Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2020) 44: 1595 – 1611
© TÜBİTAK
doi:10.3906/mat-2003-20

On basicity of the system of eigenfunctions of one discontinuous spectral problem for second order differential equation for grand-Lebesgue space

Yusuf ZEREN[1,*], **Migdad ISMAILOV**[2], **Fatih ŞİRİN**[3]
[1]Department of Mathematics, Faculty of Science and Literature, Yıldız Technical University, İstanbul, Turkey
[2]Baku State University, Institute of Mathematics and Mechanics of the NAS of Azerbaijan, Baku, Azerbaijan
[3]Department of Mathematics, Institute of Science, Yıldız Technical University, İstanbul, Turkey

Received: 04.03.2020 • **Accepted/Published Online:** 10.06.2020 • **Final Version:** 21.09.2020

Abstract: Basicity of the system of eigenfunctions of some discontinuous spectral problem for a second order differential equation with spectral parameter in boundary condition for grand-Lebesgue space $L_{p)}(-1;1)$ is studied in this work. Since the space is nonseparable, a subspace suitable for the spectral problem is defined. The subspace $G_{p)}(-1;1)$ of $L_{p)}(-1;1)$ generated by shift operator is considered. Basicity of the system of eigenfunctions for the space $G_{p)}(-1;1)\oplus C$, $1<p<+\infty$, is proved. It is shown that the system of eigenfunctions of considered problem forms a basis for $G_{p)}(-1;1)$, $1<p<+\infty$, after removal of any of its even-numbered functions.

Key words: Grand Lebesgue space, eigenfunctions, basicity, completeness, minimality, discontinuous spectral problem

1. Introduction

It is known from [34] that the solution of the problem of vibrating string fixed at one or both ends with a mass in the middle is reduced to the solution of discontinuous spectral problem of the form

$$y''(x)+\lambda y(x)=0, x\in(-1;0)\bigcup(0;1), \tag{1.1}$$

$$\left.\begin{array}{l} y(-1)=y(1)=0,\\ y(-0)=y(+0),\\ y'(-0)-y'(+0)=\lambda m y(0), m\neq 0. \end{array}\right\} \tag{1.2}$$

The problem (1.1)–(1.2) has the following series of eigenvalues [17]:

$$\lambda_{1,n}=(\pi n)^2, n=1,2,...,$$

$$\lambda_{2,n}=\rho_{2,n}^2, n=0,1,2,...,$$

where $\rho_{2,n}$ has asymptotics $\rho_{2,n}=\pi n+\frac{2}{\pi m n}+O(\frac{1}{n^2})$, and the corresponding eigenfunctions have the form of

$$u_{2n-1}(x)=\sin\pi n x, n=1,2,...,$$

*Correspondence: yzeren@yildiz.edu.tr, yusufzeren@hotmail.com
2010 *AMS Mathematics Subject Classification:* 46E30, 46E35, 34B05, 34B24, 34L20

$$u_{2n}(x)=\begin{cases}\sin\rho_{2,n}(1+x), x\in[-1;0]\\ \sin\rho_{2,n}(1-x), x\in[0;1]\end{cases}, n=0,1,2,....$$

It is also proved in [17] that the system of vectors

$$\hat{u}_{2n-1}(x)=(u_{2n-1}(x);0), n=1,2,...,$$

$$\hat{u}_{2n}(x)=(u_{2n}(x);m\sin\rho_{2,n}), n=0,1,2,...,$$

forms a basis for the space $L_p(-1;1)\oplus C$, $1<p<+\infty$, and for $p=2$ this basis becomes a Riesz basis. Besides, a biorthogonal system is constructed in [17] for the system $\{u_n\}_{n=0}^{\infty}$. Further developments with different methods have been proposed in [6, 16, 18, 19, 25, 32, 33] by different methods. In particular, in [16] the spectral problem (1.1)–(1.2) was considered in weighted Lebesgue spaces. Method of the theory of basis perturbations is usually used to solve such problems. Basicity problems of some perturbed trigonometric systems in Morrey-type spaces have been considered in [2–5, 7, 9–12, 27, 28].

Recently, there has been an increase in interest in various nonstandard spaces in the context of applications to different areas of mathematics. Among those spaces, we can mention Lebesgue spaces with variable summability index, Morrey spaces, grand-Lebesgue spaces, etc. Many classical facts of harmonic analysis such as boundedness problems of singular operator with a Cauchy kernel, maximal function, Hilbert transform have been extended to these spaces (for more details see, e.g., [1, 8, 14, 15, 20, 26, 29–31, 35–39], etc.). These results stimulate consideration of the problems of theory of differential equations, theory of partial differential equations, etc. in these spaces. To do so, of course you need to study the basis properties of eigenfunctions of corresponding differential operators in the considered spaces. In case of nonseparable spaces, you need to consider separable subspaces suited for your differential equation.

This work deals with the basis properties of the system of eigenfunctions of the problem (1.1)–(1.2) in the spaces $L_{p)}(-1;1)$. Due to the nonseparability of the space $L_{p)}(-1;1)$, a suitable subspace is considered for the spectral problem. In $L_{p)}(-1;1)$, $G_{p)}(-1;1)$ subspace where the set of continuous functions are dense is defined. Using the method of [17], the basicity of the system of vectors $\{\hat{u}_n\}_{n=0}^{\infty}$ for the space $G_{p)}(-1;1)\oplus C$, $1<p<+\infty$, is established. Also, it is proved that the system $\{u_n\}_{n=0}^{\infty}$ becomes a basis for $G_{p)}(-1;1)$ if any of its even-numbered functions is excluded.

2. Useful preliminaries and auxiliary results

We will use the following notations. N is the set of positive integers, $Z_+=\{0\}\bigcup N$ denotes a set of nonnegative integers, C is a set of complex numbers, δ_{nk} is a Kronecker symbol, $\overline{M}$ is a closure of the set M in a corresponding space, and $L(M)$ denotes a linear span of M.

Let X and Y be Banach spaces with the norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. The value of the functional $f:X\to C$ at the point $x\in X$ will be denoted by $<x,f>$. By $D(A)$ and $\mathrm{Z}(A)$ we denote the domain and the kernel, respectively, of the linear operator $A:X\to Y$. The operator $A:X\to Y$ is said to be densely defined if $\overline{D(A)}=X$. By $X\oplus Y$ we denote a direct sum of the spaces X and Y. The space $X\oplus Y$ is a Banach space equipped with the norm

$$\|(x,y)\|_{X\oplus Y}=\|x\|_X+\|y\|_Y, (x,y)\in X\oplus Y.$$

The linear operator $A : X \to Y$ is called closed if its graph $G(A) = \{(x, Ax), x \in D(A)\}$ is a closed set in $X \oplus Y$. The closedness of the linear operator $A : X \to Y$ is equivalent to saying that, for any $x_n \in D(A)$, x_n tending to x and Ax_n tending to y imply $x \in D(A)$ and $Ax = y$. Also, the closedness of the linear operator $A : X \to Y$ is equivalent to the density of $D(A)$ with respect to the graph norm

$$\|x\|_A = \|x\|_X + \|Ax\|_Y , x \in D(A).$$

Let the linear operator $B : X \to Y$ be an extension of the linear operator $A : X \to Y$. If $\dim D(B)/D(A) = m$, then the operator B is called an m-multiple extension of the operator A and denoted as $A \subset {}^m B$, where $D(B)/D(A)$ is a factor-space of the space $D(B)$ with respect to the subspace $D(A)$. The linear functional $f : D(A) \to C$, continuous with respect to the graph norm of the linear operator $A : X \to Y$, is called an A-boundary form. The space of A-boundary forms is denoted by $D(A)'$.

We will need the following result.

Theorem 2.1 *[23] Let X and Y be Banach spaces, and the linear operator $A : X \to Y$ be closed. Let U' be a subspace of the space $D(A)'$ of dimension $m < +\infty$, not containing nonzero functionals continuous with respect to the norm $\|\cdot\|_X$. Let*

$$D = \{x \in D(A) : \forall f \in U', f(x) = 0\} .$$

Then the restriction $A|_D$ of the operator A to D is a closed densely defined operator and $A|_D \subset {}^m A$.

More details on these facts can be found in the monograph [23].

Let us recall some concepts and facts from the theory of grand-Lebesgue spaces. Denote by $L_{p)}(-\pi; \pi)$, $1 < p < +\infty$, a grand-Lebesgue space of measurable functions f on $[-\pi; \pi]$ satisfying the condition

$$\|f\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t)|^{p-\varepsilon} \, dt \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

The space $L_{p)}(-\pi; \pi)$ is a complete normed space with the norm $\|f\|_{p)}$ (see [13]). The space $L_{p)}(-\pi; \pi)$ is nonseparable. In fact, consider a family of functions

$$f_\alpha(t) = \begin{cases} 0, & 0 \le t \le \alpha, \\ (t-\alpha)^{-\frac{1}{p}}, & \alpha < t \le 1, \end{cases} .$$

$\alpha \in [0; 1)$. We have $\{f_\alpha\} \subset L_{p)}(0; 1)$. In fact,

$$\|f_\alpha\|_{p)} = \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_{\alpha}^{1} (t-\alpha)^{-1+\frac{\varepsilon}{p}} \, dt \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon<p-1} \left(p (t-\alpha)^{\frac{\varepsilon}{p}} |_{\alpha}^{1} \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} \left(p (1-\alpha)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{p-\varepsilon}} < +\infty.$$

For any different $\alpha, \beta \in [0; 1)$ with $\alpha < \beta$ we have

$$\|f_\alpha - f_\beta\|_{p)} \ge \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_{\alpha}^{\beta} (t-\alpha)^{-1+\frac{\varepsilon}{p}} \, dt \right)^{\frac{1}{p-\varepsilon}} = \sup_{0<\varepsilon<p-1} \left(p (t-\alpha)^{\frac{\varepsilon}{p}} |_{\alpha}^{\beta} \right)^{\frac{1}{p-\varepsilon}} =$$

$$= \sup_{0<\varepsilon<p-1} \left(p(\beta-\alpha)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{p-\varepsilon}} \geq \lim_{\varepsilon \to +0} \left(p(\beta-\alpha)^{\frac{\varepsilon}{p}} \right)^{\frac{1}{p-\varepsilon}} = p^{\frac{1}{p}}.$$

This directly implies the nonseparability of the space $L_{p)}(0;1)$.

Let us consider a separable subspace of the space $L_{p)}(-\pi;\pi)$ as follows. Consider a shift operator for $\forall\delta > 0$

$$T_\delta f(x) = \begin{cases} f(x+\delta), x+\delta \in [-\pi;\pi], \\ 0, x+\delta \in R\backslash[-\pi;\pi], \end{cases} \quad f \in L_{p)}(-\pi;\pi),$$

and a linear manifold $\tilde{G}_{p)}(-\pi;\pi)$ of functions $f \in L_{p)}(-\pi;\pi)$ satisfying the condition

$$\|T_\delta f - f\|_{p)} \to 0, \delta \to 0.$$

Let $G_{p)}(-\pi;\pi)$ be a closure of $\tilde{G}_{p)}(-\pi;\pi)$ in $L_{p)}(-\pi;\pi)$. We prove the following lemma.

Lemma 2.2 *Continuous imbedding $L_p(-\pi;\pi) \subset G_{p)}(-\pi;\pi)$ holds, and this imbedding is strict, i.e.*

$$G_{p)}(-\pi;\pi)\backslash L_p(-\pi;\pi) \neq \emptyset.$$

Proof Obviously, $L_p(-\pi;\pi) \subset G_{p)}(-\pi;\pi)$. The continuity of this imbedding follows from the inequality

$$\|f\|_{G_{p)}} = \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}} \|f\|_{L_{p-\varepsilon}} \leq \|f\|_{L_p} \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{p-\varepsilon}} 2^{\frac{\varepsilon}{p(p-\varepsilon)}} \leq 2^{-\frac{1}{p}}(p-1)\|f\|_{L_p}.$$

It only remains to prove the validity of the relation $G_{p)}(-\pi;\pi)\backslash L_p(-\pi;\pi) \neq \emptyset$. To do so, it suffices to find a function from $G_{p)}(0;1)$ not contained in $L_p(0;1)$. Consider a sequence of functions

$$f_n(t) = \begin{cases} t^{-\frac{1}{p}}, t \in [\exp\{-n^{2p}\};1] \\ 0, t \notin [\exp\{-n^{2p}\};1]. \end{cases}$$

For the norms $\|f_n\|_{p)}$ and $\|f_n\|_p$ we have

$$\|f_n\|_{p)} \leq \sup_{0<\varepsilon<p-1} \left(\varepsilon \int_0^1 t^{-1+\frac{\varepsilon}{p}} dt \right)^{\frac{1}{p-\varepsilon}} = p,$$

$$\|f_n\|_p = \left(\int_{\exp\{-n^{2p}\}}^1 t^{-1} dt \right)^{\frac{1}{p}} = n^2.$$

Then it is clear that the convergence of the series $\sum_{n=1}^{\infty} \frac{\|f_n\|_{p)}}{n^2}$ implies the convergence in $L_{p)}(0;1)$ of the series $\sum_{n=1}^{\infty} \frac{f_n(t)}{n^2}$. Let $f(t)$ be the sum of this series. Obviously, $f \in G_{p)}(0;1)$. Let us show that $f \notin L_p(0;1)$. Let

$$S_m(t) = \sum_{n=1}^{m} \frac{f_n(t)}{n^2}.$$

As S_m converges to f in $L_{p-\varepsilon}(0;1)$ as $m \to \infty$, from the monotonic nondecreasing of S_m it follows that S_m converges almost everywhere to f as $m \to \infty$. Then $|S_m|^p$ converges almost everywhere to $|f|^p$ as $m \to \infty$. By Levi theorem, we have

$$\int_0^{2\pi} |S_m(t)|^p \, dt \to \int_0^{2\pi} |f(t)|^p \, dt, m \to \infty.$$

On the other hand, from

$$\int_0^{2\pi} |S_m(t)|^p \, dt = \int_0^{2\pi} \left| \sum_{n=1}^{m} \frac{f_n(t)}{n^2} \right|^p dt \geq \sum_{n=1}^{m} \frac{\int_0^{2\pi} |f_n(t)|^p \, dt}{n^{2p}} = \sum_{n=1}^{m} \frac{n^{2p}}{n^{2p}} = m$$

it follows that $\int_0^{2\pi} |S_m(t)|^p \, dt \to +\infty$ as $m \to \infty$. Consequently, $\int_0^{2\pi} |f(t)|^p \, dt = +\infty$. The lemma is proved. □

The following lemma is also true.

Lemma 2.3 *The space* $C_0^{\infty}[-\pi;\pi]$ *is dense in* $G_{p)}(-\pi;\pi)$, $1 < p < +\infty$.

Proof Consider an arbitrary number $\eta > 0$ and an arbitrary function $f \in G_{p)}(-\pi;\pi)$. Denote by $\omega_\eta(t)$ the following kernel:

$$\omega_\eta(t) = \begin{cases} c_\eta \exp(-\frac{\eta^2}{\eta^2 - t^2}), |t| \leq \eta, \\ 0, |t| > \eta, \end{cases} ,$$

where the constant c_η is such that $\int_{-\infty}^{+\infty} \omega_\eta(t) dt = 1$. Let the function $f_\eta(\cdot)$ be a convolution with kernel $\omega_\eta(\cdot)$, i.e.

$$f_\eta(t) = \int_{-\infty}^{+\infty} f(t-s)\omega_\eta(s) ds = \int_{-\infty}^{+\infty} \omega_\eta(t-s) f(s) ds.$$

The correctness of such definition follows from the inclusion $L_{p)}(-\pi;\pi) \subset L_1(-\pi;\pi)$. Obviously, $f_\eta(t)$ is an infinitely differentiable function. Using Minkowski's inequality, we obtain

$$\|f - f_\eta\|_{p)} = \left\| \int_{-\infty}^{+\infty} f(\cdot)\omega_\eta(s) ds - \int_{-\infty}^{+\infty} f(\cdot - s)\omega_\eta(s) ds \right\|_{p)} =$$

$$= \left\| \int_{-\infty}^{+\infty} [f(\cdot - s) - f(\cdot)]\omega_\eta(s) ds \right\|_{p)} =$$

$$= \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} \left| \int_{-\infty}^{+\infty} [f(t-s) - f(t)]\omega_\eta(s) ds \right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \int_{-\infty}^{+\infty} \omega_\eta(s) \sup_{0<\varepsilon<p-1} \left(\frac{\varepsilon}{2\pi} \int_{-\pi}^{\pi} |f(t-s) - f(t)|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} ds \leq$$

$$\leq e_p \sup_{|s|\leq\eta} \|f(\cdot - s) - f(\cdot)\|_{p)} \to 0, \eta \to 0.$$

Consequently, $C^{\infty}[-\pi;\pi]$ is dense in $G_{p)}(-\pi;\pi)$.

Consider an arbitrary $\eta > 0$ and an arbitrary function $f \in G_{p)}(-\pi;\pi)$. As proved above, there exists $g \in C^{\infty}[-\pi;\pi]$ such that

$$\|f - g\|_{p)} < \frac{\eta}{3}. \tag{2.1}$$

Let us choose the number $\delta > 0$ such that $\delta < \pi\left(\frac{\eta}{3e_p\|g\|_{\infty}}\right)^p$, $e_p = p-1$. Consider the intervals $E_{\delta}^{+} = (\pi - \delta;\pi)$ and $E_{\delta}^{-} = (-\pi;-\pi+\delta)$ of length δ and define the function

$$g_{\delta}(t) = \begin{cases} g(t), & t \in (-\pi;\pi)\backslash(E_{\delta}^{+} \bigcup E_{\delta}^{-}), \\ 0, & t \in \ E_{\delta}^{+} \bigcup E_{\delta}^{-}. \end{cases}$$

We have

$$\|g - g_{\delta}\|_{p)} = \sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}\int_{E_{\delta}^{+}\bigcup E_{\delta}^{-}}|g(t)|^{p-\varepsilon}dt\right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \|g\|_{\infty}\sup_{0<\varepsilon<p-1}\left(\frac{\varepsilon}{2\pi}2\delta\right)^{\frac{1}{p-\varepsilon}} < \frac{\|g\|_{\infty}\,e_p\delta^{\frac{1}{p}}}{\pi^{\frac{1}{p}}} < \frac{\eta}{3}. \tag{2.2}$$

Let

$$g_{\delta,\tau}(t) = \int_{-\infty}^{+\infty} g_{\delta}(t-s)\omega_{\tau}(s)ds, \tau \in R.$$

Obviously, for $\tau < \frac{\delta}{2}$ we have $g_{\delta,\tau} \in C_0^{\infty}[-\pi;\pi]$. As $\|g_{\delta} - g_{\delta,\tau}\|_{p)} \to 0$ for $\tau \to 0$, there exists $\tau < \frac{\delta}{2}$ such that

$$\|g_{\delta} - g_{\delta,\tau}\|_{p)} < \frac{\eta}{3}. \tag{2.3}$$

Consequently, using (2.1), (2.2) and (2.3), we obtain

$$\|f - g_{\delta,\tau}\|_{p)} \leq \|f - g\|_{p)} + \|g - g_{\delta}\|_{p)} + \|g_{\delta} - g_{\delta,\tau}\|_{p)} < \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta,$$

i.e. $C_0^{\infty}[-\pi;\pi]$ is dense in $G_{p)}(-\pi;\pi)$. The lemma is proved. □

3. Main results

Denote by $GW_{p)}^2(a;b)$, $1 < p < +\infty$, the subspace of grand-Sobolev space $W_{p)}^2(a;b)$ (see [31]) of functions $f \in W_{p)}^2(a;b)$ such that $f'' \in G_{p)}(a;b)$. Let

$$GW_{p)}^2((-1;0)\bigcup(0;1)) = GW_{p)}^2(-1;0) \oplus GW_{p)}^2(0;1).$$

We then prove the following:

Lemma 3.1 *Dirac delta functional $\delta_x(u) = u(x)$, $x \in (-1;1)$ is linear and bounded in $GW_{p)}^2(-1;1)$ and unbounded in $G_{p)}(-1;1)$, $1 < p < +\infty$.*

Proof Fix the point $x \in (-1;1)$. Then for $\forall t \in (-1;1)$ we have

$$|\delta_x(u)| = |u(x)| = \left| u(t) + \int_t^x u^{'}(s)ds \right| \leq |u(t)| + \int_t^x \left| u^{'}(s) \right| ds.$$

Integrating both sides of the last inequality with respect to t along the interval $[-1;1]$, we obtain

$$2\,|\delta_x(u)| \leq \int_{-1}^1 |u(t)|\,dt + \int_{-1}^1 \left(\int_t^x \left| u^{'}(s) \right| ds \right) dt \leq$$

$$\leq \int_{-1}^1 |u(t)|\,dt + 2\int_{-1}^1 \left| u^{'}(t) \right| dt.$$

Hence, for $\forall \varepsilon \in (0; p-1)$, using Hölder's inequality with the index $p-\varepsilon$, we obtain

$$|\delta_x(u)| \leq 2^{-\frac{1}{p-\varepsilon}} \left(\int_{-1}^1 |u(t)|^{p-\varepsilon}\, dt \right)^{\frac{1}{p-\varepsilon}} + 2^{1-\frac{1}{p-\varepsilon}} \left(\int_{-1}^1 \left| u^{'}(t) \right|^{p-\varepsilon} dt \right)^{\frac{1}{p-\varepsilon}} \leq$$

$$\leq \varepsilon^{-\frac{1}{p-\varepsilon}} (\|u\|_{p)} + 2\left\| u^{'} \right\|_{p)}) \leq 2\varepsilon^{-\frac{1}{p-\varepsilon}} (\|u\|_{p)} + \left\| u^{'} \right\|_{p)}) =$$

$$\leq 2\varepsilon^{-\frac{1}{p-\varepsilon}} (\|u\|_{p)} + \left\| u^{'} \right\|_{p)} + \left\| u^{''} \right\|_{p)}) = 2\varepsilon^{-\frac{1}{p-\varepsilon}} \|u\|_{GW^2_{p)}},$$

i.e. the functional δ_x is linear and bounded in $GW^2_{p)}(-1;1)$.

Now let us establish the unboundedness of the functional δ_x in $G_{p)}(-1;1)$. Suppose the contrary, i.e. let δ_x be bounded in $G_{p)}(-1;1)$. Then there exists the number $M > 0$ such that

$$|\delta_x(u)| \leq M\,\|u\|_{p)}\,, \forall u \in G_{p)}(-1;1). \tag{3.1}$$

It is easy to show that $\forall u \in L_p(-1;1)$ the relation

$$\|u\|_{p)} \leq 2^{-\frac{1}{p}} (p-1)\,\|u\|_p \tag{3.2}$$

holds. Then from (3.1) and (3.2) we obtain

$$|\delta_x(u)| \leq 2^{-\frac{1}{p}} (p-1) M\,\|u\|_p\,, \forall u \in L_p(-1;1),$$

i.e. δ_x is bounded in $L_p(-1;1)$. But this contradicts the unboundedness of the functional δ_x in $L_p(-1;1)$. Thus, the functional δ_x is unbounded in $G_{p)}(-1;1)$. The lemma is proved. □

Consider in $G_{p)}(-1;1) \oplus C$ the operator A defined by

$$A(\hat{u}) = (-u^{''}, u^{'}(-0) - u^{'}(+0))$$

with domain

$$D(A) = \left\{ \hat{u} = (u,\alpha) : u \in GW^2_{p)}((-1;0) \bigcup (0;1)) \right\}.$$

We show that A is a closed operator. Let $\hat{u}_n = (u_n, \alpha_n) \in D(A)$ be an arbitrary sequence such that

$$\|\hat{u}_n - \hat{u}\|_{G_{p)} \oplus C} \to 0, n \to \infty,$$

$$\|A\hat{u}_n - \hat{v}\|_{G_{p)} \oplus C} \to 0 n \to \infty.$$

where $\hat{u} = (u, \alpha)$ and $\hat{v} = (v, \beta)$, respectively. Then it is clear that the sequence u_n converges in $G_{p)}(-1;1)$ to u, and the sequence $u_n^{''}$ converges in $G_{p)}(-1;1)$ to $-v$. Consequently, the sequences u_n and $u_n^{''}$ converge, respectively, to u and $-v$ in the space $L_1(-1;1)$. As u_n and $u_n^{'}$ are absolutely continuous in $[-1;0]$, there exists $u^{'}$, and the functions u and $u^{'}$ are absolutely continuous in $[-1;0]$. Moreover, the sequence $u_n^{''}$ converges in $L_1(-1;0)$ to $u^{''}$. So $u^{''}(x) = -v(x)$ almost everywhere in $[-1;0]$, and therefore, $u^{''} \in G_{p)}(-1;0)$, i.e. $u \in GW_{p)}^2(-1;0)$. It can be similarly shown that $u \in GW_{p)}^2(0;1)$. From the obtained inclusions it follows that $\hat{u} \in D(A)$. On the other hand, the equality $\beta = u^{'}(-0) - u^{'}(+0)$ holds. Thus, $A(\hat{u}) = \hat{v}$, i.e. A is closed in $G_{p)}(-1;1) \oplus C$. Define the operator L in $G_{p)}(-1;1) \oplus C$ by the formula

$$L(\hat{u}) = (-u^{''}, u^{'}(-0) - u^{'}(+0)) \tag{3.3}$$

with domain

$$D(L) = \left\{ \hat{u} = (u, mu(0)) : u \in GW_{p)}^2((-1;0) \bigcup (0;1)), u(-1) = u(1) = 0, u(-0) = u(+0) \right\}. \tag{3.4}$$

Let us show that L is a closed densely defined operator. For $\forall \hat{u} = (u, \alpha) \in G_{p)}(-1;1) \oplus C$, let

$$F(\hat{u}) = mu(0) - \alpha,$$

$$U_1(\hat{u}) = U(u) = u(-1),$$

$$U_2(\hat{u}) = U(u) = u(1),$$

$$U_3(\hat{u}) = U(u) = u(-0) - u(+0).$$

Domain $D(L)$ can be written as follows:

$$D(L) = \left\{ \hat{u} \in G_{p)} \oplus C : \hat{u} = (u, \alpha) \in GW_{p)}^2((-1;0) \bigcup (0;1)), F(u) = 0, U_i(u) = 0, i = \overline{1,3} \right\}.$$

By Lemma 3.1, the linear functionals F and U_i, $i = 1, 2, 3$, are bounded in $GW_{p)}^2(-1;1) \oplus C$, but unbounded in $G_{p)}(-1;1) \oplus C$. Then, due to the closedness of the operator A, by Theorem 2.1 its restriction L to $D(L)$ is a closed densely defined operator in $G_{p)}(-1;1) \oplus C$. Thus, we have proved the following theorem.

Theorem 3.2 *Let the operator L be defined by the formula (3.3) with domain $D(L)$ defined by (3.4). Then the operator L is a closed densely defined operator in the space $G_{p)}(-1;1) \oplus C$, $1 < p < +\infty$.*

Remark 3.1 Note that Theorem 3.2 can also be proved with the use of the statement which says that L is a closed densely defined operator in $L_p(-1;1) \oplus C$, $1 < p < +\infty$ (see Lemma 3, [17]). This kind of proof requires the knowledge of corresponding properties of Dirac functional in these spaces, which is provided by Lemma 3.1 . This is of independent interest.

The eigenfunctions of the operator L coincide with those of the problem (1.1)–(1.2), and the corresponding eigenvectors are [17]

$$\hat{u}_{2n-1}(x) = (u_{2n-1}(x), 0), n \in N$$

$$\hat{u}_{2n}(x) = (u_{2n}(x), m \sin \rho_{2,n}), n \in Z_+.$$

We state and prove the following main theorem on the basicity of the system $\{\hat{u}_n\}_{n \in Z_+}$ for the space $G_{p)}(-1;1) \oplus C$.

Theorem 3.3 *The system of eigenvectors $\{\hat{u}_n\}_{n \in Z_+}$ of the operator L forms a basis for the space $G_{p)}(-1;1) \oplus C$, $1 < p < +\infty$.*

Proof We prove the conditions of basicity criterion for systems (see [32]). Let us first prove the completeness of the system $\{\hat{u}_n\}_{n \in Z_+}$. Consider an arbitrary vector $\hat{u} \in G_{p)}(-1;1) \oplus C$ and an arbitrary number $\eta > 0$. From Lemma 2.2 it follows the space $L_p(-1;1) \oplus C$ is dense in $G_{p)}(-1;1) \oplus C$. Consequently, there exists a vector $\hat{v} \in L_p(-1;1) \oplus C$ such that

$$\|\hat{u} - \hat{v}\|_{G_{p)} \oplus C} < \eta. \tag{3.5}$$

From the completeness of the system $\{\hat{u}_n\}_{n \in Z_+}$ in $L_p(-1;1) \oplus C$ (see Theorem 1, [1]) it follows that there exists a vector $\hat{w} \in L\left(\{\hat{u}_n\}_{n \in Z_+}\right)$ such that

$$\|\hat{v} - \hat{w}\|_{L_p \oplus C} < \eta. \tag{3.6}$$

From (3.2) and (3.6) we obtain

$$\|\hat{v} - \hat{w}\|_{G_{p)} \oplus C} \leq 2^{-\frac{1}{p}} (p-1) \|\hat{v} - \hat{w}\|_{L_p \oplus C} < 2^{-\frac{1}{p}} (p-1)\eta. \tag{3.7}$$

Then, applying the triangle inequality and taking into account the inequalities (3.5) and (3.7), we obtain

$$\|\hat{u} - \hat{w}\|_{G_{p)} \oplus C} \leq \|\hat{u} - \hat{v}\|_{G_{p)} \oplus C} + \|\hat{v} - \hat{w}\|_{G_{p)} \oplus C} \leq$$

$$\leq \eta + 2^{1-\frac{1}{p}} (p-1)\eta = M\eta,$$

i.e. the system $\{\hat{u}_n\}_{n \in Z_+}$ is complete in $G_{p)}(-1;1) \oplus C$.

We show the minimality of the system $\{\hat{u}_n\}_{n \in Z_+}$ in $G_{p)}(-1;1) \oplus C$. As shown in [17], the system $\{\hat{u}_n\}_{n \in Z_+}$ has a biorthogonal conjugate vector system $\{\hat{v}_n\}_{n \in Z_+}$: $\hat{v}_n(x) = (v_n(x), mv(0))$ in $L_p(-1;1) \oplus C$, where the functions $v_n(x)$, $n \in Z_+$ are the eigenfunctions of the corresponding conjugate spectral problem

$$v''(x) + \lambda v(x) = 0, x \in (-1;0) \bigcup (0;1),$$

$$\left.\begin{array}{l} v(-1) = v(1) = 0 \\ v(-0) = v(+0) \\ v'(-0) - v'(+0) = \lambda \bar{m} v(0) \end{array}\right\}$$

and have the form

$$v_{2n-1}(x) = \sin \pi n x, n \in N,$$

$$v_{2n}(x) = \begin{cases} c_{2n} \sin \bar{\rho}_{2,n}(1+x), x \in [-1;0] \\ c_{2n} \sin \bar{\rho}_{2,n}(1-x), x \in [0;1] \end{cases}, n \in Z_+, \tag{3.8}$$

with the normalized numbers c_{2n} satisfying

$$c_{2n} = 1 + O\left(\frac{1}{n^2}\right). \tag{3.9}$$

Moreover there exists a constant $a > 0$ independent of p and $n \in Z_+$ such that

$$|< \hat{u}, \hat{v}_n >| \leq a \left\| \hat{u} \right\|_{L_p \oplus C}.$$

Let us show that $\{\hat{v}_n\}_{n \in Z_+}$ is a biorthogonally conjugate system to $\{\hat{u}_n\}_{n \in Z_+}$ in $G_{p)}(-1;1) \oplus C$, defined by the formula

$$< \hat{u}, \hat{v}_n >= \int_{-1}^{1} u(x) \overline{v_n}(x) dx + \alpha m v_n(0), \hat{u} =< u, \alpha >.$$

In fact, for every fixed $\varepsilon \in (0; p-1)$ we have

$$|< \hat{u}, \hat{v}_n >| \leq a \left\| \hat{u} \right\|_{L_{p-\varepsilon} \oplus C} \leq a_1 \left\| \hat{u} \right\|_{G_{p)} \oplus C}. \tag{3.10}$$

Thus, the system $\{\hat{u}_n\}_{n \in Z_+}$ is minimal in $G_{p)}(-1;1) \oplus C$.

It remains to show the uniform boundedness of the sequence of projectors

$$S_n(\hat{f}) = \sum_{k=0}^{n} < \hat{f}, \hat{v}_n > \hat{u}_n, \hat{f} \in G_{p)}(-1;1) \oplus C,$$

in $G_{p)}(-1;1) \oplus C$. For fixed $\hat{f} \in G_{p)}(-1;1) \oplus C$, $\hat{f} = (f, \beta)$, consider the equation

$$L\hat{u} - \lambda \hat{u} = \hat{f}. \tag{3.11}$$

The equation (3.11) can be rewritten in the form of the following problem:

$$\begin{cases} u''(x) - \lambda u(x) = f(x) \\ u'(-0) - u'(+0) - \lambda m u(0) = \beta \\ U_i(u) = 0, i = 1,2,3 \end{cases}. \tag{3.12}$$

The problem (3.12) has a solution [Lemma4,[1]]

$$u(x,\rho) = \frac{\beta \sin \rho(1+x)}{\rho(2\cos\rho - \rho m \sin\rho)} - \frac{1}{\rho}\int_{-1}^{x} f(\xi)\sin\rho(x-\xi)d\xi +$$

$$+\frac{1}{\rho}\int_{x}^{0} f(\xi)\sin\rho(x-\xi)d\xi + \frac{1}{\Delta(\rho)}\int_{-}^{0} f(\xi)\sin\rho(1+x)\sin\rho(1+\xi)d\xi -$$

$$-\frac{1}{\rho\sin\rho}\int_{-1}^{x} f(\xi)\sin\rho(1+x)\sin\rho\xi d\xi-\frac{1}{\rho\sin\rho}\int_{x}^{0} f(\xi)\sin\rho x\sin\rho(1+\xi)d\xi+$$

$$+\frac{1}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1+x)\sin\rho(1-\xi)d\xi, x\in[-1;0]; \tag{3.13}$$

$$u(x,\rho)=\frac{\beta\sin\rho(1-x)}{\rho(2\cos\rho-\rho m\sin\rho)}-\frac{1}{\rho}\int_{0}^{x} f(\xi)\sin\rho(x-\xi)d\xi+$$

$$+\frac{1}{\rho}\int_{x}^{1} f(\xi)\sin\rho(x-\xi)d\xi+\frac{1}{\Delta(\rho)}\int_{0}^{1} f(\xi)\sin\rho(1-x)\sin\rho(1-\xi)d\xi+$$

$$+\frac{1}{\rho\sin\rho}\int_{0}^{x} f(\xi)\sin\rho x\sin\rho(1-\xi)d\xi+\frac{1}{\rho\sin\rho}\int_{x}^{1} f(\xi)\sin\rho(1-x)\sin\rho\xi d\xi+$$

$$+\frac{1}{\Delta(\rho)}\int_{-1}^{0} f(\xi)\sin\rho(1-x)\sin\rho(1+\xi)d\xi, x\in[0;1], \tag{3.14}$$

where $\lambda=\rho^2$ belongs to the resolvent set of the operator L. Consequently,

$$u(0,\rho)=\frac{1}{\rho(2\cos\rho-\rho m\sin\rho)}\times$$

$$\times\left[\beta\sin\rho+\int_{-1}^{0} f(\xi)\sin\rho(1+\xi)d\xi+\int_{0}^{1} f(\xi)\sin\rho(1-\xi)d\xi\right]. \tag{3.15}$$

Let

$$\Gamma_{2n-1}=\left\{\rho:|\rho-\rho_{1,n}|=\frac{1}{2\pi mn}\right\}, n\in N;$$

$$\Gamma_{2n}=\left\{\rho:|\rho-\rho_{2,n}|=\frac{1}{2\pi mn}\right\}, n\in Z_+;$$

$$C_n=\left\{\rho:|\rho|=\pi(n+\frac{1}{2}), 0\le\arg\rho\le\pi\right\}, n\in Z_+$$

and Γ'_{2n-1}, Γ'_{2n} and C'_n be the corresponding images in the mapping $\lambda=\rho^2$. Then, using the resolvent $R(\lambda)$ of the operator L, we can rewrite the operator S_n as follows:

$$S_n(\hat{f})=\sum_{k=0}^{n} E_k(\hat{f}),$$

where the sequence of projectors $\{E_n\}_{n\in Z_+}$ is defined by the formulas

$$E_{2n-1}\hat{f}=\frac{1}{2\pi i}\int_{\Gamma'_{2n-1}} R(\lambda)\hat{f}d\lambda=\frac{1}{\pi i}\int_{\Gamma_{2n-1}}\rho R(\rho^2)\hat{f}d\rho, n\in N;$$

$$E_{2n}\hat{f}=\frac{1}{2\pi i}\int_{\Gamma'_{2n}} R(\lambda)\hat{f}d\lambda=\frac{1}{\pi i}\int_{\Gamma_{2n}}\rho R(\rho^2)\hat{f}d\rho, n\in Z_+.$$

In view of $R(\rho^2)\hat{f} = \hat{u}(x,\rho) = (u(x,\rho); mu(0,\rho))$, we obtain

$$S_{2n-1}\hat{f}(x) = \frac{1}{2\pi i}\int_{C'_n} R(\lambda)\hat{f}(x)d\lambda = \frac{1}{\pi i}\int_{C_n} \rho R(\rho^2)\hat{f}(x)d\rho = \frac{1}{\pi i}\int_{C'_n} \rho\hat{u}(x,\rho)d\rho =$$

$$= \frac{1}{\pi i}\left(\int_{C_n} \rho u(x,\rho)d\rho; \int_{C_n} m\rho u(0,\rho)d\rho\right) = \frac{1}{\pi i}(J_n(x), mJ_n(0)), n \in Z_+.$$

It is known (see [24]) that

$$|\sin\rho| \le M_0 e^{|\rho|\sin\varphi}, \rho = |\rho|\sin\varphi, 0 \le \varphi \le \pi.$$

Furthermore, for sufficiently large values of $|\rho|$ outside the circles of some radius δ centered at the zeros of $\Delta_1(\rho) = 2\cos\rho - m\rho\sin\rho$, we have the following inequality:

$$|\Delta_1(\rho)| \ge M_1 |\rho| e^{|\rho|\sin\varphi}.$$

From these inequalities it follows that for $\rho = |\rho|\sin\varphi$, $0 \le \varphi \le \pi$, and sufficiently great $|\rho|$ outside the circles of some radius δ centered at the zeros of $\Delta_1(\rho)$, the following relation holds:

$$\left|\frac{\sin\rho(1+x)}{\Delta_1(\rho)}\right| \le \frac{M_0 e^{|\rho|x\sin\varphi}}{M_1|\rho|} \le \frac{M_2}{|\rho|}, \forall x \in [-1;0]. \tag{3.16}$$

By the equality (3.13), we have

$$J_n(0) = \beta\int_{C_n} \frac{\sin\rho}{\rho\Delta_1(\rho)}d\rho + \int_{C_n}\left(\int_{-1}^{0} f(\xi)\frac{\sin\rho(1+\xi)}{\rho\Delta_1(\rho)}d\xi\right)d\rho +$$

$$+ \int_{C_n}\left(\int_{0}^{1} f(\xi)\frac{\sin\rho(1-\xi)}{\rho\Delta_1(\rho)}d\xi\right)d\rho =$$

$$= \beta\int_{C_n} \frac{\sin\rho}{\rho\Delta_1(\rho)}d\rho + \int_{C_n}\left(\int_{-1}^{0} f(\xi)\frac{\sin\rho(1+\xi)}{\rho\Delta_1(\rho)}d\xi\right)d\rho + \int_{C_n}\left(\int_{-1}^{0} f(-\xi)\frac{\sin\rho(1+\xi)}{\rho\Delta_1(\rho)}d\xi\right)d\rho.$$

Hence, using (3.16) and Hölder's inequality with the index $p-\varepsilon$, $\varepsilon \in (0; p-1)$, we obtain

$$|J_n(0)| \le |\beta|\int_{C_n}\left|\frac{\sin\rho}{\rho\Delta_1(\rho)}\right||d\rho| + \int_{C_n}\left(\int_{-1}^{0}|f(\xi)|\left|\frac{\sin\rho(1+\xi)}{\rho\Delta_1(\rho)}\right|d\xi\right)|d\rho| +$$

$$+ \int_{C_n}\left(\int_{-1}^{0}|f(-\xi)|\left|\frac{\sin\rho(1+\xi)}{\rho\Delta_1(\rho)}\right|d\xi\right)|d\rho| \le$$

$$\le M_2\int_{C_n}\frac{1}{|\rho|}|d\rho|\left(|\beta| + \int_{-1}^{0}|f(\xi)|d\xi + \int_{0}^{1}|f(\xi)|d\xi\right) \le$$

$$\le \pi M_2\left(|\beta| + \left(\int_{-1}^{0}|f(\xi)|^{p-\varepsilon}d\xi\right)^{\frac{1}{p-\varepsilon}} + \left(\int_{0}^{1}|f(\xi)|^{p-\varepsilon}d\xi\right)^{\frac{1}{p-\varepsilon}}\right) \le$$

$$\leq 2\pi M_2\left(|\beta| + \|f\|_{L_{p-\varepsilon}(-1;1)}\right) \leq 2\pi M_2\left(|\beta| + \left(\frac{\varepsilon}{2}\right)^{-\frac{1}{p-\varepsilon}}\|f\|_{G_{p)}(-1;1)}\right) \leq$$

$$\leq M_3\left(|\beta| + \|f\|_{G_{p)}(-1;1)}\right). \tag{3.17}$$

Also, from the results obtained in [17] it follows, by (3.15), that there exists a constant $M_4 > 0$ independent of f and p such that for sufficiently large values of n the relation

$$|J_n(x)| \leq M_4 \int_{-1}^{0} \frac{|f(\xi)|}{-x-\xi} d\xi = M_4 H\left(g\right)(-x), \forall x \in [-1;0], \tag{3.18}$$

holds, where $g(x) = |f(-x)|$. As the Hilbert transform H is bounded in grand-Lebesgue space $L_{p)}(0;1)$ (see [31]), there exists a positive constant $M_5 > 0$ such that

$$\|H\left(g\right)\|_{G_{p)}(0;1)} \leq M_5\|g\|_{G_{p)}(0;1)} = M_5\|f\|_{G_{p)}(-1;0)}. \tag{3.19}$$

Thus, from (3.18) and (3.19) we obtain

$$\|J_n\|_{G_{p)}(-1;0)} \leq M_4\|H(g)\|_{G_{p)}(0;1)} \leq M_6\|f\|_{G_{p)}(-1;0)}, \tag{3.20}$$

where $M_6 = M_4 M_5$. By a similar argument, we can demonstrate that there exists a constant $M_7 > 0$ independent of f and p such that

$$\|J_n\|_{G_{p)}(0;1)} \leq M_7\|f\|_{G_{p)}(0;1)}. \tag{3.21}$$

Then, taking into account (3.20) and (3.21), we obtain

$$\|J_n\|_{G_{p)}(-1;1)} \leq \|J_n\|_{G_{p)}(-1;0)} + \|J_n\|_{G_{p)}(0;1)} \leq$$

$$\leq M_6\|f\|_{G_{p)}(-1;0)} + M_7\|f\|_{G_{p)}(0;1)} \leq M_8\|f\|_{G_{p)}(-1;1)}, \tag{3.22}$$

where $M_8 = M_6 + M_7$.

We now find a bound for the norm $\left\|S_{2n-1}(\hat{f})\right\|_{G_{p)}\oplus C}$. Using the relations (3.17) and (3.22), we obtain

$$\left\|S_{2n-1}(\hat{f})\right\|_{G_{p)}(-1;1)\oplus C} = \frac{1}{\pi}\left(\|J_n\|_{G_{p)}(-1;1)} + |mJ_n(0)|\right) \leq$$

$$\leq \frac{1}{\pi}\left(M_8\|f\|_{G_{p)}(-1;1)} + |m|\, M_3\left(|\beta| + \|f\|_{G_{p)}(-1;1)}\right)\right) \leq M\left\|\hat{f}\right\|_{G_{p)}\oplus C}, \tag{3.23}$$

where $M = \frac{2(M_8+|m|M_4)}{\pi}$. It remains to estimate $\left\|S_{2n}(\hat{f})\right\|_{G_{p)}\oplus C}$. Let us rewrite $S_{2n}\hat{f}$ in the form

$$S_{2n}\hat{f} = S_{2n-1}\hat{f} + < \hat{f}, \hat{v}_{2n} > \hat{u}_{2n}.$$

Obviously, $a_0 = \sup_n \|\hat{u}_n\|_{G_{p)}(-1;1)\oplus C} < +\infty$. Consequently, using (3.10) and (3.23), by triangle inequality we obtain

$$\left\|S_{2n}(\hat{f})\right\|_{G_{p)}(-1;1)\oplus C} \leq \left\|S_{2n-1}(\hat{f})\right\|_{G_{p)}(-1;1)\oplus C} + \left|< \hat{f}, v_n >\right| \|\hat{u}_{2n}\|_{G_{p)}(-1;1)\oplus C} \leq$$

$$\leq (M + a_0 a_1) \left\| \hat{f} \right\|_{G_{p)}(-1;1) \oplus C} = K \left\| \hat{f} \right\|_{G_{p)}(-1;1) \oplus C} . \tag{3.24}$$

From (3.23) and (3.24) it follows that the sequence $\{S_n\}_{n \in Z_+}$ is uniformly bounded. Thus, the system $\{\hat{u}_n\}_{n \in Z_+}$ forms a basis for the space $G_{p)}(-1;1) \oplus C$. The theorem is proved. □

The theorem implies that the system $\{u_n\}_{n \in Z_+}$ of eigenvectors of the problem (1.1)–(1.2) is complete in the space $G_{p)}(-1;1)$, $1 < p < +\infty$. In fact, if otherwise, then there exists nonzero linear continuous functional v in $G_{p)}(-1;1)$ such that

$$< u_n, v >= 0, \forall n \in Z_+.$$

Denote by $\hat{v}$ a functional in $G_{p)}(-1;1) \oplus C$ defined by the formula

$$< \hat{u}, \hat{v} >=< u, v > .$$

Obviously, $\hat{v}$ is a nonzero linear continuous functional in $G_{p)}(-1;1) \oplus C$ and

$$< \hat{u}_n, \hat{v} >= 0, \forall n \in Z_+.$$

This contradicts the completeness of the system $\{\hat{u}_n\}_{n \in Z_+}$ in the space $G_{p)}(-1;1) \oplus C$. Let us consider the basicity of the system $\{u_n\}_{n \in Z_+}$ in the space $G_{p)}(-1;1)$.

We prove the following theorem.

Theorem 3.4 *For every $k_0 \in Z_+$, the system $\{u_n\}_{n \in Z_+, n \neq 2k_0}$ forms a basis for $G_{p)}(-1;1)$, $1 < p < +\infty$.*

Proof Consider an arbitrary $f \in G_{p)}(-1;1)$. From (3.9) it follows that for the system $\{v_n\}_{n \in Z_+}$, biorthogonal to $\{u_n\}_{n \in Z_+}$, the relation

$$v_{2n}(0) = c_{2n} \sin \bar{\rho}_{2,n} \neq 0, n \in Z_+,$$

holds. Decomposing the vector $\hat{f} = (f; \beta)$, $\beta = -\frac{<f, v_{2k_0}>}{m\overline{v_{2k_0}(0)}}$, into the basis $\{\hat{u}_n\}_{n \in Z_+}$, we obtain

$$\hat{f} = \sum_{n=0}^{+\infty} < \hat{f}, \hat{v}_n > \hat{u}_n = \sum_{n=0}^{+\infty} (< f, v_n > + \beta m \overline{v_n(0)}) \hat{u}_n =$$

$$= \sum_{n=0}^{+\infty} (< f, v_n > - \frac{< f, v_{2k_0} >}{m \overline{v_{2k_0}(0)}} m \overline{v_n(0)}) \hat{u}_n = \sum_{n=0, n \neq 2k_0}^{+\infty} (< f, v_n > - \frac{< f, v_{2k_0} >}{\overline{v_{2k_0}(0)}} \overline{v_n(0)}) \hat{u}_n =$$

$$= \sum_{n=0, n \neq 2k_0}^{+\infty} (< f, v_n > - \frac{< f, v_{2k_0} >}{\overline{v_{2k_0}(0)}} \overline{v_n(0)}) \hat{u}_n = \sum_{n=0, n \neq 2k_0}^{+\infty} (< f, v_n - \frac{v_n(0)}{v_{2k_0}(0)} v_{2k_0} > \hat{u}_n.$$

Hence it follows

$$f = \sum_{n=0, n \neq 2k_0}^{+\infty} < f, v_n - \frac{v_n(0)}{v_{2k_0}(0)} v_{2k_0} > u_n = \sum_{n=0, n \neq 2k_0}^{+\infty} < f, v_n^* > u_n, \tag{3.25}$$

where $v_n^* = v_n - \frac{v_n(0)}{v_{2k_0}(0)} v_{2k_0}$, $n \neq 2k_0$. On the other hand, the systems $\{v_n^*\}_{n \in Z_+, n \neq 2k_0}$ and $\{u_n\}_{n \in Z_+, n \neq 2k_0}$ are biorthogonal. In fact,

$$< u_k, v_n^* >=< u_k, v_n > - \frac{\overline{v_n(0)}}{v_{2k_0}(0)} < u_k, v_{2k_0} >=< u_k, v_n >= \delta_{nk}, n, k \in Z_+ \setminus \{2k_0\}.$$

Therefore, $f \in G_{p)}(-1;1)$ has a unique decomposition (3.25) with respect to the system $\{u_n\}_{n \in Z_+, n \neq 2k_0}$. Consequently, the system $\{u_n\}_{n \in Z_+, n \neq 2k_0}$ forms a basis for $G_{p)}(-1;1)$. The theorem is proved. □

Acknowledgment

This work is supported by the Scientific and Technological Research Council of Turkey [TÜBİTAK 2542 - Bilateral Cooperation Program with Azerbaijan National Academy of Sciences (ANAS)], Project Number: 118F447 and Project Number: 19042020 (for Azerbaijan National Academy of Sciences).

References

[1] Adams DR. Morrey Spaces. Switzerland: Springer International Publishing, 2015.

[2] Bilalov BT. Basicity of some systems of exponents, cosines and sines. Differents Uravneniya 1990; 26 (1): 10-16 (in Russian).

[3] Bilalov BT. On basicity of perturbed system of exponents. Siberian Mathematical Journal 2019; 45 (2): 264-273 (in Russian).

[4] Bilalov BT. On basicity of systems of exponents, cosines and sines for L_p. Doklady RAN 2001; 365 (2): 7-9 (in Russian).

[5] Bilalov BT. Basis properties of some systems of exponents, cosines and sines. Siberian Mathematical Journal 2004; 45 (2): 264-273 (in Russian).

[6] Bilalov BT, Gasymov TB. On basicity of eigenfunctions of second order discontinuous differential operator. Ufa Mathematical Journal 2017; 9 (1): 109-122.

[7] Bilalov BT, Gasymov TB, Guliyeva AA. On the solvability of the Riemann boundary value problem in Morrey-Hardy classes. Turkish Journal of Mathematics 2016; 40 (50): 1085-1101. doi: 10.3906/mat-1507-10

[8] Bilalov BT, Guseynov ZG. Basicity criterion of perturbed system of exponents in Lebesgue spaces with variable summability. Doklady RAN 2011; 436 (5): 586-589.

[9] Bilalov BT, Guseynov ZG. Basicity of a system of exponents with a piece-wise linear phase in variable spaces. Mediterranean Journal of Mathematics 2012; 9 (3): 487-498. doi: 10.1007/s00009-011-0135-7

[10] Bilalov BT, Huseynli AA, El-Shabrawy SR. Basis properties of trigonometric systems in weighted Morrey spaces. Azerbaijan Journal of Mathematics 2019; 9 (2): 166-192.

[11] Bilalov BT, Seyidova F. Basicity of a system of exponents with a piecewise linear phase in Morrey-type spaces. Turkish Journal of Mathematics 2019; 43 (4): 1850-1866. doi: 10.3906/mat-1901-113

[12] Bilalov BT, Quliyeva AA. On basicity of exponential systems in Morrey-type spaces. International Journal of Mathematics 2014; 25 (6): 1-10. doi: 10.1142/S0129167X14500542

[13] Castilo RE, Rafeiro H. An Introductory Course in Lebesgue Spaces. Switzerland: Springer International Publishing, 2016.

[14] Cruz-Uribe DV, Fiorenza A. Variable Lebesgue spaces: Foundations and Harmonic Analysis. New York, NY, USA: Springer Science and Business Media, 2013.

[15] Fiorenza A, Krbec M. On the domain and range of the maximal operator. Nagoya Mathematical Journal 2000; 158: 43-61. doi: 10.1017/S0027763000007285

[16] Gasymov TB, Akhtyamov AM, Ahmedzade NR. On the basicity in weighted Lebesgue spaces of eigenfunctions of a second-order differential operator with a diccontinuty point. Proceedding of the Institute of Mathematics and Mechanics, National Academy of Sciences of Azerbaijan 2020; 46 (1): 32-44.

[17] Gasymov TB, Mammadova SJ. On convergence of spectral expansions for one discontinuous problem with spectral parameter in the boundary. Transactions of NAS of Azerbaijan 2006; 4 (26): 103-116.

[18] Kapustin NYu, Moiseev EI. On spectral problems with spectral parameters in the boundary condition. Differents Uravneniya 1997; 33 (1): 115-119 (in Russian).

[19] Kapustin NYu. Oscillation properties of the solution of a nonself-adjoint spectral problem with spectral parameter in the boundary condition. Differents Uravneniya 1999; 35 (8): 1024-1027 (in Russian).

[20] Kokilashvili VM, Meskhi A, Rafeiro H, Samko S. Integral Operators in Non-standard Function Spaces. Volume 1: Variable Exponent Lebesgue and Amalgam Spaces. Basel, Switzerland: Birkhauser, 2016.

[21] Kokilashvili VM, Meskhi A, Rafeiro H, Samko S. Integral Operators in Non-Standart Function Spaces. Volume 2: Variable Exponent Holder, Morrey-Companato and Grand Spaces. Basel, Switzerland: Birkhauser, 2016.

[22] Krein SG. Functional Analysis. Moscow, Russia: Wolters-Noordhoff, 1972 (in Russian).

[23] Lyantse VE, Storozh OG. Methods of Unbounded Operators Theory. Kiev, Ukraine: Nauk. dumka, 1983 (in Russian).

[24] Levitan BM, Sargsyan IS. Introduction to Spectral Theory (Self-adjoint Differential Operators). Moscow, Russia: Nauka, 1970 (in Russian).

[25] Moiseev EI, Kapustin NYu. The basis property in Lp of the systems of eigenfuctions corresponding to two problems with a spectral parameter in the boundary condition. Differents Uravneniya 2000; 36 (108): 1498-1501 (in Russian).

[26] Morrey CB. On the solutions of quasi-linear elliptic partial differential equations. Transactions of the American Mathematical Society 1938; 43 (4): 207-226.

[27] Najafov TI, Nasibova NP. On the noetherness of the Riemann problem in a generalized weighted Hardy classes. Azerbaijan Journal of Mathematics 2015; 5 (2): 109-139.

[28] Sadigova SR. The general solution of the homogeneous Riemann problem in the weighted Smirnov classes with general weight. Azerbaijan Journal of Mathematics 2019; 9 (2): 117-130.

[29] Samko N. Weight Hardy and singular operators in Morrey spaces. Journal of Mathematical Analysis and Applications 2009; 35 (1): 183-188. doi: 10.1016/j.jmaa.2008.09.021

[30] Sharapudinov II. On direct and inverse theorems of approximation theory in variable Lebesgue and Sobolev spaces. Azerbaijan Journal of Mathematics 2014; 4 (1): 55-72.

[31] Sharapudinov II. On the topology of the space $L^{p(\cdot)}(0;1)$. Matematicheskie Zametki 1979; 26 (4): 613-632 (in Russian).

[32] Shkalikov AA. On basicity of eigenfunctions ordinary differential operators with integral boundary contitions. Vestnik Moscow University Mathematics and Mechanics Series 1982; 6 : 12-21 (in Russian).

[33] Shkalikov AA. Boundary value problems for ordinary differential equations with parameter in the boundary conditions. Functional Analysis and Its Applications 1983; 9: 190-229 (in Russian).

[34] Tichonoff AN, Samarskii AA. Equations of Mathematical Physics. Moscow, Russia: Nauka, 1977 (in Russian).

[35] Xianling F, Dun Z. On the spaces $L^{p(x)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. Journal of Mathematical Analysis and Applications 2001; 263 (2): 424-446. doi: 10.1006/jmaa.2000.7617

[36] Zorko CT. Morrey spaces. Proceedings of the American Mathematical Society 1986; 98 (4): 586-592. doi: 10.1090/S0002-9939-1986-0861756-X

[37] Zeren Y. Some questions of harmonic analysis in weighted Morrey type spaces. Afrika Mathematica 2018; 30 (4): 128-150. doi: 10.1007/s13370-018-0622-0

[38] Zeren Y, Abdullayeva RH, Cetin S. On Morrey-type classes of harmonic functions. Rendiconti del Circolo Matematico di Palermo 2018; 67 (3): 501–516. doi: 10.1007/s12215-018-0331-4

[39] Zeren Y. On the Riesz potential in Morrey spaces, associated with the Laplace–Bessel differential operator. Integral Transforms and Special Functions 2008; 19 (8): 607-612. doi: 10.1080/10652460802030524