresponding potentials outside of the set that defined as above

$$E = \left\{u \in u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \qquad ; \qquad K(u) < 0, \qquad J(u) < d\right\},$$

$$E_\delta = \left\{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}) \qquad ; \qquad K_\delta(u) < 0, \qquad J(u) < d(\delta)\right\}$$

for any $0 < \delta$.

Definition 4.3 $u = u(z,t) \in L^\infty(0,T;\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ *with* $u_t \in L^\infty(0,T;L_2^{\frac{n+1}{2}}(\mathbb{E}))$ *is called a weak solution of the problem 1.1 on* $int\mathbb{E} \times [0,T)$ *if*

$$(u_t, v)_2 + \int_0^t (\nabla_{\mathbb{E}} u, \nabla_{\mathbb{E}} v)_2 d\tau + \int_0^t (V(z)u, v)_2 d\tau + \int_0^t (\gamma u_t, v)_2 d\tau = \int_0^t (f(z,u), v)_2 d\tau + (u_1, v)_2$$

for all $v \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ *and* $t \in (o,T)$. $u(z,0) = u_0$ *in* $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ *and hold the following energy inequality*

$$I(t) + \gamma \int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \leq I(0), \qquad \forall t \in [0,T),$$

where $0 \leq T \leq \infty$ *and*

$$I(t) = \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + J(u).$$

We note, since $u \in L^\infty(0,T;\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ and $u_t \in L^\infty(0,T;L_2^{\frac{n}{2}}(\mathbb{E}))$ from the first equation of the problem 1.1 as similar in [14], one can obtain that $u_{tt} \in L^\infty(0,T;\mathcal{H}_{2,0}^{-1,\frac{n+1}{2}}(\mathbb{E}))$.

Now we discuss the invariance of some sets corresponding to the problem 1.1.

Proposition 4.4 *Let* $0 < J(u) < d$ *for* $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. *Suppose that* $\delta_1 < \frac{2}{3}C_2(p+1) < \delta_2$ *be roots of equation* $d(\delta) = J(u)$. *Then* $K_\delta(u)$ *has no change in its sign for* $\delta \in (\delta_1, \delta_2)$.

Proof We assume that there exists a $\delta_0 \in (\delta_1, \delta_2)$ for which $K_{\delta_0}(u) = 0$. Hence, by definition of $d(\delta)$ we have $J(u) \geq d(\delta)$. But, we have two cases the following for δ_0

$$\begin{cases} \delta_1 < \delta_0 < \frac{2}{3}C_2(p+1) < \delta_2 \\ \delta_1 < \frac{2}{3}C_2(p+1) < \delta_0 < \delta_2 \end{cases}$$

Now, by Remark 4.2 We get that $d(\delta_1) < d(\delta_0)$ or $d(\delta_2) < d(\delta_0)$ then we obtain that $d(\delta_1) = d(\delta_2) = J(u) < d(\delta_0)$ that this is contradiction . □

Theorem 4.5 *Let* $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $0 < e < d$. *Suppose that* $\delta_1 < \delta_2$ *are roots of equations* $d(\delta) = e$ *then*

i) all solutions of problem 1.1 with $0 < J(u_0) \leq e$ *belong to set* W_δ *for* $\delta_1 < \delta < \delta_2$ *provided* $K(u_0) > 0$ *or* $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$.

ii) all solutions of problem 1.1 with $0 < J(u_0) \leq e$ *belong to* E_δ *for* $\delta \in (\delta_1, \delta_2)$ *provided* $K(u_0) < 0$.

Proof *i)* Let $u(t)$ be a solution of the problem 1.1 with initial value u_0 for which satisfies in conditions $0 < J(u_0) \leq e < d$, $K(u_0) > 0$ or $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$. Let T be existence time for solution $u(t)$. If $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$, then since u_0 has compact support $u_0 = 0$, so by definition of W_δ we obtain that $u_0 \in W_\delta$. If $K(u_0) > 0$ then by assumption we have

$$0 < J(u_0) \leq e = d(\delta_1) = d(\delta_2) < d(\delta) \leq d$$

for $\delta_1 < \delta < \delta_2$. Hence, $K_\delta(u_0(t)) > 0$ for $\delta_1 < \delta < \delta_2$, by Proposition 4.4. Therefore, by definition of W_δ, $u_0 \in W_\delta$ for $\delta_1 < \delta < \delta_2$. Now, we have to show that for $\delta_1 < \delta < \delta_2$ and $0 < t < T$, $u(t) \in W_\delta$. Suppose that, there exist $t_0 \in (0,T)$ such that for $\delta_1 < \delta < \delta_2$, $u(t_0) \in \partial W_\delta$. Then we can imply that , $K_\delta(u(t_0)) = 0$ and $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, or by definition of W_δ, $J(u(t_0)) = d(\delta)$. Since $u(t_0)$ is a solution of problem 1.1, so it satisfies in energy inequality i.e.

$$\begin{aligned} \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \quad &+ \quad J(u(t)) \ + \gamma \int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\leq \quad I(0) = J(u_0) \leq e < d(\delta), \end{aligned}$$

for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0,T)$. Therefore, the equality $J(u(t_0)) = d(\delta)$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0,T)$ is not possible. If $K_\delta(u(t_0)) = 0$ and $\|\nabla_{\mathbb{E}} u_0\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then by definition of $d(\delta)$ we get that $d(\delta) \leq J(u_0(t))$, that is in contradiction with energy inequality. Therefore, $u(t) \in W_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0,T)$.

ii) Similar to first case it can be prove that $u_0 \in E_\delta$ for $\delta \in (\delta_1, \delta_2)$ provided $K_\delta(u_0) < 0$. Now, we should prove $u(t) \in E_\delta$ for any $\delta \in (\delta_1, \delta_2)$ and $t \in (0,T)$. Suppose that there exist $t_0 \in (0,T)$, such that for $t \in [0\ ,\ t_0)$, $u(t) \in E_\delta$ and $u(t_0) \in \partial E_\delta$, that is, $K_\delta(u_0) = 0$ or $J(u(t_0)) = d(\delta)$ for $\delta \in (\delta_1, \delta_2)$. According to energy inequality the equality $J(u(t_0)) = d(\delta)$ is not possible similar to first case. Hence, we assume that $K_\delta(u(t_0)) = 0$, then $K_\delta(u(t)) < 0$ for $t \in (0, t_0)$, since for $t \in [0, t_0)$, $u(t) \in E_\delta$, then by definition of E_δ, $K_\delta(u(t)) < 0$. Now, using the Proposition 3.6 we obtain that $\|\nabla_{\mathbb{E}} u(t)\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > l(\delta)$ and $\|\nabla_{\mathbb{E}} u(t_0)\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > l(\delta) \neq 0$. Hence by definition of $d(\delta)$, $J(u(t_0)) \geq d(\delta)$ which is in contradiction with energy inequality. □

Remark 4.6 *Suppose that all assumptions in Theorem 4.5 hold. Then for any $\delta \in (\delta_1, \delta_2)$ both seta W_δ and E_δ are invariant. Moreover, both sets*

$$W_{\delta_1\delta_2} = \bigsqcup_{\delta_1<\delta<\delta_2} W_\delta, \qquad\qquad E_{\delta_1\delta_2} = \bigsqcup_{\delta_1<\delta<\delta} E_\delta$$

are invariant respectively under flow of the problem 1.1. Hence, we can get for all weak solutions of the problem 1.1

$$u(t) \notin \mathcal{N}_{\delta_1\delta_2} = \bigsqcup_{\delta_1<\delta<\delta_2} \mathcal{N}_\delta.$$

To discuss about the invariant of the solutions with negative level energy, we introduce the following results.

Proposition 4.7 *Let $u_0 \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$ and $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. Suppose that $I(0) = 0$ and $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. Then all weak solutions of the problem 1.1 satisfy*

$$\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq M = \left(\frac{(p' + 1)(1 + C_*^2)}{2C_{**}^{p'+1} c_0} \right)^{\frac{1}{p'-1}}.$$

Proof Let us consider $u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ as a weak solution of the problem 1.1. According to the Definition 4.3

$$I(t) + \gamma \int_0^t \|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \leq I(0) = 0.$$

Therefore, by definition of constants C_* and C_{**}

$$\begin{aligned}
\frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &+ \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{c_0 C_{**}^{p'+1}}{p'+1}\|\nabla_{\mathbb{E}} u\|^{q+1}_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\leq \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{c_0}{p'+1}\int_{\mathbb{E}} r^q \mid u \mid^{q+1} d\mu \\
&\leq \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \int_{\mathbb{E}} r^q F(z,u) d\mu \\
&\leq \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{C_*^2}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \int_{\mathbb{E}} r^q F(z,u) d\mu \\
&\leq I(t) + \gamma \int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

Hence,

$$\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq \left(\frac{(p'+1)(1+C_*^2)}{2C_{**}^{p'+1}c_0}\right)^{\frac{1}{p'-1}} = M.$$

□

Theorem 4.8 *Let $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$ and assumption (A_{3-1}) holds. Suppose that either $I(0) < 0$ or $I(0) = 0$ and $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$. Then all weak solutions of the problem 1.1 belong to E_δ for any $\delta \in (0\ ,\ \ \frac{p+1}{2}(1+\frac{p-1}{p+1}C_*^2)$.*

Proof Let $u(t)$ be an arbitrary weak solution of the problem 1.1 with expressed assumptions in face of the Theorem and T be the existence time of $u(t)$. From Definition 4.3, for every

$$\delta \in \left(0\ ,\ \ \frac{p+1}{2}(1+\frac{p-1}{p+1}C_*^2)\right)$$

and $t \in [0,T)$, we can obtain

$$
\begin{aligned}
\frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &+ a(\delta)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}K_\delta(u) \\
&= \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \left(\frac{1}{2} - \frac{\delta}{p+1} + \frac{p-1}{2(p+1)}C_*^2\right)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1}\left(\delta\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \int_{\mathbb{E}} r^q u f(z,u) d\mu\right) \\
&= \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{p-1}{2(p+1)}C_*^2\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1}\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q u f(z,u) d\mu \\
&\leq \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \left[(\frac{1}{2} - \frac{1}{p+1}) + \frac{1}{p+1}\right]\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\
&- \frac{1}{p+1}\int_{\mathbb{E}} r^q u f(x,u) d\mu \ \leq \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ J(u) \ + \ \gamma\int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \leq I(0). \qquad (4.1)
\end{aligned}
$$

If $I(0) < 0$, then 4.1 implies that $K_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for every

$$\delta \in \left(0\ ,\ \ \frac{p+1}{2}(1 + \frac{p-1}{p+1}C_*^2)\right)$$

and $t \in [0,T)$. If $I(0) = 0$ and $\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \neq 0$, then Proposition 4.7 gives

$$\|\nabla_{\mathbb{E}}u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \geq M$$

for $t \in [0,T)$. Again by the relation 4.1 we get $K_\delta(u) < 0$ and $J(u) < 0 < d(\delta)$ for

$$\delta \in \left(0\ ,\ \ \frac{p+1}{2}(1 + \frac{p-1}{p+1}C_*^2)\right)$$

and $t \in [0,T)$. Therefore, for two cases discussed above, for every $\delta \in (0\ ,\ \ \frac{p+1}{2}(1 + \frac{p-1}{p+1}C_*^2))$ and $t \in [0,T)$, we have $u \in E_\delta$. □

5. Existence and nonexistence results

In this section we prove the global existence and nonexistence of solutions and give a sharp condition for global existence of solutions for problem 1.1 with $I(0) < d$.

Theorem 5.1 *Let $\gamma \geq 0$, $u_0 \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$ and $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. Suppose that $I(0) < d$, $K(u_0) > 0$ or $\|\nabla_{\mathbb{E}}u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$. Then under assumptions $(A_1 - A_2 - A_3)$, problem 1.1 admits a global weak solution $u(t) \in L^\infty(0,\infty;\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}))$ with $u_t \in L_2^{\frac{n+1}{2}}(\mathbb{E})$ and $u(t) \in W$ for $t \in [0,\infty)$.*

Proof By Proposition 3 in [7], we can choose $\{\varphi_j(x)\}$ as orthonormal basis of space $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$. Then we construct approximation solution $u_m(z,t)$ similar to [19] as following:

$$u_m(z,t) = \sum_{j=1}^{m} h_{jm}(t)\varphi_j(z),$$

for $m = 1, 2, ...$ that satisfies in problem 1.1 i.e.

$$\begin{aligned}(u_{ttm}, \varphi_k)_2 + (\nabla_{\mathbb{E}} u_m, \nabla_{\mathbb{E}}\varphi_k)_2 \quad &+ \quad (V(z)u_m, \varphi_k)_2 + \gamma\ (u_{tm}, \varphi_k)_2 \\ &= \quad (f(z, u_m), \varphi_k)_2, \end{aligned} \tag{5.1}$$

$$u_m(z,0) = \sum_{j=1}^{m} h_{jm}(0)\varphi_j(z) \to u_0(z), \tag{5.2}$$

in $\mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$ and

$$u_{tm}(z,0) = \sum_{j=1}^{m} h_{jm}(0)\varphi_j(z) \to u_1(z), \tag{5.3}$$

in $L_2^{\frac{n+1}{2}}(\mathbb{E})$. Multiplying 5.1, 5.2 and 5.3 by $h'_{km}(t)$ and forming the sum on $k = 1, 2, ...,$

$$\begin{aligned}\sum_{k=1}^{m}(u_{ttm}, \varphi_k)_2 h'_{km}(t) \quad &+ \quad (\nabla_{\mathbb{E}} u_m, \nabla_{\mathbb{E}}\varphi_k)_2 h'_{km}(t) + (V(z)u_m, \varphi_k)_2 h'_{km}(t) \\ &+ \quad \sum_{k=1}^{m}\gamma\ (u_{tm}, \varphi_k)_2 h'_{km}(t) = \sum_{k=1}^{m}(f(z, u_m), \varphi_k)_2 h'_{km}(t),\end{aligned}$$

for $m = 1, 2, 3, ...$. Therefore,

$$\begin{aligned}\int_{\mathbb{E}} r^q u_{ttm} u_{tm} d\mu \quad &+ \quad \int_{\mathbb{E}} r^q \nabla_{\mathbb{E}} u_m \nabla_{\mathbb{E}} u_{tm} d\mu + \int_{\mathbb{E}} r^a V(z) u_m u_{tm} d\mu \\ &+ \quad \gamma \int_{\mathbb{E}} r^q u_{tm} u_{tm} d\mu = \int_{\mathbb{E}} r^q f(z, u_m) u_{tm} d\mu.\end{aligned} \tag{5.4}$$

Using the Leibniz rule one can get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} r^q |u_{tm}|^2 d\mu \quad &+ \quad \frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} r^q |\nabla_{\mathbb{E}} u_m|^2 d\mu + \frac{1}{2}\frac{d}{dt}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu \\ &+ \quad \gamma \int_{\mathbb{E}} r^q |u_{tm}|^2 d\mu = \frac{d}{dt}\int_{\mathbb{E}} r^q F(z, u_m) d\mu.\end{aligned} \tag{5.5}$$

By integration of the relation 5.5 with respect to t

$$\begin{aligned}
\frac{1}{2}\|u_{tm}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &+ \frac{1}{2}\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu + \gamma\int_0^t \|u_{\tau m}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \\
&- \int_{\mathbb{E}} r^q F(z,u_m) d\mu \\
&= I_m(t) + \gamma\int_0^t \|u_{\tau m}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \le I_m(0) < d, \qquad (5.6)
\end{aligned}$$

where the last equal is upon definition 4.3. Using 5.6 and definition of functional K,

$$\begin{aligned}
J(u_m) &= \frac{1}{2}\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu - \int_{\mathbb{E}} r^q F(z,u_m) d\mu \\
&\ge \frac{1}{2}\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu \\
&- \frac{1}{p+1}\left(\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu - K(u_m)\right) \\
&= \frac{p-1}{2(p+1)}\left[\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u_m|^2 d\mu\right] \\
&\ge \frac{p-1}{2(p+1)}(1+C_*^2)\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}.
\end{aligned}$$

Then

$$\begin{aligned}
\int_0^t \frac{1}{2}\|u_{tm}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau &+ \frac{p-1}{2(p+1)}(1+C_*^2)\|\nabla_{\mathbb{E}}u_m\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&\le I_m(t) + \gamma\int_0^t \|u_{\tau m}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \le I_m(0) < d. \qquad (5.7)
\end{aligned}$$

for $t \in [0,\infty)$ and sufficiently large m. Therefore, for any $t \in [0,\infty)$,

$$\frac{d}{dt}I_m(t) + \gamma\|u_{tm}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0 \qquad (5.8)$$

and $I_m(t) + \gamma\int_0^t \|u_{\tau m}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau = I_m(0) < 0$ where, $I_m(t) = \frac{1}{2}\|u_{tm}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + J(u_m)$. Hence, for sufficiently large m and $0 \le t < \infty$ we obtain that $u_m \in W$ by Proposition 4.5. From 5.8 and by the same argument in [19] it implies that there exists a u and subsequence $\{u_i\}$ of $\{u_m\}$ such that $u_i \to u$ in $L^\infty(0,T;\mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E}))$ weakly star and a.e. in $\mathbb{E}\times[0,\infty)$. Moreover, $u_i \to u$ in $L_{p'+1}^{\frac{n}{p'+1}}(\mathbb{E})$ strongly and for each $t > 0$ and $u_{ti} \to u_t$ in $L^\infty(0,\infty;L_2^{\frac{n+1}{2}}(\mathbb{E}))$ weakly star. Also it satisfies on all conditions of Definition 4.3. Now, we prove that

$$\lim_{i\to\infty}\int_{\mathbb{E}} r^q F(z,u_i) d\mu = \int_{\mathbb{E}} r^q F(z,u) d\mu.$$

To this end, we have the following estimations

$$\begin{aligned}\left|\int_{\mathbb{E}} r^q F(z,u_i)d\mu \right. &- \left. \int_{\mathbb{E}} r^q F(z,u)d\mu\right| \\ &\leq \int_{\mathbb{E}} r^q \mid F(z,u_i)-F(z,u) \mid d\mu \\ &\leq \int_{\mathbb{E}} r^q \mid f(z,u+\mu_i(u_i-u)) \mid \mid u_i-u \mid d\mu \\ &\leq \|f(z,u+\mu_i(u_i-u))\|^r_{L_s^{\frac{n+1}{s}}(\mathbb{E})} \|u_i-u\|^{p'+1}_{L_{p'+1}^{\frac{n}{p'+1}}(\mathbb{E})} \qquad (5.9)\end{aligned}$$

where, $0<\mu_i<1,\ s=\frac{p'+1}{p'}$. On the other hand,

$$\begin{aligned}\|f(z,u+\mu_i(u_i-u))\|^s_{L_s^{\frac{n+1}{s}}(\mathbb{E})} &= \int_{\mathbb{E}} r^q \mid f(z,u+\mu_i(u_i-u)) \mid^s d\mu \\ &\leq c_0^s \int_{\mathbb{E}} r^q \mid u+\mu_i(u_i-u) \mid^{p' s} d\mu \\ &= c_0^s \|u+\mu_i(u_i-u)\|^{p'+1}_{L_{p'+1}^{\frac{n+1}{p'+1}}(\mathbb{E})} < \infty. \qquad (5.10)\end{aligned}$$

Then one get that

$$\lim_{i\to\infty} \int_{\mathbb{E}} r^q F(z,u_i)d\mu = \int_{\mathbb{E}} r^q F(z,u)d\mu.$$

Therefore, from 5.8

$$\begin{aligned}\frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &+ \frac{1}{2}\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z) \mid u \mid^2 d\mu + \gamma \int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \\ &\leq \liminf_{i\to\infty} \frac{1}{2}\|u_{ti}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \liminf_{i\to\infty} \frac{1}{2}\|\nabla_{\mathbb{E}} u_i\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \liminf_{i\to\infty} \frac{1}{2}\int_{\mathbb{E}} r^q V(z) \mid u_i \mid^2 d\mu \\ &+ \liminf_{i\to\infty} \gamma \int_0^t \|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \\ &\leq \liminf_{i\to\infty} \left(\frac{1}{2}\|u_{ti}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}} u_i\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\int_{\mathbb{E}} r^q V(z) \mid u_i \mid^2 d\mu \right. \\ &+ \left. \gamma \int_0^t \|u_{\tau i}\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \right) = \liminf_{i\to\infty} \left(I_i(0) + \int_{\mathbb{E}} r^q F(z,u_i)d\mu \right) \\ &= \lim_{i\to\infty} \left(I_i(0) + \int_{\mathbb{E}} r^q F(z,u_i)d\mu \right) = I(0) + \int_{\mathbb{E}} r^q F(z,u)d\mu. \qquad (5.11)\end{aligned}$$

This implies the energy inequality in Definition 4.3. Finally, by Theorem 4.5, $u \in W$ for $0 \leq t < \infty$. □

Corollary 5.2 *If we replace the assumption* $I(0) < d,\ K(u_0) > 0$ *by* $0 < I(0) < d,\ K_{\delta_2}(u_0) > 0$ *where* (δ_1, δ_2) *is the maximal interval including* $\delta = \frac{2}{3}C_2(p+1)$*, where* C_2 *introduced in Remark 4.2, such that* $I(0) < d(\delta)$ *for* $\delta \in (\delta_1, \delta_2)$*. Then problem 1.1 admits a global weak solution* $u(t) \in L^\infty(0,\infty; \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ *with* $u_t \in L^\infty(0,\infty; L_2^{\frac{n+1}{2}}(\mathbb{E}))$ *and* $u(t) \in W_\delta$ *for* $\delta \in (\delta_1, \delta_2),\ t \in [0,\infty)$*.*

Proof It is immediately implied form Theorems 4.5 and 5.1. □

Corollary 5.3 *If we replace the assumption* $K_{\delta_2}(u_0) > 0$ *or* $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$*, by* $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < l(\delta_2)$*, then problem 1.1 admits a global weak solution* $u(t) \in L^\infty(0,\infty; \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ *with* $u_t(t) \in L^\infty(0,\infty; L_2^{\frac{n+1}{2}}(\mathbb{E}))$ *satisfying*

$$\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \le \frac{I(0)}{a(\delta_1)}, \quad \|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \le 2I(0), \quad 0 \le t \le \infty \tag{5.12}$$

Proof From assumption $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} < l(\delta_2)$, we can get that $K_{\delta_2}(u_0) > 0$ or $\|\nabla_{\mathbb{E}} u_0\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} = 0$. Then it follows from Corollary 5.2 that problem 1.1 admits a global weak solution such that for any $\delta_1 < \delta < \delta_2$, $0 \le t < \infty$, $u(t) \in L^\infty(0,\infty; \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}))$ with $u_t \in L^\infty(0,\infty; L_2^{\frac{n+1}{2}}(\mathbb{E}))$ and $u(t) \in W_\delta$. Moreover, similar of the proof Theorem 4.8 for every $\delta_1 < \delta < \delta_2,\ 0 \le t < \infty$,

$$\frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + a(\delta)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}K_\delta(u) \le I(0).$$

If we tend δ to δ_1 then we achieve 5.12. □

Now we discuss the global nonexistence of solutions of the problem 1.1.

Theorem 5.4 *Let* $0 \le \gamma \le (p-1)(1+C_*^2)\lambda_1$, $u_0 \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E})$, $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$*. Suppose that* $I(0) < d$ *and* $K(u_0) < 0$*. Then the existence time of solution for problem 1.1 is finite, where* λ_1 *is the first eigenvalue in Proposition* **??** *i.e.*

$$\lambda_1 = \inf_{u \in \mathcal{H}_{2,0}^{1,\frac{n+1}{2}}(\mathbb{E}), u \neq 0} \frac{\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}{\|u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})}}.$$

Proof Let $u(t)$ be any weak solution of problem 1.1 with $I(0) < d$ and $K(u_0) < 0$, T be the maximal existence time of $u(t)$. We will prove $T < \infty$ by contradiction. Let $M(t) := \|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}$, then

$$\dot{M}(t) = \frac{d}{dt}\int_E r^q |u(z,t)|^2 d\mu = 2(u_t, u)_2,$$

from definition of functional K,

$$\ddot{M}(t) = 2\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + 2(u_{tt}, u)_2 = 2\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(u_t, u)_2 - 2K(u). \tag{5.13}$$

Using proof of Theorem 4.8 we can get,

$$\begin{aligned}
\frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} &+ a(1)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{p+1}K(u) \\
&= \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \left(\frac{1}{2} - \frac{1}{p+1} + \frac{p-1}{2(p+1)}C_*^2\right)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1}\left(\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \int_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \int_{\mathbb{E}} r^q u f(z,u) d\mu\right) \\
&= \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{p-1}{2(p+1)}C_*^2\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&+ \frac{1}{p+1}\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu - \frac{1}{p+1}\int_{\mathbb{E}} r^q u f(z,u) d\mu \\
&\le \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{1}{2}\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \left[\left(\frac{1}{2} - \frac{1}{p+1}\right) + \frac{1}{p+1}\right]\int_{\mathbb{E}} r^q V(z)|u|^2 d\mu \\
&- \int_{\mathbb{E}} r^q F(z,u) d\mu \ \le \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + J(u) \ + \ \gamma\int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} d\tau \\
&= I(t) + \gamma\int_0^t \|u_\tau\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \le I(0). \qquad (5.14)
\end{aligned}$$

Thus inequality 5.14 implies that

$$\begin{aligned}
\ddot{M}(t) &\ge 2\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(u_t,u)_2 \\
&- 2(p+1)\left[I(0) - \frac{1}{2}\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - \frac{p-1}{2(p+1)}(1+C_*^2)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\right] \\
&= (p+3)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)(1+C_*^2)\|\nabla_{\mathbb{E}}u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\
&- 2\gamma(u_t,u)_2 - 2(p+1)I(0). \qquad (5.15)
\end{aligned}$$

In first, let us consider $I(0) \le 0$. Then,

$$\ddot{M}(t) \ge (p+3)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)(1+C_*^2)\lambda_1\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(u_t,u)_2.$$

condition $\gamma < (p-1)(1+C_*^2)\lambda_1$ implies that, there exists a constant $\epsilon \in \Big(0 \quad , \quad (p-1)(1+C_*^2)\Big)$ such that

$$\gamma^2 < (p-1-\epsilon)(1+C_*^2)\lambda_1^2.$$

Therefore,

$$\begin{aligned}
\ddot{M}(t) &\ge (4+\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1-\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(u_t \quad , \quad u)_2 \\
&+ (p-1)(1+C_*^2)\lambda_1^2\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \qquad (5.16)
\end{aligned}$$

On the other hand,

$$\begin{aligned} 2\gamma(u_t \quad , \quad u)_2 &\leq (p-1-\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\gamma^2}{p-1-\epsilon}\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\leq (p-1-\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1)(1+C_*^2)\lambda_1^2\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned} \tag{5.17}$$

From 5.16 and 5.17, we can get that

$$\ddot{M}(t) \geq (4+\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \tag{5.18}$$

By cone Hölder inequality we get

$$M(t)\ddot{M}(t) - \frac{4+\epsilon}{4}\dot{M}(t) \geq (4+\epsilon)\Big(\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - (u_t \quad , \quad u)_2\Big) \geq 0,$$

$$(M^{-\alpha})'' = \frac{-\alpha}{M^{\alpha+2}(t)}\Big(M(t)\ddot{M}(t) - (\alpha+1)\dot{M}(t)^2\Big) \leq 0,$$

for $\alpha = \frac{\epsilon}{4}$ and $0 \leq t < \infty$. Hence, there exists a $T_1 > 0$ such that

$$\lim_{t\to T_1} M^{-\alpha}(t) = 0$$

and $\lim_{t\to T_1} M(t) = +\infty$, which is contradicts $T = +\infty$.

In second case, we consider $0 < I(0) < d$. In this case from Theorem 4.5 we have $u \in E_\delta$ for $0 \leq t < \infty$ and $\delta \in (\frac{2}{3}C_2(p+1) \quad , \quad \delta_2)$ (see Remark 4.2)where $(\delta_1 \quad , \quad \delta_2)$ is the maximal interval including $\delta = \frac{2}{3}C_2(p+1)$ such that $d(\delta) > I(0)$ for $\delta \in (\delta_1 \quad , \quad \delta_2)$. Therefore, $K_\delta(u) < 0$ and $\|\nabla_{\mathbb{E}} u\|_{L_2^{\frac{n+1}{2}}(\mathbb{E})} > l(\delta)$ for $\frac{2}{3}C_2(p+1) < \delta < \delta_2$, $0 \leq t < \infty$. Consequent, $K_\delta(u) \leq 0$ and $\|\nabla_{\mathbb{E}} u\| \geq l(\delta)$ for $0 \leq t < \infty$. From 5.13,

$$\begin{aligned} \frac{d}{dt}(e^{\gamma t}\dot{M}(t)) &= e^{\gamma t}\Big(\gamma\dot{M}(t) + \ddot{M}(t)\Big) = 2e^{\gamma t}\Big(\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\| - K(u)\Big) \\ &= 2e^{\gamma t}\Big(\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}\| + (\frac{3\delta_2}{2C_2(p+1)} - 1)\|\nabla_{\mathbb{E}} u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - K_{\frac{3\delta_2}{2C_2(p+1)}}(u)\Big) \\ &\geq 2e^{\gamma t}(\frac{3\delta_2}{2C_2(p+1)} - 1)r^2(\frac{3\delta_2}{2C_2(p+1)}) = C(\delta_2)e^{\gamma t}. \end{aligned}$$

Hence,

$$e^{\gamma t}\dot{M}(t) \geq C(\delta_2)\int_0^t e^{\gamma\tau}d\tau + \dot{M}(0) = \frac{C(\delta_2)}{\gamma}(e^{\gamma t} - 1) + \dot{M}(0),$$

$$\dot{M}(t) \geq \frac{C(\delta_2)}{\gamma}(1 - e^{-\gamma t}) + e^{-\gamma t}\dot{M}(0).$$

Hence there exists $t_0 > 0$ for which

$$\dot{M}(t) \geq \frac{C(\delta_2)}{2\gamma} \quad \forall t \geq t_0$$

and

$$M(t) \geq \frac{C(\delta_2)}{2\gamma}(t-t_0) + M(t_0) \geq \frac{C(\delta_2)}{2\gamma}(t-t_0), \qquad t \geq t_0. \tag{5.19}$$

From assumption $\gamma < (p-1)(1+C_*^2)\lambda_1$, it follows there exists a constant

$$\epsilon \in \Big(0 \quad , \quad (p-1)(1+C_*^2)\Big)$$

such that

$$\gamma^2 < (p-1-\epsilon) \, \Big[(p-1)(1+C_*^2)\lambda_1^2 - \epsilon\Big].$$

From 5.15,

$$\begin{aligned} \ddot{M}(t) &\geq (p+3)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(u_t \quad , \quad u)_2 + (p-1)(1+C_*^2)\lambda_1^2\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2(p+1)I(0) \\ &= \|(4+\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + (p-1-\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} - 2\gamma(u_t \quad , \quad u)_2 \\ &+ [(p-1)(1+C_*^2)\lambda_1^2 - \epsilon]\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \epsilon M(t) - 2(p+1)I(0). \end{aligned} \tag{5.20}$$

Also we can obtain

$$\begin{aligned} 2\gamma(u_t \quad , \quad u)_2 &\leq (p-1-\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \frac{\gamma^2}{p-1-\epsilon}\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} \\ &\leq (p-1-\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + [(p-1)(1+C_*^2)\lambda_1^2 - \epsilon]\|u\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}. \end{aligned} \tag{5.21}$$

From 5.20 and 5.21 we get

$$\ddot{M}(t) \geq (4+\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})} + \epsilon M(t) - 2(p+1)I(0). \tag{5.22}$$

From 5.19, it follows that there exists a $t_1 > 0$ such that

$$\epsilon(t) > 2(p+1)I(0) \qquad \forall t > t_1,$$

and then

$$\ddot{M}(t) > (4+\epsilon)\|u_t\|^2_{L_2^{\frac{n+1}{2}}(\mathbb{E})}, \qquad \forall t > t_1.$$

Now, similar to first case we can obtain a contradiction. Hence we always have $T < \infty$.

□

From Theorems 5.1 and 5.4 we can obtain the following theorem for global existence and nonexistence of solutions for problem 1.1.

Theorem 5.5 *Let $0 \leq \gamma < (p-1)(1+C_*^2)\lambda_1$, $u_0 \in \mathcal{H}^{1,\frac{n+1}{2}}_{2,0}(\mathbb{E})$ and $u_1 \in L_2^{\frac{n+1}{2}}(\mathbb{E})$. Suppose that $I(0) < 0$. Then, when $K(u_0) > 0$, problem 1.1 admits a global weak solution and when $K(u_0) < 0$, problem 1.1 does not admits any global weak solution.*

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