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Classification of some subclasses of 6−dimensional nilpotent Leibniz algebras

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Abstract: This article is a contribution to the improvement of classification theory in Leibniz algebras. We extend the method of congruence classes of matrices of bilinear forms that was used to classify complex nilpotent Leibniz algebras with one dimensional derived algebra. In this work we focus on applying this method to the classification of 6−dimensional complex nilpotent Leibniz algebras with two dimensional derived algebra.

Key words: Leibniz algebra, nilpotency, classification

1. Introduction

Leibniz algebras are introduced by Loday [9] as a nonantisymmetric generalization of Lie algebras. A. Bloh had studied these algebraic structures in 1965 and emphasizing their connections with derivations called them D-algebras [4]. A vector space $\mathcal{A}$ over $\mathbb{C}$ equipped with a bilinear product $[\,,\,]:\mathcal{A}\times\mathcal{A}\longrightarrow\mathcal{A}$ such that the left multiplication is a derivation is said to be a Leibniz algebra. Define the ideals of $\mathcal{A}$, $\mathcal{A}^1=\mathcal{A}$ and $\mathcal{A}^j=[\mathcal{A},\mathcal{A}^{j-1}]$ for $j\in\mathbb{Z}_{\geq 2}$. A Leibniz algebra $\mathcal{A}$ is nilpotent of class c if $\mathcal{A}^{c+1}=0$ but $\mathcal{A}^c\neq 0$ for some positive integer c. The most important ideal of $\mathcal{A}$ is $Leib(\mathcal{A})=\text{span}\{[a,a]\mid a\in\mathcal{A}\}$. Recall that if a Leibniz algebra A with $Leib(A)=0$, is a Lie algebra. Hence, if $Leib(A)\neq 0$, we will call A a non-Lie Leibniz algebra throughout the paper. We define the center of $\mathcal{A}$ by $Z(\mathcal{A})=\{z\in\mathcal{A}\mid [z,a]=0=[a,z] \text{ for all } a\in\mathcal{A}\}$. A Leibniz algebra $\mathcal{A}$ is said to be split if it can be written as a direct sum of two nontrivial ideals. Otherwise it is called nonsplit. Throughout this paper, we assume the vector spaces we study is over the field of complex numbers $\mathbb{C}$.

It is always an interesting problem to give the classification of any kind of algebras. The classification of nilpotent Lie algebras is a difficult problem and it is still unsolved. Since antisymmetry property is not satisfied in Leibniz algebras the problem of classification of non-Lie nilpotent Leibniz algebras is much harder. The classification of nilpotent Leibniz algebras over $\mathbb{C}$ of dimension less than or equal to four has been completed (see [1, 2, 5–7, 9, 10]). The classification of 5−dimensional non-Lie nilpotent Leibniz algebras is given in [8] with bilinear forms technique. In this paper, we apply this congruence classes of bilinear forms technique to give the classification of a subclass of 6−dimensional non-Lie nilpotent Leibniz algebras. This approach can be used to classify any $n-$dimensional nilpotent Leibniz algebras with $(n-2)-$dimensional derived algebra. Using the Mathematica program implementing Algorithm 2.6 given in [5], we verify that the classes we obtained are indeed pairwise nonisomorphic.

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Consider $\mathcal{A}$ be a $n-$dimensional complex non-Lie nilpotent Leibniz algebra with $\dim(\mathcal{A}^2) = n-2$ and $\dim(Leib(\mathcal{A})) = 1$. It is sufficient to classify nonsplit Leibniz algebras. We give the following Lemmas which are very useful. The first of them is an immediate consequence of Proposition 4.2 in [6].

Lemma 1.1 *Let $\mathcal{A}$ be a nilpotent Leibniz algebra of class c. Then we have $\mathcal{A}^c \subseteq Z(\mathcal{A})$.*

Lemma 1.2 *If $\mathcal{A}$ is a nonsplit Leibniz algebra then $Z(\mathcal{A}) \subseteq \mathcal{A}^2$.*

Proof Let $\mathcal{A}$ be a nonsplit Leibniz algebra. Assume $Z(\mathcal{A}) \nsubseteq \mathcal{A}^2$. Take a complementary subspace W to $\mathcal{A}^2$ in $\mathcal{A}$ such that $\mathcal{A} = \mathcal{A}^2 \oplus W$. Let V be a complementary to $Z(\mathcal{A}) \cap W$ in A such that $A = V \oplus (Z(\mathcal{A}) \cap W)$. Choose $I_1 = Z(\mathcal{A}) \cap W$ and $I_2 = V$.
Note that $Z(\mathcal{A}) \cap W \subseteq Z(\mathcal{A})$ hence it is an ideal. Also V is an ideal since V contains $\mathcal{A}^2$. Therefore, $\mathcal{A} = I_1 \oplus I_2$ where I_1 and I_2 are nontrivial ideals of $\mathcal{A}$. Then $\mathcal{A}$ is split, which is a contradiction. □

Lemma 1.3 *Let $\mathcal{A}$ be a nilpotent Leibniz algebra and $\dim(Leib(\mathcal{A})) = 1$. Then $Leib(\mathcal{A}) \subseteq Z(\mathcal{A})$.*

Proof Assume $[\mathcal{A}, Leib(\mathcal{A})] \neq 0$. Then using $Leib(\mathcal{A})$ is an ideal we get $Leib(\mathcal{A}) = [\mathcal{A}, Leib(\mathcal{A})]$. So
$Leib(\mathcal{A}) = [\mathcal{A}, Leib(\mathcal{A})] \subseteq [\mathcal{A}, \mathcal{A}^2] = \mathcal{A}^3 \Rightarrow Leib(\mathcal{A}) \subseteq \mathcal{A}^3$.
$Leib(\mathcal{A}) = [\mathcal{A}, Leib(\mathcal{A})] \subseteq [\mathcal{A}, \mathcal{A}^3] = \mathcal{A}^4 \Rightarrow Leib(\mathcal{A}) \subseteq \mathcal{A}^4$. By doing this repetitively we see that $Leib(\mathcal{A}) \subseteq \mathcal{A}^n$ for any natural number n. This implies $\mathcal{A}$ is not nilpotent which is a contradiction. Hence our assumption is wrong. Then $[\mathcal{A}, Leib(\mathcal{A})] = 0$ which implies that $Leib(\mathcal{A}) \subseteq Z(\mathcal{A})$ □

We omit the proofs of the following Lemmas since they are already given in [8].

Lemma 1.4 *Let $\mathcal{A}$ be a $n-$ dimensional nilpotent Leibniz algebra with $\dim(Z(\mathcal{A})) = n-k$. If $\dim(Leib(\mathcal{A})) = 1$ then $\dim(\mathcal{A}^2) \leq \frac{k^2-k+2}{2}$.*

Lemma 1.5 *Let $\mathcal{A}$ be a $n-$ dimensional nilpotent Leibniz algebra with $\dim(\mathcal{A}^2) = n-k, \dim(Leib(\mathcal{A})) = 1$ and $\dim(\mathcal{A}^3) = t$. Then*

(i) $n \leq t + \frac{k^2+k+2}{2}$

(ii) $n \leq t + \frac{k^2+k}{2}$ *if* $Leib(\mathcal{A}) \subseteq \mathcal{A}^3$

Lemma 1.6 *Let $\mathcal{A}$ be a $n-$ dimensional nilpotent Leibniz algebra with $\dim(\mathcal{A}^2) = n-k$ and $\mathcal{A}^4 \neq 0$. Then $\dim(Z(\mathcal{A})) < n-k-1$.*

Proof Note that by Lemma 1.2 and since $\mathcal{A}^3 \neq 0$ we have $Z(\mathcal{A}) \subset \mathcal{A}^2$. So it is enough to show that $\dim(Z(\mathcal{A})) \neq n-k-1$. Assume $\dim(Z(\mathcal{A})) = n-k-1$. Take a complementary subspace W to $Z(\mathcal{A})$ in A^2 such that $A^2 = Z(\mathcal{A}) \oplus W$. Using the fact that $\mathcal{A}^3 \subseteq \mathcal{A}^2$ and $\mathcal{A}^4 \neq 0$ we can see that $W \subseteq \mathcal{A}^3$. Hence

$$\mathcal{A}^3 = [\mathcal{A}, \mathcal{A}^2] = [\mathcal{A}, Z(\mathcal{A}) \oplus W] = \mathcal{A}^4$$

which is a contradiction. So $\dim(Z(\mathcal{A})) \neq n-k-1$, and therefore $\dim(Z(\mathcal{A})) < n-k-1$.

□

Choose $Leib(\mathcal{A}) = \text{span}\{u_n\}$. We can extend it to a basis $\{u_3, u_4, \ldots, u_{n-1}, u_n\}$ for $\mathcal{A}^2$ and let a subspace U be complementary to $\mathcal{A}^2$ in $\mathcal{A}$ so that $\mathcal{A} = \mathcal{A}^2 \oplus U$. So $[u, v] = \alpha_3 u_3 + \alpha_4 u_4 + \alpha_{n-1} u_{n-1} + \alpha_n u_n$ for some $\alpha_i \in \mathcal{C}$, $3 \leq i \leq n$, for any $u, v \in U$. Bilinear form $f(\ ,\ ) : U \times U \to \mathcal{C}$ can be defined by $f(u, v) = \alpha_n$ for all $u, v \in U$.

Choosing a basis $\{u_1, u_2\}$ for U we see that the matrix N of the bilinear form $f(\ ,\ ) : U \times U \to \mathcal{C}$ is one the following (see Theorem 3.1 in [11]):

$$(i)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (ii)\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad (iii)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (iv)\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}, \quad (v)\begin{pmatrix} 0 & 1 \\ c & 0 \end{pmatrix}$$

where $c \neq 1, -1$. We can assume that N cannot be the matrix (i) because the resulting algebra is a Lie algebra. It is enough to consider the matrices (ii) and (iii) because others are isomorphic to one of these as showed in Lemma 2.1 in [8] .

2. Classification of 6-dimensional nilpotent Leibniz algebras with $\dim(\mathcal{A}^2) = 4$ and $\dim(Leib(\mathcal{A})) = 1$

Let $\mathcal{A}$ be a complex nonsplit non-Lie nilpotent Leibniz algebra with $\dim(\mathcal{A}) = 6, \dim(\mathcal{A}^2) = 4$ and $\dim(Leib(\mathcal{A})) = 1$. Then by Lemma 1.3 we get $Leib(\mathcal{A}) \subseteq Z(\mathcal{A})$. Using Lemma 1.5 we have $2 \leq \dim(\mathcal{A}^3) \leq 3$. First suppose $\dim(\mathcal{A}^3) = 2$. Then fom Lemma 1.5 we obtain $Leib(\mathcal{A}) \nsubseteq \mathcal{A}^3$. Note that $\dim(\mathcal{A}^4) = 0$ or $\dim(\mathcal{A}^4) = 1$.

Theorem 2.1 *Let* $\dim(\mathcal{A}) = 6$, $\dim(\mathcal{A}^2) = 4$, $\dim(\mathcal{A}^3) = 2$, $\dim(\mathcal{A}^4) = 0$, *and* $\dim(Leib(\mathcal{A})) = 1$. *Then, up to isomorphism, the nonzero multiplications in* $\mathcal{A}$ *is given by one of the following:*

$\mathcal{A}_1$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_5 = -[\delta_3, \delta_2]$.

$\mathcal{A}_2$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_2, \delta_2] = \delta_6, [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_5 = -[\delta_3, \delta_2]$.

Proof Let $\dim(\mathcal{A}^4) = 0$. Then by Lemma 1.1 and Lemma 1.2 we get $\mathcal{A}^3 \subseteq Z(\mathcal{A}) \subset \mathcal{A}^2$. If $\dim(Z(\mathcal{A})) = 2$ then $\mathcal{A}^3 = Z(\mathcal{A})$ and by Lemma 1.3 $Leib(\mathcal{A}) \subseteq \mathcal{A}^3$, a contradiction. Hence $\dim(Z(\mathcal{A})) = 3$. Using $Leib(\mathcal{A}) \nsubseteq \mathcal{A}^3$, let $Leib(\mathcal{A}) = \text{span}\{e_6\}$ and $\mathcal{A}^3 = \text{span}\{e_4, e_5\}$. Then $Z(\mathcal{A}) = \text{span}\{e_4, e_5, e_6\}$ and $\{e_3, e_4, e_5, e_6\}$ is an extended basis of $\mathcal{A}^2$. Take $U = \text{span}\{e_1, e_2\}$.

Case 1: If the matrix $N = (ii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1, e_1] = e_6, [e_1, e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_1, e_3] = \beta_1 e_4 + \beta_2 e_5 = -[e_3, e_1], [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5 = -[e_3, e_2]$, where $\alpha_1 \neq 0, \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$.
Then the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, \delta_4 = \alpha_1(\beta_1 e_4 + \beta_2 e_5), \delta_5 = \alpha_1(\gamma_1 e_4 + \gamma_2 e_5), \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_1$.

Case 2: If the matrix $N = (iii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1, e_1] = e_6, [e_1, e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_2, e_2] = e_6, [e_1, e_3] = \beta_1 e_4 + \beta_2 e_5 = -[e_3, e_1], [e_2, e_3] = \gamma_1 e_4 + \gamma_2 e_5 = -[e_3, e_2]$, where $\alpha_1 \neq 0, \beta_1\gamma_2 - \beta_2\gamma_1 \neq 0$.
Then the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, \delta_4 = \alpha_1(\beta_1 e_4 + \beta_2 e_5), \delta_5 = \alpha_1(\gamma_1 e_4 + \gamma_2 e_5), \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_2$. □

Theorem 2.2 *Let* $\dim(\mathcal{A}) = 6$, $\dim(\mathcal{A}^2) = 4$, $\dim(\mathcal{A}^3) = 2$, $\dim(\mathcal{A}^4) = 1$, *and* $\dim(Leib(\mathcal{A})) = 1$. *Then, up to isomorphism, the nonzero multiplications in* $\mathcal{A}$ *is given by one of the following:*

$\mathcal{A}_3$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_4$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_5$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_3]=\delta_4=-[\delta_3,\delta_2],[\delta_2,\delta_4]=\delta_5=-[\delta_4,\delta_2]$.

$\mathcal{A}_6$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_5=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_4=-[\delta_3,\delta_2],[\delta_2,\delta_4]=\delta_5=-[\delta_4,\delta_2]$.

$\mathcal{A}_7$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_8$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_9$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=i\delta_4=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=i\delta_5=-[\delta_4,\delta_2]$.

$\mathcal{A}_{10}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=i\delta_4+\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=i\delta_5=-[\delta_4,\delta_2]$.

Proof Let $\dim(\mathcal{A}^4)=1$. Then by Lemma 1.1 and Lemma 1.2 we have $\mathcal{A}^4\subseteq Z(\mathcal{A})\subset\mathcal{A}^2$. Using this with Lemma 1.6 we get $1\leq dim(Z(\mathcal{A}))<3$. If $\dim(Z(\mathcal{A}))=1$ then $Leib(\mathcal{A})=Z(\mathcal{A})=\mathcal{A}^4\subset\mathcal{A}^3$, leads to a contradiction. Hence $\dim(Z(\mathcal{A}))=2$. Note that $Leib(\mathcal{A})\neq A^4$ because otherwise $Leib(\mathcal{A})=\mathcal{A}^4\subset\mathcal{A}^3$, a contradiction. From Leibniz identities $[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^3]+[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]$ and $[\mathcal{A}^3,[\mathcal{A},\mathcal{A}]]=[[\mathcal{A}^3,\mathcal{A}],\mathcal{A}]+[\mathcal{A},[\mathcal{A}^3,\mathcal{A}]]$ we get $[\mathcal{A}^2,\mathcal{A}^3]=0=[\mathcal{A}^3,\mathcal{A}^2]$. Let $Leib(\mathcal{A})=\text{span}\{e_6\},\mathcal{A}^4=\text{span}\{e_5\}$ and $\mathcal{A}^3=\text{span}\{e_4,e_5\}$. Then $Z(\mathcal{A})=\text{span}\{e_5,e_6\}$. Extend this to a basis $\{e_3,e_4,e_5,e_6\}$ of $\mathcal{A}^2$. Take $U=\text{span}\{\text{e}_1,\text{e}_2\}$.

Case 1: Let N be the matrix (ii) then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_1e_4+\beta_2e_5=-[e_3,e_1],[e_2,e_3]=\beta_3e_4+\beta_4e_5=-[e_3,e_2],[e_1,e_4]=\gamma_1e_5=-[e_4,e_1],[e_2,e_4]=\gamma_2e_5=-[e_4,e_2]$.
From Leibniz identities we obtain the following equation:

$$\beta_3\gamma_1=\beta_1\gamma_2 \tag{2.1}$$

Case 1.1: Let $\gamma_2=0$. Then $\gamma_1\neq0$ since $\dim(Z(\mathcal{A}))=2$. So $\beta_3=0$ which implies $\beta_1\neq0$. If $\beta_4=0$ then the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5,\delta_4=\alpha_1\beta_1e_4+(\alpha_1\beta_2+\alpha_2\gamma_1)e_5,\delta_5=\alpha_1\beta_1\gamma_1e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_3$. If $\beta_4\neq0$ then the base change $\delta_1=e_1,\delta_2=\frac{\beta_1\gamma_1}{\beta_4}e_2,\delta_3=\frac{\beta_1\gamma_1}{\beta_4}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1\beta_1^2\gamma_1}{\beta_4}e_4+\frac{\beta_1\gamma_1}{\beta_4}(\alpha_1\beta_2+\alpha_2\gamma_1)e_5,\delta_5=\frac{\alpha_1\beta_1^2\gamma_1^2}{\beta_4}e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_4$.

Case 1.2: Let $\gamma_2\neq0$. Then with the base change $\delta_1=\gamma_2e_1-\gamma_1e_2,\delta_2=e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=\gamma_2^2e_6$ we can make $\gamma_1=0$. Then by (2.1) we have $\beta_1=0$. So $\beta_3=\neq0$ since $\dim(\mathcal{A}^2)=4$. If $\beta_2=0$ then the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5,\delta_4=\alpha_1\beta_3e_4+(\alpha_1\beta_4+\alpha_2\gamma_2)e_5,\delta_5=\alpha_1\beta_3\gamma_2e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_5$. If $\beta_2\neq0$ then the base change $\delta_1=\frac{\beta_3\gamma_2}{\beta_2}e_1,\delta_2=e_2,\delta_3=\frac{\beta_3\gamma_2}{\beta_2}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1\beta_3^2\gamma_2}{\beta_2}e_4+\frac{\beta_3\gamma_2}{\beta_2}(\alpha_1\beta_4+\alpha_2\gamma_2)e_5,\delta_5=\frac{\alpha_1\beta_3^2\gamma_2^2}{\beta_2}e_5,\delta_6=\frac{\beta_3^2\gamma_2^2}{\beta_2^2}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_6$.

Case 2: Let N be the matrix (iii) then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5=-[e_3,e_1],[e_2,e_3]=\beta_3e_4+\beta_4e_5=-[e_3,e_2],[e_1,e_4]=\gamma_1e_5=-[e_4,e_1],[e_2,e_4]=\gamma_2e_5=-[e_4,e_2]$.
Again from Leibniz identities we obtain the equation (2.1).

Case 2.1: Let $\gamma_2=0$. Then $\gamma_1\neq 0$ since $\dim(Z(\mathcal{A}))=2$. So $\beta_3=0$ which implies $\beta_1\neq 0$. Then the nontrivial multiplications in $\mathcal{A}$ given by

$$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5=-[e_3,e_1],$$
$$[e_2,e_3]=\beta_4e_5=-[e_3,e_2],[e_1,e_4]=\gamma_1e_5=-[e_4,e_1],[e_3,e_4]=\gamma_3e_5=-[e_4,e_3]. \quad (2.2)$$

If $\beta_4=0$ then the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5,\delta_4=\alpha_1\beta_1e_4+(\alpha_1\beta_2+\alpha_2\gamma_1)e_5,\delta_5=\alpha_1\beta_1\gamma_1e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_7$. If $\beta_4\neq 0$ then the base change $\delta_1=\frac{\beta_4}{\beta_1\gamma_1}e_1,\delta_2=\frac{\beta_4}{\beta_1\gamma_1}e_2,\delta_3=\frac{\beta_4^2}{\beta_1^2\gamma_1^2}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1\beta_4^3}{\beta_1^2\gamma_1^3}e_4+\frac{\beta_4^3}{\beta_1^3\gamma_1^3}(\alpha_1\beta_2+\alpha_2\gamma_1)e_5,\delta_5=\frac{\alpha_1\beta_4^4}{\beta_1^3\gamma_1^3}e_5,\delta_6=\frac{\beta_4^2}{\beta_1^2\gamma_1^2}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_8$.

Case 2.2: Let $\gamma_2\neq 0$. If $\gamma_1=0$ then by (2.1) we have $\beta_1=0$. Then the base change $\delta_1=e_2,\delta_2=e_1,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.2). Hence $\mathcal{A}$ is isomorphic to $\mathcal{A}_7$ or $\mathcal{A}_8$. Now let $\gamma_1\neq 0$. If $\gamma_1^2+\gamma_2^2\neq 0$ then the base change $\delta_1=e_1,\delta_2=-\frac{\gamma_2}{\sqrt{\gamma_1^2+\gamma_2^2}}e_1+\frac{\gamma_1}{\sqrt{\gamma_1^2+\gamma_2^2}}e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.2). Hence $\mathcal{A}$ is isomorphic to $\mathcal{A}_7$ or $\mathcal{A}_8$. Now consider the case $\gamma_1^2+\gamma_2^2=0$. Then by (2.1) we get $\beta_1^2+\beta_3^2=0$. The base change $y_1=e_1,y_2=e_2,y_3=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5,y_4=\alpha_1\beta_1e_4+(\alpha_1\beta_2+\alpha_2\gamma_1)e_5,y_5=\alpha_1\beta_1\gamma_1e_5,y_6=e_6$ shows $\mathcal{A}$ is isomorphic to the following algebra:

$$[y_1,y_1]=y_6,[y_1,y_2]=y_3=-[y_2,y_1],[y_2,y_2]=y_6,[y_1,y_3]=y_4=-[y_3,y_1],[y_2,y_3]=iy_4+\theta y_5=-[y_3,y_2],$$
$$[y_1,y_4]=y_5=-[y_4,y_1],[y_2,e_4]=iy_5=-[y_4,y_2]$$

If $\theta=0$ then the base change $\delta_1=y_1,\delta_2=y_2,\delta_3=y_3,\delta_4=y_4,\delta_5=y_5,\delta_6=y_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_9$. If $\theta\neq 0$ then the base change $\delta_1=\theta y_1,\delta_2=\theta y_2,\delta_3=\theta^2y_3,\delta_4=\theta^3y_4,\delta_5=\theta^4y_5,\delta_6=\theta^2y_6$ shows A is isomorphic to $\mathcal{A}_{10}$. □

Now suppose $\dim(\mathcal{A}^3)=3$. Then fom Lemma 1.5 we have $Leib(\mathcal{A})\subseteq\mathcal{A}^3$. Note that $\dim(\mathcal{A}^4)=0,\dim(\mathcal{A}^4)=1$ or $\dim(\mathcal{A}^4)=2$.
Let $\dim(\mathcal{A}^4)=0$. Then by Lemma 1.1 and Lemma 1.2 we have $\mathcal{A}^3\subseteq Z(\mathcal{A})\subset\mathcal{A}^2$. Hence $\dim(Z(\mathcal{A}))=3$. Using $Leib(\mathcal{A})\subseteq\mathcal{A}^3$, let $Leib(\mathcal{A})=\text{span}\{e_6\}$ and $\mathcal{A}^3=\text{span}\{e_4,e_5,e_6\}=Z(\mathcal{A})$. Extend this to bases $\{e_3,e_4,e_5,e_6\},\{e_1,e_2,e_3,e_4,e_5,e_6\}$ of $\mathcal{A}^2$ and $\mathcal{A}$, respectively. Leibniz identity $[\mathcal{A},[\mathcal{A},\mathcal{A}^2]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^2]+[\mathcal{A},[\mathcal{A},\mathcal{A}^2]]$ implies that $[\mathcal{A}^2,\mathcal{A}^2]=0$. Then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=\theta_1e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=\theta_2e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\gamma_1e_4+\gamma_2e_5+\gamma_3e_6,[e_3,e_2]=-\gamma_1e_4-\gamma_2e_5+\gamma_4e_6$.
But then the Leibniz identities yield the equations $\beta_4=-\beta_3$ and $\gamma_4=-\gamma_3$ which implies $\dim(\mathcal{A}^3)=2$, contradiction. There is no Leibniz algebra in this case.

Let $\dim(\mathcal{A}^4)=1$. Then by Lemma 1.1 and Lemma 1.2 we have $\mathcal{A}^4 \subseteq Z(\mathcal{A}) \subset A^2$. We get $\dim(Z(\mathcal{A}))<3$ from Lemma 1.6. Then $\dim(Z(\mathcal{A}))=1$ or $\dim(Z(\mathcal{A}))=2$.
First suppose $\dim(Z(\mathcal{A}))=2$. Take a complementary subspace W to A^3 in A^2. Here $Z(\mathcal{A}) \subseteq A^3$ because otherwise $\mathcal{A}^2=\mathcal{A}^3 \oplus W$ such that $W \subseteq Z(\mathcal{A})$. But then $\mathcal{A}^3=[\mathcal{A},\mathcal{A}^2]=[\mathcal{A},\mathcal{A}^3 \oplus W]=\mathcal{A}^4$, which is a contradiction. We need to consider two cases: $\mathcal{A}^4=Leib(\mathcal{A})$ and $\mathcal{A}^4 \neq Leib(\mathcal{A})$.

Theorem 2.3 *Let* $\dim(\mathcal{A})=6$, $\dim(\mathcal{A}^2)=4$, $\dim(\mathcal{A}^3)=3$, $\dim(\mathcal{A}^4)=1$, $\dim(Z(\mathcal{A}))=2$, $\mathcal{A}^4=Leib(\mathcal{A})$ *and* $\dim(Leib(\mathcal{A}))=1$. *Then, up to isomorphism, the nonzero multiplications in* $\mathcal{A}$ *is given by one of the following:*

$\mathcal{A}_{11}$ $[\delta_1,\delta_1]=\delta_6, [\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1], [\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1], [\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2], [\delta_1,\delta_4]=\delta_6=-[\delta_4,\delta_1]$.

$\mathcal{A}_{12}$ $[\delta_1,\delta_1]=\delta_6, [\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1], [\delta_1,\delta_3]=\delta_5=-[\delta_3,\delta_1], [\delta_2,\delta_3]=\delta_4=-[\delta_3,\delta_2], [\delta_2,\delta_4]=\delta_6=-[\delta_4,\delta_2]$.

$\mathcal{A}_{13}$ $[\delta_1,\delta_1]=\delta_6, [\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1], [\delta_2,\delta_2]=\delta_6, [\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1], [\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2], [\delta_1,\delta_4]=\delta_6=-[\delta_4,\delta_1]$.

$\mathcal{A}_{14}$ $[\delta_1,\delta_1]=\delta_6, [\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1], [\delta_2,\delta_2]=\delta_6, [\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1], [\delta_2,\delta_3]=i\delta_4+\delta_5=-[\delta_3,\delta_2], [\delta_1,\delta_4]=\delta_6=-[\delta_4,\delta_1], [\delta_2,\delta_4]=i\delta_6=-[\delta_4,\delta_2]$.

Proof Let $\mathcal{A}^4=Leib(\mathcal{A})$. From Leibniz identities $[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^3]+[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]$ and $[\mathcal{A}^3,[\mathcal{A},\mathcal{A}]]=[[\mathcal{A}^3,\mathcal{A}],\mathcal{A}]+[\mathcal{A},[\mathcal{A}^3,\mathcal{A}]]$ we get $[\mathcal{A}^2,\mathcal{A}^3]=0=[\mathcal{A}^3,\mathcal{A}^2]$. Using $Leib(\mathcal{A}) \subseteq Z(\mathcal{A})$, let $Leib(\mathcal{A})=\mathcal{A}^4=\text{span}\{e_6\}$ and $Z(\mathcal{A})=\text{span}\{e_5,e_6\}$. Extend this to bases $\{e_4,e_5,e_6\},\{e_3,e_4,e_5,e_6\}$ of $\mathcal{A}^3$ and $\mathcal{A}^2$, respectively. Take $U=\text{span}\{\text{e}_1,\text{e}_2\}$.

Case 1: If the matrix $N=(ii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6, [e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1], [e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6, [e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6, [e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6, [e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6, [e_3,e_3]=\beta_9e_6, [e_1,e_4]=\gamma_1e_6, [e_4,e_1]=\gamma_2e_6, [e_2,e_4]=\gamma_3e_6, [e_4,e_2]=\gamma_4e_6$
From Leibniz identities we get the following equations:

$$\begin{cases} \alpha_1(\beta_3+\beta_4)+\alpha_2(\gamma_1+\gamma_2)=0 \\ \alpha_1(\beta_7+\beta_8)+\alpha_2(\gamma_3+\gamma_4)=0 \\ \beta_5\gamma_1-\alpha_1\beta_9-\beta_1\gamma_3=0 \\ \beta_1(\gamma_1+\gamma_2)=0 \\ \beta_1\gamma_4+\beta_5\gamma_1+\alpha_1\beta_9=0 \\ \beta_1\gamma_3+\beta_5\gamma_2-\alpha_1\beta_9=0 \\ \beta_5(\gamma_3+\gamma_4)=0 \end{cases} \tag{2.3}$$

Case 1.1: Let $\beta_5=0$. Then $\beta_1 \neq 0$ since $\dim(\mathcal{A}^3)=3$. So by (2.3) we obtain $\gamma_2=-\gamma_1, \beta_4=-\beta_3, \beta_9=0=\gamma_3=\gamma_4$ and $\beta_8=-\beta_7$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1,e_1]=e_6, [e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1], [e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6=-[e_3,e_1], [e_2,e_3]=\beta_6e_5+\beta_7e_6=-[e_3,e_2], [e_1,e_4]=\gamma_1e_6=-[e_4,e_1]$ Then the base change $\delta_1=e_1, \delta_2=\frac{1}{\alpha_1\beta_1\gamma_1}e_2, \delta_3=$

$\frac{1}{\alpha_1\beta_1\gamma_1}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{1}{\gamma_1}e_4+\frac{\beta_2}{\beta_1\gamma_1}e_5+(\frac{\beta_3}{\beta_1\gamma_1}+\frac{\alpha_2}{\alpha_1\beta_1})e_6,\delta_5=\frac{1}{\alpha_1\beta_1^2\gamma_1^2}(\beta_6e_5+\beta_7e_6),\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{11}$.

Case 1.2: Let $\beta_5\neq 0$. Then with the base change $\delta_1=\beta_5e_1-\beta_1e_2,\delta_2=e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=\beta_5^2e_6$ we can make $\beta_1=0$. So by (2.3) we obtain $\gamma_4=-\gamma_3,\beta_4=-\beta_3,\beta_9=0=\gamma_1=\gamma_2$ and $\beta_8=-\beta_7$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_2e_5+\beta_3e_6=-[e_3,e_1],[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6=-[e_3,e_2],[e_2,e_4]=\gamma_3e_6=-[e_4,e_2]$ Then the base change $\delta_1=\alpha_1\beta_5\gamma_1e_1,\delta_2=e_2,\delta_3=\alpha_1\beta_5\gamma_1(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\alpha_1^2\beta_5^2\gamma_1e_4+\alpha_1^2\beta_5\beta_6\gamma_1e_5+(\alpha_1^2\beta_5\beta_7\gamma_1+\alpha_1\alpha_2\beta_5\gamma_1^2)e_6,\delta_5=\alpha_1^3\beta_5^2\gamma_1^2(\beta_2e_5+\beta_3e_6),\delta_6=\alpha_1^2\beta_5^2\gamma_1^2e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{12}$.

Case 2: If the matrix $N=(iii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6,[e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6,[e_3,e_3]=\beta_9e_6,[e_1,e_4]=\gamma_1e_6,[e_4,e_1]=\gamma_2e_6,[e_2,e_4]=\gamma_3e_6,[e_4,e_2]=\gamma_4e_6$
Again from Leibniz identities we obtain the equations (2.3).

Case 2.1: Let $\beta_5=0$. Then $\beta_1\neq 0$ since $\dim(\mathcal{A}^3)=3$. So by (2.3) we obtain $\gamma_2=-\gamma_1,\beta_4=-\beta_3,\beta_9=0=\gamma_3=\gamma_4$ and $\beta_8=-\beta_7$. Then the nontrivial multiplications in $\mathcal{A}$ given by

$$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6=-[e_3,e_1],$$
$$[e_2,e_3]=\beta_6e_5+\beta_7e_6=-[e_3,e_2],[e_1,e_4]=\gamma_1e_6=-[e_4,e_1]. \quad (2.4)$$

Then the base change $\delta_1=(\frac{1}{\alpha_1\beta_1\gamma_1})^{1/2}e_1,\delta_2=(\frac{1}{\alpha_1\beta_1\gamma_1})^{1/2}e_2,\delta_3=\frac{1}{\alpha_1\beta_1\gamma_1}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1}{(\alpha_1\beta_1\gamma_1)^{3/2}}(\beta_1e_4+\beta_2e_5+\beta_3e_6)+\frac{\alpha_2\gamma_1}{(\alpha_1\beta_1\gamma_1)^{3/2}},\delta_5=\frac{\alpha_1}{(\alpha_1\beta_1\gamma_1)^{3/2}}(\beta_6e_5+\beta_7e_6),\delta_6=\frac{1}{\alpha_1\beta_1\gamma_1}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{13}$.

Case 2.2: Let $\beta_5\neq 0$. If $\beta_1=0$ then by 2.3 we have $\gamma_4=-\gamma_3,\beta_4=-\beta_3,\beta_9=0=\gamma_1=\gamma_2$ and $\beta_8=-\beta_7$. Then the base change $\delta_1=e_2,\delta_2=e_1,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.4). Hence $\mathcal{A}$ is isomorphic to $\mathcal{A}_{13}$. Now let $\beta_1\neq 0$. If $\beta_1^2+\beta_5^2\neq 0$ then the base change $\delta_1=\beta_1e_1+\beta_5e_2,\delta_2=\beta_5e_1-\beta_1e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=(\beta_1^2+\beta_5^2)e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.4). Hence $\mathcal{A}$ is isomorphic to $\mathcal{A}_{13}$. Now consider the case $\beta_1^2+\beta_5^2=0$. Then by (2.3) we get $\gamma_1^2+\gamma_3^2=0$. Then the base change $\delta_1=(\frac{1}{\alpha_1\beta_1\gamma_1})^{1/2}e_1,\delta_2=(\frac{1}{\alpha_1\beta_1\gamma_1})^{1/2}e_2,\delta_3=\frac{1}{\alpha_1\beta_1\gamma_1}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1}{(\alpha_1\beta_1\gamma_1)^{3/2}}(\beta_1e_4+\beta_2e_5+\beta_3e_6)+\frac{\alpha_2\gamma_1}{(\alpha_1\beta_1\gamma_1)^{3/2}},\delta_5=\frac{\alpha_1}{(\alpha_1\beta_1\gamma_1)^{3/2}}((\beta_6-i\beta_2)e_5+(\beta_7-i\beta_3)e_6),\delta_6=\frac{1}{\alpha_1\beta_1\gamma_1}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{14}$. □

Theorem 2.4 *Let* $\dim(\mathcal{A})=6$, $\dim(\mathcal{A}^2)=4$, $\dim(\mathcal{A}^3)=3$, $\dim(\mathcal{A}^4)=1$, $\dim(Z(\mathcal{A}))=2$, $\mathcal{A}^4\neq Leib(\mathcal{A})$ *and* $\dim(Leib(\mathcal{A}))=1$. *Then, up to isomorphism, the nonzero multiplications in* $\mathcal{A}$ *is given by one of the following:*

$\mathcal{A}_{15}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_6=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_{16}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_5+\delta_6=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_{17}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_6=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_4=-[\delta_3,\delta_2],[\delta_2,\delta_4]=\delta_5=-[\delta_4,\delta_2]$.

$\mathcal{A}_{18}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_1,\delta_3]=\delta_5+\delta_6=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_4=-[\delta_3,\delta_2],[\delta_2,\delta_4]=\delta_5=-[\delta_4,\delta_2]$.

$\mathcal{A}_{19}(\alpha)$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\alpha\delta_5+\delta_6=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1]$.

$\mathcal{A}_{20}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=i\delta_4+\delta_6=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=i\delta_5=-[\delta_4,\delta_2]$.

$\mathcal{A}_{21}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=i\delta_4+\delta_5+\delta_6=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=i\delta_5=-[\delta_4,\delta_2]$.

Proof Let $\mathcal{A}^4\neq Leib(\mathcal{A})$. From Leibniz identities $[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^3]+[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]$ and $[\mathcal{A}^3,[\mathcal{A},\mathcal{A}]]=[[\mathcal{A}^3,\mathcal{A}],\mathcal{A}]+[\mathcal{A},[\mathcal{A}^3,\mathcal{A}]]$ we get $[\mathcal{A}^2,\mathcal{A}^3]=0=[\mathcal{A}^3,\mathcal{A}^2]$. Let $Leib(\mathcal{A})=\text{span}\{e_6\}$ and $\mathcal{A}^4=\text{span}\{e_5\}$. Then $Z(\mathcal{A})=\text{span}\{e_5,e_6\}$. Extend this to bases $\{e_4,e_5,e_6\},\{e_3,e_4,e_5,e_6\}$ of $\mathcal{A}^3$ and $\mathcal{A}^2$, respectively. Take $U=\text{span}\{\text{e}_1,\text{e}_2\}$.

Case 1: Let N be the matrix (ii) then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6,[e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6,[e_3,e_3]=\beta_9e_6,[e_1,e_4]=\gamma_1e_5=-[e_4,e_1],[e_2,e_4]=\gamma_2e_5=-[e_4,e_2]$
From Leibniz identities we get the following equations:

$$\begin{cases}\beta_3+\beta_4=0\\ \beta_7+\beta_8=0\\ \beta_9=0\\ \beta_5\gamma_1-\beta_1\gamma_2=0\end{cases}\tag{2.5}$$

Case 1.1: Let $\gamma_2=0$. Then $\gamma_1\neq0$ since $\mathcal{A}^4\neq0$. So by (2.5) we have $\beta_5=0$. Then $\beta_1,\beta_7\neq0$ since $\dim(\mathcal{A}^3)=3$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6=-[e_3,e_1],[e_2,e_3]=\beta_6e_5+\beta_7e_6=-[e_3,e_2],[e_1,e_4]=\gamma_1e_5=-[e_4,e_1]$
If $\beta_6=0$ then the base change $\delta_1=\alpha_1\beta_7e_1,\delta_2=e_2,\delta_3=\alpha_1\beta_7(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\alpha_1^3\beta_1\beta_7^2e_4+(\alpha_1^3\beta_2\beta_7^2+\alpha_1^2\alpha_2\gamma_1\beta_7^2)e_5+\alpha_1^3\beta_3\beta_7^2e_6,\delta_5=\alpha_1^4\beta_1\gamma_1\beta_7^3e_5,\delta_6=\alpha_1^2\beta_7^2e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{15}$.
If $\beta_6\neq0$ then the base change $\delta_1=\frac{1}{(\alpha_1\beta_7)^{1/3}}(\frac{\beta_6}{\beta_1\gamma_1})^{2/3}e_1,\delta_2=(\frac{\beta_6}{\alpha_1^2\beta_7^2\beta_1\gamma_1})^{1/3}e_2,\delta_3=\frac{\beta_6}{\alpha_1\beta_7\beta_1\gamma_1}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=(\frac{\beta_6}{\beta_1\gamma_1})^{5/3}\frac{\alpha_1}{(\alpha_1\beta_7)^{4/3}}(\beta_1e_4+\beta_2e_5+\beta_3e_6)+(\frac{\beta_6}{\beta_1\gamma_1})^{5/3}\frac{\alpha_2\gamma_1}{(\alpha_1\beta_7)^{4/3}}e_5,\delta_5=(\frac{\beta_6}{\beta_1\gamma_1})^{7/3}\frac{\alpha_1\beta_1\gamma_1}{(\alpha_1\beta_7)^{5/3}}e_5,\delta_6=\frac{1}{(\alpha_1\beta_7)^{2/3}}(\frac{\beta_6}{\beta_1\gamma_1})^{4/3}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{16}$.

Case 1.2: Let $\gamma_2\neq0$. Then with the base change $\delta_1=\gamma_2e_1-\gamma_1e_2,\delta_2=e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=\gamma_2^2e_6$ we can make $\gamma_1=0$. Then by (2.5) we have $\beta_1=0$. Then $\beta_5,\beta_3\neq0$ since $\dim(\mathcal{A}^3)=3$.
If $\beta_2=0$ then the base change $\delta_1=e_1,\delta_2=\frac{1}{\alpha_1\beta_3}e_2,\delta_3=\frac{1}{\alpha_1\beta_3}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{1}{\alpha_1\beta_3^2}(\beta_1e_4+\beta_2e_5+$

$\beta_3 e_6) + \frac{\alpha_2\gamma_2}{(\alpha_1\beta_3)^2}e_5, \delta_5 = \frac{\alpha_1\beta_5\gamma_2}{(\alpha_1\beta_3)^3}e_5, \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{17}$.

If $\beta_2 \neq 0$ then the base change $\delta_1 = \frac{\beta_5\gamma_2}{\alpha_1^2\beta_2\beta_3^2}e_1, \delta_2 = \frac{1}{\alpha_1\beta_3}e_2, \delta_3 = \frac{\beta_5\gamma_2}{\alpha_1^3\beta_2\beta_3^3}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \frac{\beta_5\gamma_2}{\alpha_1^3\beta_2\beta_3^3}(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \frac{\alpha_2\beta_5\gamma_2^2}{\alpha_1^4\beta_2\beta_3^4}e_5, \delta_5 = \frac{\alpha_1\beta_5^2\gamma_2^2}{(\alpha_1\beta_3)^3}e_5, \delta_6 = (\frac{\beta_5\gamma_2}{\alpha_1^2\beta_2\beta_3^2})^2 e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{18}$.

Case 2: Let N be the matrix (iii) then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1] = e_6, [e_1,e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 = -[e_2,e_1], [e_2,e_2] = e_6, [e_1,e_3] = \beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6, [e_3,e_1] = -\beta_1 e_4 - \beta_2 e_5 + \beta_4 e_6, [e_2,e_3] = \beta_5 e_4 + \beta_6 e_5 + \beta_7 e_6, [e_3,e_2] = -\beta_5 e_4 - \beta_6 e_5 + \beta_8 e_6, [e_3,e_3] = \beta_9 e_6, [e_1,e_4] = \gamma_1 e_5 = -[e_4,e_1], [e_2,e_4] = \gamma_2 e_5 = -[e_4,e_2]$
From Leibniz identities again we get the equations (2.5).

Case 2.1: Let $\gamma_2 = 0$. Then $\gamma_1 \neq 0$ since $\mathcal{A}^4 \neq 0$. So by (2.5) we obtain $\beta_5 = 0$ and $\beta_1, \beta_7 \neq 0$ since $\dim(\mathcal{A}^3) = 3$. Then the nontrivial multiplications in $\mathcal{A}$ given by

$$[e_1,e_1] = e_6, [e_1,e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 = -[e_2,e_1], [e_2,e_2] = e_6, [e_1,e_3] = \beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6 = -[e_3,e_1],$$
$$[e_2,e_3] = \beta_6 e_5 + \beta_7 e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1 e_5 = -[e_4,e_1] \quad (2.6)$$

The base change $\delta_1 = \frac{1}{\alpha_1\beta_7}e_1, \delta_2 = \frac{1}{\alpha_1\beta_7}e_2, \delta_3 = \frac{1}{(\alpha_1\beta_7)^2}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \frac{1}{\alpha_1^2\beta_7^3}(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \frac{\alpha_2\gamma_1}{\alpha_1^3\beta_7^3}e_5, \delta_5 = \frac{\beta_1\gamma_1}{\alpha_1^3\beta_7^4}e_5, \delta_6 = \frac{1}{(\alpha_1\beta_7)^2}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{19}(\alpha)$.

Case 2.2: Let $\gamma_2 \neq 0$. If $\gamma_1 = 0$ then by 2.5 we have $\beta_1 = 0$. Then the base change $\delta_1 = e_2, \delta_2 = e_1, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.6). So $\mathcal{A}$ is isomorphic to $\mathcal{A}_{19}(\alpha)$. Now let $\gamma_1 \neq 0$. If $\gamma_1^2 + \gamma_2^2 \neq 0$ then the base change $\delta_1 = \gamma_1 e_1 + \gamma_2 e_2, \delta_2 = \gamma_2 e_1 - \gamma_1 e_2, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = (\gamma_1^2 + \gamma_2^2)e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.6). Hence again $\mathcal{A}$ is isomorphic to $\mathcal{A}_{19}(\alpha)$. Now consider the case $\gamma_1^2 + \gamma_2^2 = 0$. Then by (2.5) we get $\beta_1^2 + \beta_5^2 = 0$. Take $\theta_1 = \frac{\beta_6 - i\beta_2}{\beta_1\gamma_1}$ and $\theta_2 = \alpha_1(\beta_7 - i\beta_3)$. The base $y_1 = e_1, y_2 = e_2, y_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, y_4 = \alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_2\gamma_1 e_5, y_5 = \alpha_1\beta_1\gamma_1 e_5, y_6 = e_6$ shows $\mathcal{A}$ is isomorphic to the following algebra:
$[y_1,y_1] = y_6, [y_1,y_2] = y_3 = -[y_2,y_1], [y_2,y_2] = y_6, [y_1,y_3] = y_4 = -[y_3,y_1], [y_2,y_3] = iy_4 + \theta_1 y_5 + \theta_2 y_6 = -[y_3,y_2], [y_1,y_4] = y_5 = -[y_4,y_1], [y_2,y_4] = y_5 = -[y_4,y_2]$
Note that $\theta_2 \neq 0$ since $\dim(\mathcal{A}^3) = 3$.
If $\theta_1 = 0$ then the base change $\delta_1 = \frac{1}{\theta_2}y_1, \delta_2 = \frac{1}{\theta_2}y_2, \delta_3 = \frac{1}{\theta_2^2}y_3, \delta_4 = \frac{1}{\theta_2^3}y_4, \delta_5 = \frac{1}{\theta_2^4}y_5, \delta_6 = \frac{1}{\theta_2^2}y_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{20}$.
If $\theta_1 \neq 0$ then with suitable change of basis $\mathcal{A}$ isomorphic to $\mathcal{A}_{21}$. □

Now suppose $\dim(Z(\mathcal{A})) = 1$. Then we have $\mathcal{A}^4 = Z(\mathcal{A}) = Leib(\mathcal{A})$.

Theorem 2.5 *Let* $\dim(\mathcal{A}) = 6$, $\dim(\mathcal{A}^2) = 4$, $\dim(\mathcal{A}^3) = 3$, $\dim(\mathcal{A}^4) = 1$, $\dim(Z(\mathcal{A})) = 1$, *and* $\dim(Leib(\mathcal{A})) = 1$. *Then, up to isomorphism, the nonzero multiplications in* $\mathcal{A}$ *is given by one of the following:*

$\mathcal{A}_{22}$ $[\delta_1,\delta_1] = \delta_6, [\delta_1,\delta_2] = \delta_3 = -[\delta_2,\delta_1], [\delta_1,\delta_3] = \delta_4 = -[\delta_3,\delta_1], [\delta_2,\delta_3] = \delta_5 = -[\delta_3,\delta_2], [\delta_2,\delta_4] = \delta_6 = -[\delta_4,\delta_2], [\delta_1,\delta_5] = \delta_6 = -[\delta_5,\delta_1]$.

$\mathcal{A}_{23}(\alpha)$ $[\delta_1,\delta_1] = \delta_6, [\delta_1,\delta_2] = \delta_3 = -[\delta_2,\delta_1], [\delta_1,\delta_3] = \delta_4 = -[\delta_3,\delta_1], [\delta_2,\delta_3] = \delta_5 = -[\delta_3,\delta_2], [\delta_1,\delta_4] = \delta_6 = -[\delta_4,\delta_1], [\delta_2,\delta_4] = \alpha\delta_6 = -[\delta_4,\delta_2], [\delta_2,\delta_5] = \delta_6 = -[\delta_5,\delta_2]$.

$\mathcal{A}_{24}$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2],[\delta_2,\delta_4]=\delta_6=-[\delta_4,\delta_2],[\delta_1,\delta_5]=\delta_6=-[\delta_5,\delta_1]$.

$\mathcal{A}_{25}(\alpha,\beta)$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\alpha\delta_6=-[\delta_4,\delta_1],[\delta_1,\delta_5]=\beta\delta_6=-[\delta_5,\delta_1],[\delta_2,\delta_5]=\delta_6=-[\delta_5,\delta_2]$.

Proof Note that $\mathcal{A}^4=Z(\mathcal{A})=Leib(\mathcal{A})$. From Leibniz identities $[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^3]+[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]$ and $[\mathcal{A}^3,[\mathcal{A},\mathcal{A}]]=[[\mathcal{A}^3,\mathcal{A}],\mathcal{A}]+[\mathcal{A},[\mathcal{A}^3,\mathcal{A}]]$ we get $[\mathcal{A}^2,\mathcal{A}^3]=0=[\mathcal{A}^3,\mathcal{A}^2]$. Let $Leib(\mathcal{A})=\text{span}\{e_6\}=Z(\mathcal{A})=\mathcal{A}^4$. Extend this to bases $\{e_4,e_5,e_6\},\{e_3,e_4,e_5,e_6\}$ of $\mathcal{A}^3$ and $\mathcal{A}^2$, respectively. Take $U=\text{span}\{\text{e}_1,\text{e}_2\}$.

Case 1: If the matrix $N=(ii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6,[e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6,[e_3,e_3]=\beta_9e_6,[e_1,e_4]=\gamma_1e_6,[e_4,e_1]=\gamma_2e_6,[e_2,e_4]=\gamma_3e_6,[e_4,e_2]=\gamma_4e_6,[e_1,e_5]=\theta_1e_6,[e_5,e_1]=\theta_2e_6,[e_2,e_5]=\theta_3e_6,[e_5,e_2]=\theta_4e_6$
From Leibniz identities we get the following equations:

$$\begin{cases}\alpha_1(\beta_3+\beta_4)+\alpha_2(\gamma_1+\gamma_2)+\alpha_3(\theta_1+\theta_2)=0\\ \alpha_1(\beta_7+\beta_8)+\alpha_2(\gamma_3+\gamma_4)+\alpha_3(\theta_3+\theta_4)=0\\ \beta_9=0\\ \beta_5\gamma_1+\beta_6\theta_1-\beta_1\gamma_3-\beta_2\theta_3=0\\ \beta_1(\gamma_1+\gamma_2)+\beta_2(\theta_1+\theta_2)=0\\ \beta_5\gamma_1+\beta_6\theta_1+\beta_1\gamma_4+\beta_2\theta_4=0\\ \beta_5\gamma_2+\beta_6\theta_2+\beta_1\gamma_3+\beta_2\theta_3=0\\ \beta_5(\gamma_3+\gamma_4)+\beta_6(\theta_3+\theta_4)=0\end{cases}\tag{2.7}$$

Notice that if $\beta_5\neq0$, with the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=e_3,\delta_4=e_5,\delta_5=\beta_5e_4+\beta_6e_5,\delta_6=e_6$ we can make $\beta_5=0$. So let $\beta_5=0$. Then $\beta_1,\beta_6\neq0$ since $\dim(\mathcal{A}^3)=3$. So by (2.7) we have $\theta_3+\theta_4=0=\theta_1+\theta_2=\gamma_3+\gamma_4=\gamma_1+\gamma_2=\beta_3+\beta_4=\beta_7+\beta_8$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6=-[e_3,e_1],[e_2,e_3]=\beta_6e_5+\beta_7e_6=-[e_3,e_2],[e_1,e_4]=\gamma_1e_6=-[e_4,e_1],[e_2,e_4]=\gamma_3e_6=-[e_4,e_2],[e_1,e_5]=\theta_1e_6=-[e_5,e_1],[e_2,e_5]=\theta_3e_6=-[e_5,e_2]$
If $\theta_3=0$ then $\theta_1\neq0$ since $\dim(Z(\mathcal{A}))=1$. Notice that if $\gamma_3=0$ then $\theta_1e_4-\gamma_1e_5\in Z(\mathcal{A})$, contradiction. So $\gamma_3\neq0$. Then with the base change $\delta_1=\gamma_3e_1-\gamma_1e_2,\delta_2=e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=\gamma_3^2e_6$ we can make $\gamma_1=0$. The base change $\delta_1=e_1,\delta_2=\frac{1}{(\alpha_1\beta_1\gamma_3)^{1/2}}e_2,\delta_3=\frac{1}{(\alpha_1\beta_1\gamma_3)^{1/2}}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1}{(\alpha_1\beta_1\gamma_3)^{1/2}}(\beta_1e_4+\beta_2e_5+\beta_3e_6)+\frac{\alpha_3\theta_1}{(\alpha_1\beta_1\gamma_3)^{1/2}}e_6,\delta_5=\frac{1}{\beta_1\gamma_3}(\beta_6e_5+\beta_7e_6)+\frac{\alpha_2}{\alpha_1\beta_1}e_6,\delta_6=e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{22}$.
If $\theta_3\neq0$ then with the base change $\delta_1=\theta_3e_1-\theta_1e_2,\delta_2=e_2,\delta_3=e_3,\delta_4=e_4,\delta_5=e_5,\delta_6=\theta_3^2e_6$ we can make $\theta_1=0$. Note that if $\gamma_1=0$ then $\theta_3e_4-\gamma_3e_5\in Z(\mathcal{A})$, contradiction. Hence $\gamma_1\neq0$. Then the base change $\delta_1=\frac{(\beta_6\theta_3)^{1/4}}{\alpha_1^{1/2}(\beta_1\gamma_1)^{3/4}}e_1,\delta_2=\frac{1}{(\alpha_1^2\beta_1\beta_6\gamma_1\theta_3)^{1/4}}e_2,\delta_3=\frac{1}{\alpha_1\beta_1\gamma_1}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{(\beta_6\theta_3)^{1/4}}{\alpha_1^{1/2}(\beta_1\gamma_1)^{7/4}}(\beta_1e_4+\beta_2e_5+\beta_3e_6)+\frac{\alpha_3(\beta_6\theta_3)^{1/4}\theta_1}{\alpha_1^{3/2}(\beta_1\gamma_1)^{7/4}}e_6,\delta_5=\frac{1}{\alpha_1^{3/2}(\beta_1\gamma_1)^{5/4}(\beta_6\theta_3)^{1/4}}[\alpha_1(\beta_6e_5+\beta_7e_6)+\alpha_2\gamma_3e_6],\delta_6=\frac{(\beta_6\theta_3)^{1/2}}{\alpha_1(\beta_1\gamma_1)^{3/2}}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{23}(\alpha)$.

Case 2: If the matrix $N=(iii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6,[e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6,[e_3,e_3]=\beta_9e_6,[e_1,e_4]=\gamma_1e_6,[e_4,e_1]=\gamma_2e_6,[e_2,e_4]=\gamma_3e_6,[e_4,e_2]=\gamma_4e_6,[e_1,e_5]=\theta_1e_6,[e_5,e_1]=\theta_2e_6,[e_2,e_5]=\theta_3e_6,[e_5,e_2]=\theta_4e_6$
Again Leibniz identities yield the equations (2.7). Notice that if $\beta_5=0$, with the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=e_3,\delta_4=e_5,\delta_5=\beta_5e_4+\beta_6e_5,\delta_6=e_6$ we can make $\beta_5=0$. Let $\beta_5=0$. Then $\beta_1,\beta_6\neq 0$ since $\dim(A^3)=3$. So by (2.7) we have $\theta_3+\theta_4=0=\theta_1+\theta_2=\gamma_3+\gamma_4=\gamma_1+\gamma_2=\beta_3+\beta_4=\beta_7+\beta_8$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_2,e_2]=e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6=-[e_3,e_1],[e_2,e_3]=\beta_6e_5+\beta_7e_6=-[e_3,e_2],[e_1,e_4]=\gamma_1e_6=-[e_4,e_1],[e_2,e_4]=\gamma_3e_6=-[e_4,e_2],[e_1,e_5]=\theta_1e_6=-[e_5,e_1],[e_2,e_5]=\theta_3e_6=-[e_5,e_2]$
If $\theta_3=0$ then $\theta_1\neq 0$ since $\dim(Z(\mathcal{A}))=1$. Notice that if $\gamma_3=0$ then $\theta_1e_4-\gamma_1e_5\in Z(\mathcal{A})$, contradiction. So $\gamma_3\neq 0$. Then with the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=e_3,\delta_4=\theta_1e_4-\gamma_1e_5,\delta_5=e_5,\delta_6=e_6$ we can make $\gamma_1=0$. The base change $\delta_1=\frac{1}{(\alpha_1\beta_1\gamma_3)^{1/2}}e_1,\delta_2=\frac{1}{(\alpha_1\beta_1\gamma_3)^{1/2}}e_2,\delta_3=\frac{1}{\alpha_1\beta_1\gamma_3}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{\alpha_1}{(\alpha_1\beta_1\gamma_3)^{1/2}}(\beta_1e_4+\beta_2e_5+\beta_3e_6)+\frac{\alpha_3\theta_1}{(\alpha_1\beta_1\gamma_3)^{1/2}}e_6,\delta_5=\frac{1}{\alpha_1^{1/2}(\beta_1\gamma_3)^{3/2}}(\beta_6e_5+\beta_7e_6)+\frac{\alpha_2}{(\alpha_1\beta_1)^{3/2}\gamma_3^{1/2}}e_6,\delta_6=\frac{1}{\alpha_1\beta_1\gamma_3}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{24}$.
If $\theta_3\neq 0$ then with the base change $\delta_1=e_1,\delta_2=e_2,\delta_3=e_3,\delta_4=\theta_3e_4-\gamma_3e_5,\delta_5=e_5,\delta_6=e_6$ we can make $\gamma_3=0$. Note that $\gamma_1\neq 0$ because otherwise $e_4\in Z(\mathcal{A})$, contradiction. Then the base change $\delta_1=\frac{1}{(\alpha_1\beta_6\theta_3)^{1/2}}e_1,\delta_2=\frac{1}{(\alpha_1\beta_6\theta_3)^{1/2}}e_2,\delta_3=\frac{1}{\alpha_1\beta_6\theta_3}(\alpha_1e_3+\alpha_2e_4+\alpha_3e_5),\delta_4=\frac{1}{(\alpha_1\beta_6\theta_3)^{3/2}}[\alpha_1(\beta_1e_4+\beta_2e_5+\beta_3e_6)+(\alpha_2\gamma_1+\alpha_3\theta_1)e_6],\delta_5=\frac{1}{(\alpha_1\beta_6\theta_3)^{3/2}}[\alpha_1(\beta_6e_5+\beta_7e_6)+\alpha_3\theta_3e_6],\delta_6=\frac{1}{\alpha_1\beta_6\theta_3}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{25}(\alpha,\beta)$. □

Finally consider the case $\dim(\mathcal{A}^4)=2$. Then $\dim(\mathcal{A}^5)=0$ or $\dim(\mathcal{A}^5)=1$. Suppose $\dim(\mathcal{A}^5)=0$. Then by Lemma 1.1 and Lemma 1.2 we have $\mathcal{A}^4\subseteq Z(\mathcal{A})\subset\mathcal{A}^2$. If $\dim(Z(\mathcal{A}))=3$, take $\mathcal{A}^2=\mathcal{A}^3\oplus W$ such that $W\subseteq Z(\mathcal{A})$. Then $\mathcal{A}^3=[\mathcal{A},\mathcal{A}^2]=[\mathcal{A},\mathcal{A}^3\oplus W]=\mathcal{A}^4$, contradiction. Hence $\dim(Z(\mathcal{A}))=2$. This implies that $\mathcal{A}^4=Z(\mathcal{A})$.
From Leibniz identities $[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^3]+[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]$ and $[\mathcal{A}^3,[\mathcal{A},\mathcal{A}]]=[[\mathcal{A}^3,\mathcal{A}],\mathcal{A}]+[\mathcal{A},[\mathcal{A}^3,\mathcal{A}]]$ we get $[\mathcal{A}^2,\mathcal{A}^3]=0=[\mathcal{A}^3,\mathcal{A}^2]$.
Let $Leib(\mathcal{A})=\operatorname{span}\{e_6\}$. Extend this to bases $\{e_5,e_6\},\{e_4,e_5,e_6\},\{e_3,e_4,e_5,e_6\},\{e_1,e_2,e_3,e_4,e_5,e_6\}$ of $Z(\mathcal{A})=\mathcal{A}^4,\mathcal{A}^3,\mathcal{A}^2$ and $\mathcal{A}$, respectively. Then the nontrivial multiplications in $\mathcal{A}$ given by
$[e_1,e_1]=\theta_1e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5+\alpha_4e_6,[e_2,e_1]=-\alpha_1e_3-\alpha_2e_4-\alpha_3e_5+\alpha_5e_6,[e_2,e_2]=\theta_2e_6,[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6,[e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6,[e_3,e_3]=\theta_3e_6,[e_1,e_4]=\gamma_1e_5+\gamma_2e_6,[e_4,e_1]=-\gamma_1e_5+\gamma_3e_6,[e_2,e_4]=\gamma_4e_5+\gamma_5e_6,[e_4,e_2]=-\gamma_4e_5+\gamma_6e_6$
From Leibniz identities we get the following equations:

$$\begin{cases}\beta_1\gamma_4-\beta_5\gamma_1=0\\ \beta_5\gamma_2-\alpha_1\theta_3-\beta_1\gamma_5=0\\ \beta_5\gamma_2+\alpha_1\theta_3+\beta_1\gamma_6=0\\ \beta_5\gamma_3-\alpha_1\theta_3+\beta_1\gamma_5=0\\ \beta_5(\gamma_5+\gamma_6)=0\end{cases}\tag{2.8}$$

Let $\beta_5\neq 0$. If $\beta_1=0$ then by (2.8) we have $\gamma_1=0=\gamma_2=\gamma_3=\gamma_5+\gamma_6$. This implies that $\dim(\mathcal{A}^4)=1$,

contradiction. Note that if $\beta_1 \neq 0$ then with the base change $\delta_1 = e_1, \delta_2 = \beta_5 e_1 - \beta_1 e_2, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ we can make $\beta_5 = 0$. Without loss of generality, assume $\beta_5 = 0$. Then $\beta_1 \neq 0$ since $\dim(\mathcal{A}^3) = 3$. Then from (2.8) we have $\gamma_4 = 0 = \gamma_5 = \gamma_6 = \gamma_2 + \gamma_3$. This implies that $\dim(\mathcal{A}^4) = 1$, contradiction. Hence there is no Leibniz algebra when $\dim(\mathcal{A}^5) = 0$!

Theorem 2.6 *Let* $\dim(\mathcal{A}) = 6$, $\dim(\mathcal{A}^2) = 4$, $\dim(\mathcal{A}^3) = 3$, $\dim(\mathcal{A}^4) = 2$ $\dim(\mathcal{A}^5) = 1$ *and* $\dim(Leib(\mathcal{A})) = 1$. *Then, up to isomorphism, the nonzero multiplications in* $\mathcal{A}$ *is given by one of the following:*

$\mathcal{A}_{26}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{27}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_6 = -[\delta_3, \delta_2], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{28}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_5 = -[\delta_3, \delta_2], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_2, \delta_4] = \delta_6 = -[\delta_4, \delta_2], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{29}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_2, \delta_5] = \delta_6 = -[\delta_5, \delta_2]$.

$\mathcal{A}_{30}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_5 = -[\delta_3, \delta_2], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_2, \delta_5] = \delta_6 = -[\delta_5, \delta_2]$.

$\mathcal{A}_{31}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_2, \delta_3] = \delta_4 = -[\delta_3, \delta_2], [\delta_2, \delta_4] = \delta_5 = -[\delta_4, \delta_2], [\delta_2, \delta_5] = \delta_6 = -[\delta_5, \delta_2]$.

$\mathcal{A}_{32}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_6 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_4 = -[\delta_3, \delta_2], [\delta_2, \delta_4] = \delta_5 = -[\delta_4, \delta_2], [\delta_2, \delta_5] = \delta_6 = -[\delta_5, \delta_2]$.

$\mathcal{A}_{33}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \delta_5 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_4 = -[\delta_3, \delta_2], [\delta_1, \delta_4] = \delta_6 = -[\delta_4, \delta_1], [\delta_2, \delta_4] = \delta_5 = -[\delta_4, \delta_2], [\delta_2, \delta_5] = \delta_6 = -[\delta_5, \delta_2]$.

$\mathcal{A}_{34}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_2, \delta_3] = \delta_4 = -[\delta_3, \delta_2], [\delta_2, \delta_4] = \delta_5 = -[\delta_4, \delta_2], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{35}$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_2, \delta_3] = \delta_4 = -[\delta_3, \delta_2], [\delta_2, \delta_4] = \delta_5 = -[\delta_4, \delta_2], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{36}(\alpha)$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_1, \delta_3] = \alpha\delta_5 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_4 = -[\delta_3, \delta_2], [\delta_2, \delta_4] = \delta_5 = -[\delta_4, \delta_2], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1], [\delta_2, \delta_5] = \delta_6 = -[\delta_5, \delta_2]$.

$\mathcal{A}_{37}(\alpha)$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_2, \delta_2] = \delta_6, [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \alpha\delta_6 = -[\delta_3, \delta_2], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{38}(\alpha, \beta)$ $[\delta_1, \delta_1] = \delta_6, [\delta_1, \delta_2] = \delta_3 = -[\delta_2, \delta_1], [\delta_2, \delta_2] = \alpha\delta_6, [\delta_1, \delta_3] = \delta_4 = -[\delta_3, \delta_1], [\delta_2, \delta_3] = \delta_5 + \beta\delta_6 = -[\delta_3, \delta_2], [\delta_1, \delta_4] = \delta_5 = -[\delta_4, \delta_1], [\delta_2, \delta_4] = \delta_6 = -[\delta_4, \delta_2], [\delta_1, \delta_5] = \delta_6 = -[\delta_5, \delta_1]$.

$\mathcal{A}_{39}(\alpha)$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_1,\delta_5]=\alpha\delta_6=-[\delta_5,\delta_1],[\delta_2,\delta_5]=\delta_6=-[\delta_5,\delta_2]$.

$\mathcal{A}_{40}(\alpha,\beta)$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\alpha\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=\beta\delta_6=-[\delta_4,\delta_2],[\delta_1,\delta_5]=\beta\delta_6=-[\delta_5,\delta_1],[\delta_2,\delta_5]=\delta_6=-[\delta_5,\delta_2]$.

$\mathcal{A}_{41}(\alpha,\beta)$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=i\delta_4+\alpha\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=i\delta_5+\beta\delta_6=-[\delta_4,\delta_2],[\delta_1,\delta_5]=\delta_6=-[\delta_5,\delta_1]$.

$\mathcal{A}_{42}(\alpha,\beta)$ $[\delta_1,\delta_1]=\delta_6,[\delta_1,\delta_2]=\delta_3=-[\delta_2,\delta_1],[\delta_2,\delta_2]=\delta_6,[\delta_1,\delta_3]=\delta_4=-[\delta_3,\delta_1],[\delta_2,\delta_3]=i\delta_4+\alpha\delta_5=-[\delta_3,\delta_2],[\delta_1,\delta_4]=\delta_5=-[\delta_4,\delta_1],[\delta_2,\delta_4]=i\delta_5+\beta\delta_6=-[\delta_4,\delta_2],[\delta_1,\delta_5]=\delta_6=-[\delta_5,\delta_1],[\delta_2,\delta_5]=i\delta_6=-[\delta_5,\delta_2]$.

Proof Here $\dim(\mathcal{A}^5)=1$. Then by Lemma 1.1 and Lemma 1.2 we have $\mathcal{A}^5\subseteq Z(\mathcal{A})\subset\mathcal{A}^2$. If $\dim(Z(\mathcal{A}))=3$, take $\mathcal{A}^2=\mathcal{A}^3\oplus W$ such that $W\subseteq Z(\mathcal{A})$. Then $\mathcal{A}^3=[\mathcal{A},\mathcal{A}^2]=[\mathcal{A},\mathcal{A}^3\oplus W]=\mathcal{A}^4$, contradiction. Now suppose $\dim(Z(\mathcal{A}))=2$. If $Z(\mathcal{A})\subseteq\mathcal{A}^3$ then $\mathcal{A}^3=\mathcal{A}^4\oplus W$ such that $W\subseteq Z(\mathcal{A})$. This implies that $\mathcal{A}^4=[\mathcal{A},\mathcal{A}^3]=[\mathcal{A},\mathcal{A}^4\oplus W]=\mathcal{A}^5$, and otherwise $\mathcal{A}^2=\mathcal{A}^3\oplus W$ such that $W\subseteq Z(\mathcal{A})$. Then $\mathcal{A}^3=[\mathcal{A},\mathcal{A}^2]=[\mathcal{A},\mathcal{A}^3\oplus W]=\mathcal{A}^4$. So we arrived contradictions on both cases. Hence $\dim(Z(\mathcal{A}))=1$. Therefore $\mathcal{A}^5=Z(\mathcal{A})=Leib(\mathcal{A})$.
From Leibniz identities $[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]=[[\mathcal{A},\mathcal{A}],\mathcal{A}^3]+[\mathcal{A},[\mathcal{A},\mathcal{A}^3]]$ and $[\mathcal{A}^3,[\mathcal{A},\mathcal{A}]]=[[\mathcal{A}^3,\mathcal{A}],\mathcal{A}]+[\mathcal{A},[\mathcal{A}^3,\mathcal{A}]]$ we get $[\mathcal{A}^2,\mathcal{A}^3]=0=[\mathcal{A}^3,\mathcal{A}^2]$. Let $Leib(\mathcal{A})=\mathrm{span}\{e_6\}=Z(\mathcal{A})=\mathcal{A}^5$. Extend this to bases $\{e_5,e_6\},\{e_4,e_5,e_6\},\{e_3,e_4,e_5,e_6\}$ of $\mathcal{A}^4,\mathcal{A}^3$ and $\mathcal{A}^2$, respectively. Take $U=\mathrm{span}\{\mathrm{e}_1,\mathrm{e}_2\}$.

Case 1: If the matrix $N=(ii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1]=e_6,[e_1,e_2]=\alpha_1e_3+\alpha_2e_4+\alpha_3e_5=-[e_2,e_1],[e_1,e_3]=\beta_1e_4+\beta_2e_5+\beta_3e_6,[e_3,e_1]=-\beta_1e_4-\beta_2e_5+\beta_4e_6,[e_2,e_3]=\beta_5e_4+\beta_6e_5+\beta_7e_6,[e_3,e_2]=-\beta_5e_4-\beta_6e_5+\beta_8e_6,[e_3,e_3]=\beta_9e_6,[e_1,e_4]=\gamma_1e_5+\gamma_2e_6,[e_4,e_1]=-\gamma_1e_5+\gamma_3e_6,[e_2,e_4]=\gamma_4e_5+\gamma_5e_6,[e_4,e_2]=-\gamma_4e_5+\gamma_6e_6,[e_1,e_5]=\theta_1e_6,[e_5,e_1]=\theta_2e_6,[e_2,e_5]=\theta_3e_6,[e_5,e_2]=\theta_4e_6$.
From Leibniz identities we get the following equations:

$$\begin{cases}\alpha_1(\beta_3+\beta_4)+\alpha_2(\gamma_2+\gamma_3)+\alpha_3(\theta_1+\theta_2)=0\\ \alpha_1(\beta_7+\beta_8)+\alpha_2(\gamma_5+\gamma_6)+\alpha_3(\theta_3+\theta_4)=0\\ \beta_5\gamma_1-\beta_1\gamma_4=0\\ \beta_5\gamma_2+\beta_6\theta_1-\alpha_1\beta_9-\beta_1\gamma_5-\beta_2\theta_3=0\\ \gamma_4\theta_1-\gamma_1\theta_3=0\\ \beta_1(\gamma_2+\gamma_3)+\beta_2(\theta_1+\theta_2)=0\\ \beta_5\gamma_2+\beta_6\theta_1+\alpha_1\beta_9+\beta_1\gamma_6+\beta_2\theta_4=0\\ \gamma_1(\theta_1+\theta_2)=0\\ \gamma_4\theta_1+\gamma_1\theta_4=0\\ \beta_5\gamma_3+\beta_6\theta_2+\beta_1\gamma_5+\beta_2\theta_3-\alpha_1\beta_9=0\\ \beta_5(\gamma_5+\gamma_6)+\beta_6(\theta_3+\theta_4)=0\\ \gamma_4(\theta_3+\theta_4)=0\end{cases}\tag{2.9}$$

Case 1.1: Let $\beta_5=0$. Then $\beta_1\neq0$ since $\dim(\mathcal{A}^3)=3$. Using the equations (2.9) we obtain

$\gamma_4 = 0 = \beta_9 = \theta_1 + \theta_2 = \theta_3 + \theta_4 = \gamma_2 + \gamma_3 = \beta_3 + \beta_4 = \beta_7 + \beta_8 = \gamma_5 + \gamma_6$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1, e_1] = e_6, [e_1, e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_1, e_3] = \beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6 = -[e_3, e_1], [e_2, e_3] = \beta_6 e_5 + \beta_7 e_6 = -[e_3, e_2], [e_1, e_4] = \gamma_1 e_5 + \gamma_2 e_6 = -[e_4, e_1], [e_2, e_4] = \gamma_5 e_6 = -[e_4, e_2], [e_1, e_5] = \theta_1 e_6 = -[e_5, e_1], [e_2, e_5] = \theta_3 e_6 = -[e_5, e_2]$.

Case 1.1.1: Let $\theta_3 = 0$. Then $\theta_1 \neq 0$ since $\dim(\mathcal{A}^5) = 1$.
First suppose $\gamma_5 = 0$. Then from (2.9) we have $\beta_6 = 0$. If $\beta_7 = 0$ then the base change $\delta_1 = e_1, \delta_2 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_1} e_2, \delta_3 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_1}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_1}[\alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_2(\gamma_1 e_5 + \gamma_2 e_6) + \alpha_3\theta_1 e_6], \delta_5 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_1}[\alpha_1\beta_1(\gamma_1 e_5 + \gamma_2 e_6) + \theta_1(\alpha_1\beta_2 + \alpha_2\beta_1)e_6], \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{26}$. If $\beta_7 \neq 0$ then the base change $\delta_1 = \frac{\beta_7^{1/5}}{\alpha_1^{1/5}(\beta_1\gamma_1\theta_1)^{2/5}} e_1, \delta_2 = \frac{1}{\alpha_1^{3/5}\beta_7^{2/5}(\beta_1\gamma_1\theta_1)^{1/5}} e_2, \delta_3 = \frac{1}{\alpha_1^{4/5}\beta_7^{1/5}(\beta_1\gamma_1\theta_1)^{3/5}}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_1}[\alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_2(\gamma_1 e_5 + \gamma_2 e_6) + \alpha_3\theta_1 e_6], \delta_5 = \frac{\beta_7^{1/5}}{\alpha_1^{6/5}(\beta_1\gamma_1\theta_1)^{7/5}}[\alpha_1\beta_1(\gamma_1 e_5 + \gamma_2 e_6) + \theta_1(\alpha_1\beta_2 + \alpha_2\beta_1)e_6], \delta_6 = \frac{\beta_7^{2/5}}{\alpha_1^{2/5}(\beta_1\gamma_1\theta_1)^{4/5}} e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{27}$. Now let $\gamma_5 \neq 0$. Then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{28}$.

Case 1.1.2: Let $\theta_3 \neq 0$. Then with the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = e_3, \delta_4 = \theta_3 e_4 - \gamma_5 e_5, \delta_5 = e_5, \delta_6 = e_6$ we can make $\gamma_5 = 0$. Also with the base change $\delta_1 = \theta_3 e_1 - \theta_1 e_2, \delta_2 = e_2, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ we can make $\theta_1 = 0$. If $\beta_6 = 0$ then with the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = \theta_3 e_3 - \beta_7 e_5, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ we can make $\beta_7 = 0$. Then the base change $\delta_1 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_3} e_1, \delta_2 = e_2, \delta_3 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_3}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^2}[\alpha_1(\beta_1 e_4 + \beta_2 e_5 + \beta_3 e_6) + \alpha_2(\gamma_1 e_5 + \gamma_2 e_6)], \delta_5 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^3}\alpha_1\beta_1(\gamma_1 e_5 + \gamma_2 e_6), \delta_6 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^2} e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{29}$. If $\beta_6 \neq 0$ then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{30}$.

Case 1.2: Let $\beta_5 \neq 0$. Then with the base change $\delta_1 = \beta_5 e_1 - \beta_1 e_2, \delta_2 = e_2, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ we can make $\beta_1 = 0$. Using the equations (2.9) we obtain $\gamma_1 = 0 = \beta_9 = \theta_1 + \theta_2 = \theta_3 + \theta_4 = \gamma_2 + \gamma_3 = \beta_3 + \beta_4 = \beta_7 + \beta_8 = \gamma_5 + \gamma_6$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1, e_1] = e_6, [e_1, e_2] = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5 = -[e_2, e_1], [e_1, e_3] = \beta_2 e_5 + \beta_3 e_6 = -[e_3, e_1], [e_2, e_3] = \beta_5 e_4 + \beta_6 e_5 + \beta_7 e_6 = -[e_3, e_2], [e_1, e_4] = \gamma_2 e_6 = -[e_4, e_1], [e_2, e_4] = \gamma_4 e_5 + \gamma_5 e_6 = -[e_4, e_2], [e_1, e_5] = \theta_1 e_6 = -[e_5, e_1], [e_2, e_5] = \theta_3 e_6 = -[e_5, e_2]$.

Case 1.2.1: Let $\theta_1 = 0$. Then $\theta_3 \neq 0$ since $\dim(\mathcal{A}^5) = 1$.
First suppose $\gamma_2 = 0$. Then from (2.9) we have $\beta_2 = 0$. If $\beta_3 = 0$ then the base change $\delta_1 = \alpha_1\beta_5\gamma_4\theta_3 e_1, \delta_2 = e_2, \delta_3 = \alpha_1\beta_5\gamma_4\theta_3(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \alpha_1\beta_5\gamma_4\theta_3[\alpha_1(\beta_5 e_4 + \beta_6 e_5 + \beta_7 e_6) + \alpha_2(\gamma_4 e_5 + \gamma_5 e_6) + \alpha_3\theta_3 e_6], \delta_5 = \alpha_1\beta_5\gamma_4\theta_3[\alpha_1\beta_5(\gamma_4 e_5 + \gamma_5 e_6) + \theta_3(\alpha_1\beta_6 + \alpha_2\gamma_4)e_6], \delta_6 = (\alpha_1\beta_5\gamma_4\theta_3)^2 e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{31}$. If $\beta_3 \neq 0$ then the base change $\delta_1 = \alpha_1\beta_5\gamma_4\theta_3 e_1, \delta_2 = \frac{1}{\alpha_1\beta_3} e_2, \delta_3 = \frac{\beta_5\gamma_4\theta_3}{\beta_3}(\alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5), \delta_4 = \frac{\beta_5\gamma_4\theta_3}{\alpha_1\beta_3^2}[\alpha_1(\beta_5 e_4 + \beta_6 e_5 + \beta_7 e_6) + \alpha_2(\gamma_4 e_5 + \gamma_5 e_6) + \alpha_3\theta_3 e_6], \delta_5 = \frac{\beta_5\gamma_4\theta_3}{\alpha_1^2\beta_3^3}[\alpha_1\beta_5(\gamma_4 e_5 + \gamma_5 e_6) + \theta_3(\alpha_1\beta_6 + \alpha_2\gamma_4)e_6], \delta_6 = (\alpha_1\beta_5\gamma_4\theta_3)^2 e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{32}$. Now let $\gamma_2 \neq 0$. Then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{33}$.

Case 1.2.2: Let $\theta_1 \neq 0$. Then with the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = e_3, \delta_4 = \theta_1 e_4 - \gamma_2 e_5, \delta_5 = e_5, \delta_6 = e_6$ we can make $\gamma_2 = 0$. Suppose $\theta_3 = 0$. If $\beta_2 = 0$ then with the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = \theta_1 e_3 - \beta_3 e_5, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ we can make $\beta_3 = 0$. Then the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = \alpha_1 e_3 + \alpha_2 e_4 + \alpha_3 e_5, \delta_4 = \alpha_1(\beta_5 e_4 + \beta_6 e_5 + \beta_7 e_6) + \alpha_2(\gamma_4 e_5 + \gamma_5 e_6) + \alpha_3\theta_3 e_6, \delta_5 = \alpha_1\beta_5(\gamma_4 e_5 + \gamma_5 e_6) + (\alpha_1\beta_6 + \alpha_2\gamma_4)\theta_3 e_6, \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{34}$. If $beta_2 \neq 0$ then w.s.c.o.b. A

is isomorphic to $\mathcal{A}_{35}$. Now suppose $\theta_3 \neq 0$. Then w.s.c.o.b. $\mathcal{A}$ is isomorphic to $\mathcal{A}_{36}(\alpha)$.

Case 2: If the matrix $N = (iii)$, then the nontrivial multiplications in $\mathcal{A}$ given as follows:
$[e_1,e_1] = e_6, [e_1,e_2] = \alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5 = -[e_2,e_1], [e_2,e_2] = e_6, [e_1,e_3] = \beta_1e_4 + \beta_2e_5 + \beta_3e_6, [e_3,e_1] = -\beta_1e_4 - \beta_2e_5 + \beta_4e_6, [e_2,e_3] = \beta_5e_4 + \beta_6e_5 + \beta_7e_6, [e_3,e_2] = -\beta_5e_4 - \beta_6e_5 + \beta_8e_6, [e_3,e_3] = \beta_9e_6, [e_1,e_4] = \gamma_1e_5 + \gamma_2e_6, [e_4,e_1] = -\gamma_1e_5 + \gamma_3e_6, [e_2,e_4] = \gamma_4e_5 + \gamma_5e_6, [e_4,e_2] = -\gamma_4e_5 + \gamma_6e_6, [e_1,e_5] = \theta_1e_6, [e_5,e_1] = \theta_2e_6, [e_2,e_5] = \theta_3e_6, [e_5,e_2] = \theta_4e_6$.
From Leibniz identities we get the equations (2.9) again.

Case 2.1: Let $\beta_5 = 0$. Then $\beta_1 \neq 0$ since $\dim(A^3) = 3$. Using the equations (2.9) we obtain $\gamma_4 = 0 = \beta_9 = \theta_1 + \theta_2 = \theta_3 + \theta_4 = \gamma_2 + \gamma_3 = \beta_3 + \beta_4 = \beta_7 + \beta_8 = \gamma_5 + \gamma_6$. Then the nontrivial multiplications in $\mathcal{A}$ given by

$$[e_1,e_1] = e_6, [e_1,e_2] = \alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5 = -[e_2,e_1], [e_2,e_2] = e_6, [e_1,e_3] = \beta_1e_4 + \beta_2e_5 + \beta_3e_6 = -[e_3,e_1],$$
$$[e_2,e_3] = \beta_6e_5 + \beta_7e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1e_5 + \gamma_2e_6 = -[e_4,e_1], [e_2,e_4] = \gamma_5e_6 = -[e_4,e_2],$$
$$[e_1,e_5] = \theta_1e_6 = -[e_5,e_1], [e_2,e_5] = \theta_3e_6 = -[e_5,e_2] \quad (2.10)$$

Case 2.1.1: Let $\theta_3 = 0$. Then $\theta_1 \neq 0$ since $\dim(A^5) = 1$.
First suppose $\gamma_5 = 0$. Then from (2.9) we have $\beta_6 = 0$. Then the base change $\delta_1 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_1)^{1/3}}e_1, \delta_2 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_1)^{1/3}}e_2, \delta_3 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_1)^{2/3}}(\alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5), \delta_4 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_1}[\alpha_1(\beta_1e_4 + \beta_2e_5 + \beta_3e_6) + \alpha_2(\gamma_1e_5 + \gamma_2e_6) + \alpha_3\theta_1e_6], \delta_5 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_1)^{4/3}}[\alpha_1\beta_1(\gamma_1e_5 + \gamma_2e_6) + \theta_1(\alpha_1\beta_2 + \alpha_2\beta_1)e_6], \delta_6 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_1)^{2/3}}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{37}(\alpha)$. Now let $\gamma_5 \neq 0$. Then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{38}(\alpha,\beta)$.

Case 2.1.2: Let $\theta_3 \neq 0$. Then with the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = e_3, \delta_4 = \theta_3e_4 - \gamma_5e_5, \delta_5 = e_5, \delta_6 = e_6$ we can make $\gamma_5 = 0$. If $\beta_6 = 0$ then with the base change $\delta_1 = e_1, \delta_2 = e_2, \delta_3 = \theta_3e_3 - \beta_7e_5, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ we can make $\beta_7 = 0$. Then the base change $\delta_1 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^{1/3}}e_1, \delta_2 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^{1/3}}e_2, \delta_3 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^{2/3}}(\alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5), \delta_4 = \frac{1}{\alpha_1\beta_1\gamma_1\theta_3}[\alpha_1(\beta_1e_4 + \beta_2e_5 + \beta_3e_6) + \alpha_2(\gamma_1e_5 + \gamma_2e_6) + \alpha_3\theta_1e_6], \delta_5 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^{4/3}}[\alpha_1\beta_1(\gamma_1e_5 + \gamma_2e_6) + \theta_1(\alpha_1\beta_2 + \alpha_2\beta_1)e_6], \delta_6 = \frac{1}{(\alpha_1\beta_1\gamma_1\theta_3)^{2/3}}e_6$ shows $\mathcal{A}$ is isomorphic to $\mathcal{A}_{39}(\alpha)$. If $\beta_6 \neq 0$ then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{40}(\alpha,\beta)$.

Case 2.2: Let $\beta_5 \neq 0$. If $\beta_1 = 0$ then the base change $\delta_1 = e_2, \delta_2 = e_1, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.10). Hence A is isomorphic to $\mathcal{A}_{37}(\alpha), \mathcal{A}_{38}(\alpha,\beta), \mathcal{A}_{39}(\alpha)$ or $\mathcal{A}_{40}(\alpha,\beta)$. Now let $\beta_1 \neq 0$. If $\beta_1^2 + \beta_5^2 \neq 0$ then the base change $\delta_1 = \beta_1e_1 + \beta_5e_2, \delta_2 = \beta_5e_1 - \beta_1e_2, \delta_3 = e_3, \delta_4 = e_4, \delta_5 = e_5, \delta_6 = (\beta_1^2 + \beta_5^2)e_6$ shows $\mathcal{A}$ is isomorphic to an algebra with the nonzero multiplications given by (2.10). Thus, A is isomorphic to $\mathcal{A}_{37}(\alpha), \mathcal{A}_{38}(\alpha,\beta), \mathcal{A}_{39}(\alpha)$ or $\mathcal{A}_{40}(\alpha,\beta)$. Now consider the case $\beta_1^2 + \beta_5^2 = 0$. Then by (2.9) we get $\gamma_1^2 + \gamma_4^2 = 0$. Using the equations (2.9) we obtain $\beta_9 = 0 = \theta_1 + \theta_2 = \theta_3 + \theta_4 = \gamma_2 + \gamma_3 = \beta_3 + \beta_4 = \beta_7 + \beta_8 = \gamma_5 + \gamma_6$. Then the nontrivial multiplications in $\mathcal{A}$ given as the following:
$[e_1,e_1] = e_6, [e_1,e_2] = \alpha_1e_3 + \alpha_2e_4 + \alpha_3e_5 = -[e_2,e_1], [e_2,e_2] = e_6, [e_1,e_3] = \beta_1 + \beta_2e_5 + \beta_3e_6 = -[e_3,e_1], [e_2,e_3] = \beta_5e_4 + \beta_6e_5 + \beta_7e_6 = -[e_3,e_2], [e_1,e_4] = \gamma_1e_5 + \gamma_2e_6 = -[e_4,e_1], [e_2,e_4] = \gamma_4e_5 + \gamma_5e_6 = -[e_4,e_2], [e_1,e_5] = \theta_1e_6 = -[e_5,e_1], [e_2,e_5] = \theta_3e_6 = -[e_5,e_2]$.
If $\theta_3 = 0$ then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{41}(\alpha,\beta)$. If $\theta_3 \neq 0$ then w.s.c.o.b. $\mathcal{A}$ isomorphic to $\mathcal{A}_{42}(\alpha,\beta)$. □

3. Conclusion

We obtain 32 single algebras, 5 one-parameter infinite families and 5 two-parameter infinite families of classes of complex non-Lie nilpotent Leibniz algebras of dimension 6 with two dimensional derived algebra and Leib ideal is one dimensional. Taking into account the fact that all the isomorphism classes of $6-$dimensional complex nilpotent Lie algebras consist of only 20 single algebras, it can be deduced that the classification problem for Leibniz algebras is wild. Nonetheless extending this bilinear forms technique to higher dimensions, the classification of complex nilpotent Leibniz algebras with two dimensional derived algebra can be given.

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