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On the dynamics of certain higher-order scalar difference equation: asymptotics, oscillation, stability

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Abstract: We construct the asymptotics for solutions of the higher-order scalar difference equation that is equivalent to the linear delay difference equation $\Delta y(n) = -g(n)y(n-k)$. We assume that the coefficient of this equation oscillates at the certain level and the oscillation amplitude decreases as $n \to \infty$. Both the ideas of the centre manifold theory and the averaging method are used to construct the asymptotic formulae. The obtained results are applied to the oscillation and stability problems for the solutions of the considered equation.

Key words: Center manifold, method of averaging, asymptotics, discrete delay equation, oscillation

1. Problem statement

In this paper, we construct the asymptotics as $n \to \infty$ for the solutions of the following $(k+1)$-th order scalar linear difference equation:

$$x(n+k+1) - \frac{k+1}{k}x(n+k) + \left(\frac{1}{k} + q(n)\right)x(n) = 0, \quad k, n \in \mathbb{N}. \tag{1.1}$$

Here the real function $q(n)$ tends to zero as $n \to \infty$ in an oscillatory way. The more precise form of this function will be defined later. We now outline the problems that lead to Eq. (1.1).

In [1–3], the authors studied the oscillation problem for equation

$$\Delta y(n) = -g(n)y(n-k), \tag{1.2}$$

where $g(n) > 0$ for all $n \in \mathbb{N}$ (symbol Δ stands for the forward difference operator). The main theorems concerning this problem were proved in papers [1, 3]. We mention them here. Let us first define the expression $\ln_q t$, $q \geq 1$, by $\ln_q t = \ln(\ln_{q-1} t)$, $\ln_0 t = t$. The following theorem holds [3].

Theorem 1.1 *Let $q \in \mathbb{N}_0$ be a fixed integer, let $a \in \mathbb{N}$ be sufficiently large and*

$$0 < g(n) \leq \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k}{8(n\ln n \cdots \ln_q n)^2}\right] \tag{1.3}$$

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for every $n \geq a$. *Then there exist natural number* $a_1 \geq a$ *and the solution* $x(n)$ *of Eq.* (1.2) *such that*

$$0 < x(n) \leq \left(\frac{k}{k+1}\right)^n \sqrt{n \ln n \ln_2 n \cdots \ln_q n}$$

holds for every $n \geq a_1$.

We recall that the solution $x(n)$ is called positive (negative) for $n \geq a$ if $x(n) > 0$ ($x(n) < 0$) for all $n \geq a$. The solution $x(n)$ being neither positive nor negative for $n \geq a$ is called oscillatory. Thus, Theorem 1.1 states that Eq. (1.2) has positive solution provided inequality (1.3) holds. The following opposite, in certain sense, to Theorem 1.1 result holds [1].

Theorem 1.2 *Suppose that* $q \in \mathbb{N}_0$ *is a fixed integer,* $a \in \mathbb{N}$ *is sufficiently large,* $\theta > 1$ *and*

$$g(n) \geq \left(\frac{k}{k+1}\right)^k \left[\frac{1}{k+1} + \frac{k}{8n^2} + \frac{k}{8(n\ln n)^2} + \cdots + \frac{k\theta}{8(n\ln n \cdots \ln_q n)^2}\right] \tag{1.4}$$

for every $n \geq a$. *Then all solutions of Eq.* (1.2) *are oscillating as* $n \to \infty$.

Theorems 1.1 and 1.2 are not applicable in the case when function $g(n)$ tends to the limit value in equalities (1.3), (1.4) as $n \to \infty$ in an oscillatory way. In this situation

$$\lim_{n\to\infty} g(n) = \frac{k^k}{(k+1)^{k+1}}, \tag{1.5}$$

and neither inequality (1.3) nor inequality (1.4) holds. By setting

$$g(n) = \frac{k^k}{(k+1)^{k+1}}\Big[1 + kq(n-k)\Big] \tag{1.6}$$

in Eq. (1.2), we reduce this equation to form (1.1) by the change of variable

$$y(n) = \left(\frac{k}{k+1}\right)^n x(n). \tag{1.7}$$

Evidently, if the limit equality (1.5) holds for the function $g(n)$ in Eq. (1.2) then, due to (1.6), the function $q(n)$ in Eq. (1.1) tends to zero as $n \to \infty$. Therefore, in our paper we will study the case, when function $q(n)$ in Eq. (1.1) has the following form:

$$q(n) = \frac{p(n)}{n^\alpha}, \qquad \alpha > 0. \tag{1.8}$$

Here $p(n)$ is either the T-periodic real function or the real valued discrete trigonometric polynomial.

Another problem concerning Eq. (1.1) comes from the stability theory. The so-called critical case of the stability problem occurs in this equation. Namely, the characteristic polynomial of the unperturbed equation ($q(n) = 0$),

$$L(\lambda) = \lambda^{k+1} - \frac{k+1}{k}\lambda^k + \frac{1}{k} \tag{1.9}$$

has the multiple root $\lambda_{1,2} = 1$ and all the other roots lie inside the unit circle in $\mathbb{C}$. Here we refer to the following well-known fact. Assume that the function $g(n)$ in Eq. (1.2) identically equals to the limit value (1.5). The corresponding characteristic polynomial

$$\mu^{k+1} - \mu^k + \frac{k^k}{(k+1)^{k+1}} = 0$$

has the multiple root $\mu_{1,2} = k/(k+1)$ and all the other roots lie inside the circle $|\mu| = k/(k+1)$ in $\mathbb{C}$. This follows from the results obtained by Levin and May in [10] (see also [8, p. 491–497]). Due to (1.7), we get the mentioned above fact concerning the roots of the polynomial (1.9).

Concluding this section, we note that in the case $k = 1$ equation (1.1) may be considered as the difference Schrödinger equation with zero energy and the discrete Wigner–von Neumann type potential. The asymptotic formulae in this situation, provided the function $p(n)$ has zero mean value, were constructed in [7].

2. Asymptotic summation method

By letting

$$z_1(n) = x(n), \quad z_2(n) = x(n+1), \quad \dots \quad z_{k+1}(n) = x(n+k), \tag{2.1}$$

we write Eq. (1.1) in the form of the $(k+1)$-dimensional linear difference system

$$z(n+1) = \big[A + B(n)\big]z(n), \qquad z(n) = \big(z_1(n), \dots, z_{k+1}(n)\big)^\tau. \tag{2.2}$$

Here

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\frac{1}{k} & 0 & 0 & \dots & 0 & \frac{k+1}{k} \end{pmatrix}, \quad B(n) = -q(n)B, \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \tag{2.3}$$

and superscript $^\tau$ stands for the transpose operation. The eigenvalues of the matrix A are the roots of the characteristic polynomial (1.9). Thus, we have the eigenvalue $\lambda_{1,2} = 1$ of the algebraic multiplicity 2 and all the other eigenvalues lie inside the unit circle $|\lambda| < 1$ in $\mathbb{C}$. This is a critical case in the asymptotic summation problem. The method for constructing the asymptotics for solutions of systems having form (2.2) in this situation was proposed in [12]. We will follow it here.

The proposed method uses the fact that there exists for sufficiently large n the attractive invariant manifold (called critical manifold) in the phase space of system (2.2). It is possible to construct the system that describes the dynamics of solutions of (2.2) lying on this manifold. The asymptotic summation of this system, due to the attractivity property of the manifold, allows us to get the asymptotics for solutions of the initial Eq. (1.1). Let us describe this procedure in more detail.

Let Φ be $(k+1) \times 2$ matrix whose columns (from left to right) are the eigenvector and the generalized eigenvector corresponding to the eigenvalue $\lambda_{1,2} = 1$ of the matrix A, respectively. Moreover, let Ψ be $2\times(k+1)$ matrix whose rows (from the bottom upwards) are the eigenvector and the generalized eigenvector of the matrix A^τ corresponding to the eigenvalue $\lambda_{1,2} = 1$, respectively. We can choose matrices Φ and Ψ such that

$$\Psi\Phi = I. \tag{2.4}$$

Some easy calculations show that these matrices may be defined, for example, by the following formulae:

$$\Phi = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ \vdots & \vdots \\ 1 & k+1 \end{pmatrix}, \qquad \Psi = \frac{2}{k(k+1)} \begin{pmatrix} \frac{2k+4}{3} & \frac{2k+7}{3} & \dots & \frac{2k+(3k+1)}{3} & -\frac{k(2k+1)}{3} \\ -1 & -1 & \dots & -1 & k \end{pmatrix}. \tag{2.5}$$

Matrices Φ and Ψ are needed for the decomposition of the phase space $\mathbb{C}^{k+1}$ of system (2.2) into a direct sum of two linear subspaces:

$$\mathbb{C}^{k+1} = \mathcal{P} \oplus \mathcal{Q}. \tag{2.6}$$

The linear subspaces $\mathcal{P}$ and $\mathcal{Q}$ have the following properties:

(i) the subspace $\mathcal{P}$ is a linear span of the eigenvector and the generalized eigenvector of the matrix A corresponding to the eigenvalue $\lambda_{1,2} = 1$; therefore,

$$\mathcal{P} = \{z_{\mathcal{P}} \in \mathbb{C}^{k+1} \mid z_{\mathcal{P}} = \Phi u,\ u \in \mathbb{C}^2\}. \tag{2.7}$$

(ii) the subspace $\mathcal{Q}$ is invariant under the matrix A. Moreover, for every $z_{\mathcal{Q}} \in \mathcal{Q}$ the following inequality holds:

$$|A^n z_{\mathcal{Q}}| \le K q^n |z_{\mathcal{Q}}|, \quad n \in \mathbb{N}, \tag{2.8}$$

where $K > 0$, $0 < q < 1$ and the symbol $|\cdot|$ denotes some vector norm in $\mathbb{C}^{k+1}$. Due to the closed range theorem of Banach (see, e.g., [13]), the subspace $\mathcal{Q}$ may be defined as follows:

$$\mathcal{Q} = \{z_{\mathcal{Q}} \in \mathbb{C}^{k+1} \mid \Psi z_{\mathcal{Q}} = 0\}. \tag{2.9}$$

We are now in a position to introduce the notion of the critical manifold.

Definition 2.1 *A linear subspace $\mathcal{W}(n) \subset \mathbb{C}^{k+1}$ is said to be critical (or centre-like) manifold of system (2.2) for $n \ge n_*$ ($n_* \in \mathbb{N}$) if the following conditions hold:*

1. There exists $(k+1)\times 2$ matrix $H(n)$ whose columns belong to the subspace $\mathcal{Q}$ for $n \ge n_*$ such that $H(n) \to 0$ as $n \to \infty$;

2. For $n \ge n_*$, the subspace $\mathcal{W}(n)$ has the form

$$\mathcal{W}(n) = \Big\{z \in \mathbb{C}^{k+1} \;\Big|\; z = \Phi u + H(n)u,\ u \in \mathbb{C}^2\Big\}; \tag{2.10}$$

3. The subspace $\mathcal{W}(n)$ is positively invariant for trajectories of (2.2) for $n \ge n_*$, i.e., if $z(s) \in \mathcal{W}(s)$, $s \ge n_*$, then $z(n) \in \mathcal{W}(n)$ for $n \ge s$.

Since the subspace $\mathcal{P}$ is invariant under the matrix A, there exists 2×2 matrix D such that $A\Phi = \Phi D$. It is easy to check that

$$D = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2.11}$$

It can be shown (see [12]) that the dynamics of the solutions of system (2.2) that lie on $\mathcal{W}(n)$ for sufficiently large n is described by the following two-dimensional system of difference equations:

$$u(n+1) = \big[D + \Psi B(n)\big(\Phi + H(n)\big)\big]u(n). \tag{2.12}$$

System (2.12) is refered to as system on critical manifold. Thus, we need to calculate matrix $H(n)$ that defines the critical manifold $\mathcal{W}(n)$ according to (2.10). It turns out that matrix $H(n)$ satisfies the following nonlinear matrix difference equation:

$$H(n+1)D - AH(n) + (\Phi\Psi - I)B(n)\big(\Phi + H(n)\big) + H(n+1)\Psi B(n)\big(\Phi + H(n)\big) = 0. \tag{2.13}$$

In what follows, we will say that certain matrix $R(n)$ belongs to class ℓ_1 if

$$\sum_{l=n_0}^{\infty} \|R(l)\| < \infty$$

for some $n_0 \in \mathbb{N}$, where symbol $\|\cdot\|$ denotes some matrix norm. Actually we do not need to find the solution of Eq. (2.13) in the explicit form. We can satisfy Eq. (2.13) up to terms that belong to ℓ_1 by substituting the following expression in it:

$$\hat{H}(n) = H_1(n)n^{-\alpha} + H_2(n)n^{-2\alpha} + \cdots + H_m(n)n^{-m\alpha}. \tag{2.14}$$

Here the entries of $(k+1)\times 2$ matrices $H_i(n)$ $(i = 1,\ldots,m)$ are either T-periodic functions or discrete trigonometric polynomials depending on the corresponding property of the function $p(n)$ in (1.8). Moreover, the columns of these matrices belong to the subspace $\mathcal{Q}$. The natural number m is chosen such that

$$(m+1)\alpha > 1. \tag{2.15}$$

To find matrices $H_i(n)$ $(i = 1,\ldots,m)$ we substitute (2.14) in Eq. (2.13) and collect the terms with factors $n^{-i\alpha}$, omitting the terms from ℓ_1. We also use the trivial identity $(n+1)^{-\alpha} = \Delta(n^{-\alpha}) + n^{-\alpha}$ and the fact that function $\Delta(n^{-\alpha})$ belongs to ℓ_1. Finally, we obtain linear matrix difference equations that are uniquely solvable in the considered class of matrices.

We can now formulate the main theoretical results concerning the used asymptotic summation method (see [12]).

Theorem 2.2 *For sufficiently large n there exists a critical manifold $\mathcal{W}(n)$ of system (2.2) that is described by formula (2.10). Moreover, there exists sufficiently large $n_* \in \mathbb{N}$ such that for $n \geq n_*$ matrix $H(n)$ from (2.10) admits the following representation:*

$$H(n) = \hat{H}(n) + Z(n), \qquad n \geq n_*. \tag{2.16}$$

Here matrix $\hat{H}(n)$ is described by formula (2.14) and satisfies Eq. (2.13) up to terms from ℓ_1. Finally, $Z(n)$ is a certain $(k+1)\times 2$ matrix from ℓ_1 whose columns belong to the subspace $\mathcal{Q}$.

In our problem it is possible to give a more exact asymptotics for matrix $H(n)$ as $n \to \infty$:

$$H(n) = H_1(n)n^{-\alpha} + H_2(n)n^{-2\alpha} + \cdots + H_m(n)n^{-m\alpha} + O\big(n^{-(m+1)\alpha}\big) + O\big(n^{-(1+\alpha)}\big). \tag{2.17}$$

The reduction of the asymptotic summation problem for system (2.2) to the corresponding problem for system on critical manifold (2.12) is based on the global attractivity property of the manifold $\mathcal{W}(n)$. Namely, the following theorem holds.

Theorem 2.3 *Suppose that $z(n)$ is a solution of system (2.2) defined for $n \geq n_0$ ($n_0 \in \mathbb{N}$). Then there exists a sufficiently large $n_* \geq n_0$ such that the following asymptotic formula holds for $n \geq n_*$:*

$$z(n) = \Phi u_H(n) + H(n)u_H(n) + O\big((q+\varepsilon)^n\big), \qquad n \to \infty. \tag{2.18}$$

Here $0 < q < 1$ is defined in (2.8), $\varepsilon \in (0, 1-q)$ is an arbitrary real number and $u_H(n)$ ($n \geq n_*$) is a certain solution of the system on the critical manifold (2.12).

Let $u^{(1)}(n), u^{(2)}(n)$ be the fundamental solutions of the system on the critical manifold (2.12) and $z(n)$ is an arbitrary solution of system (2.2) defined for $n \geq n_0$. Then, by Theorem 2.3, this solution has the following asymptotic representation as $n \to \infty$:

$$z(n) = \big(\Phi + H(n)\big)\big(c_1 u^{(1)}(n) + c_2 u^{(2)}(n)\big) + O\big(q_1^n\big), \tag{2.19}$$

where c_1, c_2 are arbitrary complex constants and $0 < q_1 < 1$ is a certain real number. Due to formulae (2.1), (2.5), and (2.17), we have the following asymptotics for all solutions of the initial Eq. (1.1) as $n \to \infty$:

$$x(n) = \big(1 + o(1), 1 + o(1)\big)\big(c_1 u^{(1)}(n) + c_2 u^{(2)}(n)\big) + O\big(q_1^n\big). \tag{2.20}$$

Therefore, to obtain the asymptotic formulae for solutions of Eq. (1.1) we need to construct the asymptotics for solutions of the two-dimensional linear difference system (2.12).

With account of Theorem 2.2 and formula (2.17) the system on the critical manifold takes the following form:

$$u(n+1) = \Big[D + D_1(n)n^{-\alpha} + D_2(n)n^{-2\alpha} + \ldots + D_{m+1}(n)n^{-(m+1)\alpha} + O\big(n^{-(m+2)\alpha}\big) + O\big(n^{-(1+2\alpha)}\big)\Big]u(n). \tag{2.21}$$

Here $D_1(n), \ldots, D_{m+1}(n)$ are 2×2 matrices whose entries are either T-periodic functions or discrete trigonometric polynomials depending on the corresponding property of the function $p(n)$ in (1.8). Particularly,

$$D_1(n) = p(n)\begin{pmatrix} \beta_1 & \beta_1 \\ \beta_2 & \beta_2 \end{pmatrix}, \quad D_2(n) = p(n)\begin{pmatrix} \beta_1 h_{1,1}(n) & \beta_1 h_{1,2}(n) \\ \beta_2 h_{1,1}(n) & \beta_2 h_{1,2}(n) \end{pmatrix}, \tag{2.22}$$

where $h_{1,1}(n)$, $h_{1,2}(n)$ are the entries of the matrix $H_1(n)$ from (2.17),

$$H_1(n) = \begin{pmatrix} h_{1,1}(n) & h_{1,2}(n) \\ \vdots & \vdots \\ h_{k+1,1}(n) & h_{k+1,2}(n) \end{pmatrix} \tag{2.23}$$

and β_1, β_2 are defined as follows:

$$\beta_1 = \frac{2(2k+1)}{3(k+1)}, \qquad \beta_2 = -\frac{2}{k+1}. \tag{2.24}$$

System (2.21) belongs to the class of the dynamical systems with oscillatory decreasing coefficients. The method for asymptotic summation of this kind of systems was proposed in [7]. In the simplest but practically the most important case, it takes two steps to obtain the asymptotics for solutions of Eq. (2.21). At the first step, we utilize in (2.21) the averaging change of variable that makes it possible to exclude the oscillating coefficients from the main part of the system.

Theorem 2.4 *System (2.21) with matrix D having form (2.11) can be reduced for sufficiently large n to its averaged form*

$$v(n+1) = \Big(D + D_1 n^{-\alpha} + D_2 n^{-2\alpha} + \cdots + D_{m+1} n^{-(m+1)\alpha} + O\big(n^{-(1+\alpha)}\big) + O\big(n^{-(m+2)\alpha}\big) + O\big(n^{-(1+2\alpha)}\big)\Big) v(n) \tag{2.25}$$

with constant matrices $D_1, \ldots, D_{m+1}$ by the change of variable

$$u(n) = \Big[I + Y_1(n) n^{-\alpha} + Y_2(n) n^{-2\alpha} + \cdots + Y_{m+1}(n) n^{-(m+1)\alpha}\Big] v(n). \tag{2.26}$$

Here I is the identity matrix and matrices $Y_1(n), \ldots, Y_{m+1}(n)$ with zero mean value have the same structure as matrices $D_1(n), \ldots, D_{m+1}(n)$ in (2.21).

As a rule, to construct the asymptotics for solutions of Eq. (2.25) we need to compute only first-order and second-order approximation matrices, D_1 and D_2, respectively. Hence, we give here the explicit formulae only for these matrices. We have

$$D_1 = \mathrm{M}\big[D_1(n)\big] \qquad \Big(\mathrm{M}\big[F(n)\big] = \lim_{l\to\infty} \frac{1}{l} \sum_{j=1}^{l} F(j)\Big) \tag{2.27}$$

and

$$D_2 = \mathrm{M}\big[D_2(n) + D_1(n) Y_1(n)\big]. \tag{2.28}$$

Here matrix $Y_1(n)$ is the solutions of the matrix difference equation

$$Y_1(n+1) D - D Y_1(n) = D_1(n) - D_1 \tag{2.29}$$

having zero mean value.

At the second step, we need to reduce the averaged system (2.25) to the so-called L-diagonal form

$$w(n+1) = \big[\hat{\Lambda}(n) + R(n)\big] w(n), \tag{2.30}$$

where $\hat{\Lambda}(n)$ is a diagonal matrix and $R(n)$ is a certain matrix, playing the role of the perturbation term (in our case $R(n) \in \ell_1$). The problem of construction the asymptotics for L-diagonal systems was discussed, e.g., in paper [4] (see also [6]). The main result that can be applied to construct the asymptotics for solutions of Eq. (2.30) is the discrete analog of the fundamental Theorem of Levinson [11]. This theorem states that under certain conditions the fundamental matrix of Eq. (2.30) has the following asymptotics as $n \to \infty$:

$$W(n) = \big[I + o(1)\big] \prod_{l=n_0}^{n-1} \hat{\Lambda}(l) \tag{2.31}$$

for some $n_0 \in \mathbb{N}$. We refer the reader to the sources mentioned above and also to the excellent book [8] for details.

In what follows, due to the form of the averaged system (2.25) and the asymptotic formula (2.31), we will need to construct the asymptotics as $n \to \infty$ for the products of the form

$$f(n) = \prod_{l=n_0}^{n-1} \Big(1 + a_1 l^{-\varphi_1} + O\big(l^{-\varphi_2}\big)\Big), \tag{2.32}$$

where $a_1 \in \mathbb{C}$ and $0 < \varphi_1 < \varphi_2$. This is done easily enough. Since

$$f(n) = \exp\Big\{\sum_{l=n_0}^{n-1} \ln\Big[1 + a_1 l^{-\varphi_1} + O\big(l^{-\varphi_2}\big)\Big]\Big\},$$

by using the Taylor expansion

$$\ln\big(1+z\big) = z - \frac{1}{2}z^2 + O(z^3), \qquad z \to 0,$$

we obtain

$$f(n) = \exp\Big\{\sum_{j=n_0}^{n-1} \Big[a_1 l^{-\varphi_1} - \frac{a_1^2}{2} l^{-2\varphi_1} + O\big(l^{-3\varphi_1}\big) + O\big(l^{-\varphi_2}\big)\Big]\Big\}.$$

We apply now the Euler – Maclaurin formula (see, e.g., [9])

$$\sum_{l=n_0}^{n-1} l^{-\rho} = \int_{n_0}^{n} s^{-\rho} ds + C + O(n^{-\rho}), \qquad t \to \infty,$$

where $\rho > 0$ and C is a certain constant, to get the following asymptotics for the function $f(n)$ as $n \to \infty$:

$$f(n) = \hat{C} \exp\Big\{a_1 \int_{n_0}^{n} s^{-\varphi_1} ds - \frac{a_1^2}{2} \int_{n_0}^{n} s^{-2\varphi_1} ds + O\Big(\int_{n_0}^{n} s^{-3\varphi_1} ds\Big) + O\Big(\int_{n_0}^{n} s^{-\varphi_2} ds\Big)\Big\}\big(1 + o(1)\big). \tag{2.33}$$

3. Construction of asymptotic formulae

We start with the most simple case. Suppose that parameter α in (1.8) belongs to the interval

$$\alpha > 2. \tag{3.1}$$

This is the case when system on critical manifold

$$u(n+1) = \Big[D + O\big(n^{-\alpha}\big)\Big]u(n) \tag{3.2}$$

is an ℓ_1-perturbation of Jordan difference system. Since the perturbation term has the property that

$$O\big(n^{-\alpha}\big)n^{i-j} \in \ell_1, \quad 1 \le i, j \le 2,$$

we can use [6, Theorem 7.1] to get the following asymptotics for the fundamental matrix of system (3.2) as $n \to \infty$:

$$U(n) = \begin{pmatrix} 1 + o(1) & n\big(1 + o(1)\big) \\ o(n^{-1}) & 1 + o(1) \end{pmatrix}. \tag{3.3}$$

Hence, due to (2.20), we obtain the following asymptotics for all solutions of Eq. (1.1) as $n \to \infty$:

$$x(n) = c_1\big(1 + o(1)\big) + c_2 n\big(1 + o(1)\big) + O\big(q_1^n\big). \tag{3.4}$$

Here c_1, c_2 are arbitrary real constants and $0 < q_1 < 1$ is a certain real number.

Therefore, in what follows we will consider only the case

$$\alpha \leq 2. \tag{3.5}$$

The asymptotics for solutions of Eq. (1.1) will differ depending on whether or not the mean value of the function $p(n)$ equals zero. Denoting

$$p_0 = \mathrm{M}\big[p(n)\big], \tag{3.6}$$

we proceed to analysis of these cases.

3.1. Case $p_0 \neq 0$

By using Theorem 2.4, we reduce system on critical manifold (2.21) to the averaged form (2.25) by the change of variable (2.26). To obtain the asymptotic formulae we need to calculate only the matrix D_1. Due to (2.22) and (2.27), this matrix has the following form:

$$D_1 = p_0 \begin{pmatrix} \beta_1 & \beta_1 \\ \beta_2 & \beta_2 \end{pmatrix}. \tag{3.7}$$

Our goal is to bring system (2.25) to L-diagonal form (2.30). We write system (2.25) as follows:

$$v(n+1) = \big(I + A(n) + R(n)\big)v(n), \tag{3.8}$$

where

$$A(n) = J_0 + D_1 n^{-\alpha} + \cdots + D_{m+1} n^{-(m+1)\alpha}, \quad J_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \tag{3.9}$$

and matrix $R(n) = O\big(n^{-(m+2)\alpha}\big) + O\big(n^{-(1+\alpha)}\big)$ belongs to ℓ_1. Some easy computations show that the eigenvalues $\lambda_{1,2}(n)$ of the matrix $A(n)$ are distinct for sufficiently large n. Moreover, they have the following asymptotics as $n \to \infty$:

$$\lambda_{1,2}(n) = \pm\sqrt{p_0\beta_2}\, n^{-\frac{\alpha}{2}} + \frac{(\beta_1 + \beta_2)p_0}{2} n^{-\alpha} + O\big(n^{-\frac{3\alpha}{2}}\big). \tag{3.10}$$

Here and in what follows the symbol $\sqrt{a}$, where $a \in \mathbb{R}$ stands for the quantity

$$\sqrt{a} = \begin{cases} \sqrt{a}, & a \geq 0, \\ \mathrm{i}\sqrt{-a}, & a < 0. \end{cases} \tag{3.11}$$

We construct now the nonsingular for sufficiently large n matrix $C(n)$ such that

$$C^{-1}(n)A(n)C(n) = \Lambda(n), \qquad \Lambda(n) = \mathrm{diag}\big(\lambda_1(n), \lambda_2(n)\big). \tag{3.12}$$

This matrix can be described, for example, by the formula

$$C(n) = \begin{pmatrix} 1 & 1 \\ \sqrt{p_0\beta_2}\, n^{-\frac{\alpha}{2}} + O(n^{-\alpha}) & -\sqrt{p_0\beta_2}\, n^{-\frac{\alpha}{2}} + O(n^{-\alpha}) \end{pmatrix}, \quad n \to \infty. \tag{3.13}$$

Then

$$C^{-1}(n) = \Big(-\frac{1}{2\sqrt{p_0\beta_2}} n^{\frac{\alpha}{2}} + O(1)\Big) \begin{pmatrix} -\sqrt{p_0\beta_2}\, n^{-\frac{\alpha}{2}} + O(n^{-\alpha}) & -1 \\ -\sqrt{p_0\beta_2}\, n^{-\frac{\alpha}{2}} + O(n^{-\alpha}) & 1 \end{pmatrix}, \quad n \to \infty. \tag{3.14}$$

In (3.8), we make the change of variable

$$v(n) = C(n)w(n) \tag{3.15}$$

to obtain

$$w(n+1) = \big(C^{-1}(n+1)C(n) + C^{-1}(n+1)A(n)C(n) + C^{-1}(n+1)R(n)C(n)\big)w(n). \tag{3.16}$$

It follows from (3.13) and (3.14) that

$$C^{-1}(n)\Delta C(n) = -Gn^{-1} + O\big(n^{-\frac{\alpha}{2}-1}\big), \qquad G = \frac{\alpha}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}. \tag{3.17}$$

Since $C^{-1}(n)\Delta C(n) \to 0$ as $n \to \infty$, the following representation holds:

$$\begin{aligned} C^{-1}(n+1) = \big[\Delta C(n) + C(n)\big]^{-1} &= \Big[C(n)\big(C^{-1}(n)\Delta C(n) + I\big)\Big]^{-1} = \big[C^{-1}(n)\Delta C(n) + I\big]^{-1} C^{-1}(n) \\ &= \big[I - C^{-1}(n)\Delta C(n) + \big(C^{-1}(n)\Delta C(n)\big)^2 + \cdots\big]C^{-1}(n). \end{aligned} \tag{3.18}$$

By applying (3.12), (3.17), and (3.18) in system (3.16), we write it as follows:

$$w(n+1) = \big(I + \Lambda(n) + Gn^{-1} + R_1(n)\big)w(n). \tag{3.19}$$

Here matrix $R_1(n)$ belongs to ℓ_1. By (2.15), (3.5), it has the following asymptotic estimate:

$$R_1(n) = O\big(n^{-\frac{\alpha}{2}-1}\big) + O\big(n^{-2}\big) + O\big(n^{-(m+1)\alpha-\frac{\alpha}{2}}\big) = O\big(n^{-\frac{\alpha}{2}-1}\big), \quad n \to \infty. \tag{3.20}$$

First we consider the case

$$\alpha = 2. \tag{3.21}$$

System (3.19), according to (3.10), (3.12), and (3.20), gets the form

$$w(n+1) = \Big[I + \big(\Lambda + G\big)n^{-1} + O\big(n^{-2}\big)\Big]w(n), \tag{3.22}$$

where

$$\Lambda = \sqrt{p_0\beta_2}\,\mathrm{diag}\big(1,-1\big). \tag{3.23}$$

The eigenvalues of the matrix $\Lambda + G$ are the roots of the characteristic polynomial

$$\nu^2 - \nu - p_0\beta_2 = 0.$$

Hence,

$$\nu_{1,2} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4p_0\beta_2}. \tag{3.24}$$

We recall that quantity $\sqrt{\cdot}$ means (3.11). Evidently, if $1+4p_0\beta_2 = 0$ then we have the multiple root $\nu_{1,2} = \frac{1}{2}$. By the appropriate transformation with constant coefficient $w = Pw_1$, we reduce system (3.22) to the form

$$w_1(n+1) = \Big[I + Jn^{-1} + O\big(n^{-2}\big)\Big] w_1(n), \tag{3.25}$$

where

$$J = \begin{cases} \operatorname{diag}(\nu_1, \nu_2), & 1+4p_0\beta_2 \neq 0, \\ \begin{pmatrix} \frac{1}{2} & 1 \\ 0 & \frac{1}{2} \end{pmatrix}, & 1+4p_0\beta_2 = 0. \end{cases} \tag{3.26}$$

Therefore, we should consider two situations.

a) $1+4p_0\beta_2 \neq 0$ (i.e. $p_0 \neq \frac{k+1}{8}$).

System (3.25) has L-diagonal form (2.30) with $\hat{\Lambda}(n) = I + Jn^{-1}$ and, by [4, Lemma 2.1], we obtain asymptotics (2.31) for its fundamental matrix as $n \to \infty$. By using the computations, described at the end of Sec. 2, we get the following asymptotic formula for the fundamental matrix of system (3.25):

$$W_1(n) = \big[I + o(1)\big] \operatorname{diag}\big(\exp\{\nu_1 \ln n\}, \exp\{\nu_2 \ln n\}\big), \qquad n \to \infty.$$

We then return to system (2.21) and get the asymptotics for its fundamental matrix. Omitting the intermediate computations, we write the asymptotics for solutions of the initial Eq. (1.1) as $n \to \infty$:

$$x(n) = c_1 \exp\{\nu_1 \ln n\}\big(1 + o(1)\big) + c_2 \exp\{\nu_2 \ln n\}\big(1 + o(1)\big) + O\big(q_1^n\big). \tag{3.27}$$

Here c_1, c_2 are arbitrary real or complex constants, $0 < q_1 < 1$ is a certain real number and quantities $\nu_{1,2}$ are defined by formula (3.24). By analyzing formula (3.27), we can make the following conclusions. If $1+4p_0\beta_2 > 0$ (i.e. $p_0 < \frac{k+1}{8}$), then Eq. (1.1) has unbounded nonoscillating solutions with the asymptotic estimate $O\big(n^{\frac{1}{2}+\frac{1}{2}\sqrt{1+4p_0\beta_2}}\big)$ as $n \to \infty$. If $1+4p_0\beta_2 < 0$ (i.e. $p_0 > \frac{k+1}{8}$), then Eq. (1.1) has unbounded oscillating solutions with the asymptotic estimate $O\big(n^{\frac{1}{2}}\big)$ as $n \to \infty$ for the oscillation amplitude.

b) $1+4p_0\beta_2 = 0$ (i.e. $p_0 = \frac{k+1}{8}$).

Matrix J in system (3.25) is now defined by the lower expression in (3.26). Since it is no longer diagonal matrix, we have some additional difficulties. By the change of variable $w_1(n) = n^{\frac{1}{2}} w_2(n)$, we reduce system (3.25) to the form

$$w_2(n+1) = \Big[I + J_0 n^{-1} + O\big(n^{-2}\big)\Big] w_2(n). \tag{3.28}$$

Here matrix J_0 is defined by (3.9). System (3.28) has the same form as system (16) in [7], where the asymptotics for its fundamental matrix is obtained. Also the results from [5] are used. It can be shown that the fundamental matrix of system (3.28) has the following asymptotics as $n \to \infty$:

$$W_2(n) = \begin{pmatrix} 1 + o(1) & \ln n\big(1 + o(1)\big) \\ o(1) & 1 + o(1) \end{pmatrix}. \tag{3.29}$$

Hence, the solutions of the initial Eq. (1.1) are described by the following asymptotic formula as $n \to \infty$:

$$x(n) = c_1 n^{\frac{1}{2}}\big(1 + o(1)\big) + c_2 n^{\frac{1}{2}} \ln n\big(1 + o(1)\big) + O\big(q_1^n\big). \tag{3.30}$$

Here c_1, c_2 are arbitrary real constants and $0 < q_1 < 1$ is a certain real number.

We proceed now to the case

$$\alpha < 2. \tag{3.31}$$

In this situation, system (3.19) for sufficiently large n can be reduced to L-diagonal form (2.30) by a certain transformation $w(n) = P(n)w_1(n)$, where matrix $P(n)$ has the following properties:

(i) $P(n) \to I$, $P^{-1}(n) \to I$ as $n \to \infty$;

(ii) $\Delta P(n) \in \ell_1$;

(iii) $P^{-1}(n)\big[\Lambda(n) + Gn^{-1}\big]P(n) = \Lambda_1(n)$, where $\Lambda_1(n)$ is a diagonal matrix whose entries are the eigenvalues of the matrix $\Lambda(n) + Gn^{-1}$.

Let us clarify the fact of the existence of such matrix $P(n)$. Note that according to (3.10), (3.12) we have

$$\Lambda(n) + Gn^{-1} = \Lambda n^{-\frac{\alpha}{2}} + \frac{(\beta_1 + \beta_2)p_0}{2} I n^{-\alpha} + Gn^{-1} + O\big(n^{-\frac{3\alpha}{2}}\big),$$

where the constant matrix Λ is defined by formula (3.23). Since the eigenvalues of the matrix Λ are distinct, by [4, pp. 204-206], there exists matrix $P(n)$ with the pointed above properties such that it brings matrix $n^{\frac{\alpha}{2}}[\Lambda(n) + Gn^{-1}]$ (and, consequently, the matrix $\Lambda(n) + Gn^{-1}$ as well) to diagonal form. Moreover, since the matrix $\Lambda(n)+Gn^{-1}$ does not contain oscillating coefficients, it is easy to check that $\Delta\big(n^{\frac{\alpha}{2}}[\Lambda(n)+Gn^{-1}]\big) \in \ell_1$. This yields property (ii) of the matrix $P(n)$. It turns out that

$$\Delta P(n) = O\Big(\Delta\big(n^{\frac{\alpha}{2}}[\Lambda(n) + Gn^{-1}]\big)\Big) = O\big(n^{-\frac{\alpha}{2}-1}\big) + O\big(n^{\frac{\alpha}{2}-2}\big). \tag{3.32}$$

Therefore, we get the following L-diagonal system

$$w_1(n+1) = \big[I + \Lambda_1(n) + R_2(n)\big]w_1(n). \tag{3.33}$$

Here the diagonal matrix $\Lambda_1(n)$ has the following asymptotics as $n \to \infty$:

$$\Lambda_1(n) = \Lambda n^{-\frac{\alpha}{2}} + \frac{(\beta_1 + \beta_2)p_0}{2} I n^{-\alpha} + \frac{\alpha}{4} I n^{-1} + O\big(n^{-\frac{3\alpha}{2}}\big). \tag{3.34}$$

Moreover, matrix $R_2(n)$, with account of (3.18), (3.20), and (3.32), has the asymptotic estimate

$$R_2(n) = O\big(n^{-\frac{\alpha}{2}-1}\big) + O\big(n^{\frac{\alpha}{2}-2}\big), \quad n \to \infty.$$

We then follow the steps described at the end of Section 2 to construct the asymptotics for solutions of system (3.33). Besides, we should distinguish the following cases: $1 < \alpha < 2$, $\alpha = 1$ and $\alpha < 1$. In the first case, the term $O(n^{-\alpha})$ in (3.34) belongs to ℓ_1 and therefore does not effect the main part of the asymptotics (we can combine this term with matrix $R_2(n)$). If $\alpha = 1$ then the terms $O(n^{-\alpha})$, $O(n^{-1})$ and $O\big((n^{-\frac{\alpha}{2}})^2\big)$ are of the same asymptotic order and do not belong to ℓ_1. Finally, if $\alpha < 1$, only the terms $O(n^{-\alpha})$ and $O\big((n^{-\frac{\alpha}{2}})^2\big)$ are of same asymptotic order.

Taking into account the mentioned above facts and also (2.33), we get the following asymptotics for solutions of Eq. (1.1) as $n \to \infty$. If

$$1 < \alpha < 2,$$

then

$$x(n) = c_1 n^{\frac{\alpha}{4}} \exp\Big\{\frac{2\sqrt{p_0\beta_2}}{2-\alpha} n^{1-\frac{\alpha}{2}}\Big\}\big(1+o(1)\big) + c_2 n^{\frac{\alpha}{4}} \exp\Big\{-\frac{2\sqrt{p_0\beta_2}}{2-\alpha} n^{1-\frac{\alpha}{2}}\Big\}\big(1+o(1)\big) + O\big(q_1^n\big). \tag{3.35}$$

Suppose now that

$$\alpha = 1.$$

In this case,

$$x(n) = n^{\frac{1}{4}+\frac{p_0\beta_1}{2}}\Big[c_1 \exp\big\{2\sqrt{p_0\beta_2 n}\big\}\big(1+o(1)\big) + c_2 \exp\big\{-2\sqrt{p_0\beta_2 n}\big\}\big(1+o(1)\big)\Big] + O\big(q_1^n\big). \tag{3.36}$$

Finally, if

$$\alpha < 1,$$

we obtain

$$x(n) = n^{\frac{\alpha}{4}} \exp\Big\{\frac{p_0\beta_1}{2(1-\alpha)} n^{1-\alpha}\Big\}\Big[c_1 \exp\Big\{\frac{2\sqrt{p_0\beta_2}}{2-\alpha} n^{1-\frac{\alpha}{2}} + O\Big(\int_{n_0}^{n} s^{-\frac{3\alpha}{2}}ds\Big)\Big\}\big(1+o(1)\big)$$

$$+ c_2 \exp\Big\{-\frac{2\sqrt{p_0\beta_2}}{2-\alpha} n^{1-\frac{\alpha}{2}} + O\Big(\int_{n_0}^{n} s^{-\frac{3\alpha}{2}}ds\Big)\Big\}\big(1+o(1)\big)\Big] + O\big(q_1^n\big). \tag{3.37}$$

In all these asymptotic representations, c_1, c_2 are arbitrary real (or complex) constants and $0 < q_1 < 1$ is a certain real number. Moreover, since the quantity $O\big(n^{-\frac{3\alpha}{2}}\big)$ belongs to ℓ_1 when $\alpha > \frac{2}{3}$, the term $O\Big(\int_{n_0}^{n} s^{-\frac{3\alpha}{2}}ds\Big)$ in (3.37) may be excluded in this situation.

Let us analyze formulae (3.35)–(3.37), obtained in the case (3.31). Recalling (2.24) and also notation (3.11) for the symbol $\sqrt{\cdot}$ we conclude the following. If $p_0 < 0$, then Eq. (1.1) has unbounded nonoscillating solutions with the asymptotic estimate $O\big(\exp\big\{\frac{2\sqrt{p_0\beta_2}}{2-\alpha} n^{1-\frac{\alpha}{2}}\big(1+o(1)\big)\big\}\big)$ as $n \to \infty$. If $p_0 > 0$, then Eq. (1.1) has unbounded oscillating solutions with the asymptotic estimate for the oscillation amplitude significantly depending on α. If $1 \le \alpha < 2$, the oscillation amplitude grows at power rate $O\big(n^{\frac{\alpha}{4}}\big)$ for the case $1 < \alpha < 2$ and at power rate $O\big(n^{\frac{1}{4}+\frac{p_0\beta_1}{2}}\big)$ if $\alpha = 1$. In the case when $\alpha < 1$, the oscillation amplitude grows at exponential rate $O\big(\exp\big\{\frac{p_0\beta_1}{2(1-\alpha)} n^{1-\alpha}\big(1+o(1)\big)\big\}\big)$ as $n \to \infty$.

3.2. Case $p_0 = 0$

As in the previous case we start with the reduction of system on critical manifold (2.21) to the averaged form (2.25) by using the change of variable (2.26). With account of formulae (2.22), (2.27) we conclude that $D_1 = 0$. Hence, the averaged system takes form (3.8), where

$$A(n) = J_0 + D_2 n^{-2\alpha} + \cdots + D_{m+1} n^{-(m+1)\alpha}. \tag{3.38}$$

Here matrix J_0 is defined by (3.9) and matrix D_2 can be calculated according to (2.28). The perturbation term in (3.8) belongs to ℓ_1 and has the following asymptotic estimate:

$$R(n) = O\big(n^{-(m+2)\alpha}\big) + O\big(n^{-(1+\alpha)}\big) + O\big(n^{-(1+2\alpha)}\big), \quad n \to \infty.$$

The most simple case takes place when

$$1 < \alpha \le 2. \tag{3.39}$$

In this situation the averaged system is an ℓ_1-perturbation of Jordan difference system:

$$v(n+1) = \Big[D + O\big(n^{-2\alpha}\big) + O\big(n^{-(1+\alpha)}\big)\Big] v(n). \tag{3.40}$$

The perturbation term has the property that

$$O\big(n^{-2\alpha}\big) n^{i-j} \in \ell_1, \quad O\big(n^{-(1+\alpha)}\big) n^{i-j} \in \ell_1, \quad 1 \le i, j \le 2.$$

Therefore, to construct the asymptotics for the fundamental matrix of system (3.40) we can apply [6, Theorem 7.1]. Thus, the fundamental matrix $V(n)$ of this system has asymptotics (3.3) as $n \to \infty$. We conclude that solutions of Eq. (1.1) are described by asymptotic formula (3.4) provided inequality (3.39) holds.

Assume further that

$$\alpha \le 1. \tag{3.41}$$

We will try to improve, in certain sense, the perturbation term $R(n)$ by the appropriate transformation. First, let us clarify where the term $O\big(n^{-(1+\alpha)}\big)$, included in $R(n)$, comes from. We recall that this term appeared after the averaging of system (2.21) by the change of variable (2.26). We can obtain the explicit form for this term if we make the mentioned change of varible. By using the results from [7, p. 1577], we can show that this term is actually the matrix

$$-Y_1(n+1) D \Delta(n^{-\alpha}), \tag{3.42}$$

where $Y_1(n)$ is the solution of Eq. (2.29) having zero mean value. Since $\mathrm{M}\big[Y_1(n)\big] = 0$, the next averaging will remove the term (3.42). Namely, in system (3.8) we make the change of variable

$$v(n) = \big[I + V_1(n) n^{-\alpha-1}\big] w(n),$$

where the matrix $V_1(n)$ is the solution of equation

$$V_1(n+1) D - D V_1(n) = -Y_1(n+1) D$$

with zero mean value. We obtain the system

$$w(n+1) = \big(I + A(n) + R_1(n)\big) w(n), \tag{3.43}$$

where the matrix $A(n)$ is defined by formula (3.38) and the matrix $R_1(n)$ has the following asymptotic estimate:

$$R_1(n) = O\big(n^{-(m+2)\alpha}\big) + O\big(n^{-(2+\alpha)}\big) + O\big(n^{-(1+2\alpha)}\big), \quad n \to \infty.$$

In (3.43), we use the transformation

$$w(n) = C(n) w_1(n), \qquad C(n) = \begin{pmatrix} n^{\frac{\alpha}{2}} & 0 \\ 0 & n^{-\frac{\alpha}{2}} \end{pmatrix} \tag{3.44}$$

to get system

$$w_1(n+1) = \big(I + S_1 n^{-\alpha} + S_2 n^{-2\alpha} + \cdots + S_m n^{-m\alpha} + Q n^{-1} + R_2(n)\big) w_1(n), \tag{3.45}$$

where $S_1, \ldots, S_m, Q$ are certain constant matrices, and the matrix $R_2(n)$ from ℓ_1 has the following asymptotic estimate as $n \to \infty$:

$$R_2(n) = O\big(n^{-(m+1)\alpha}\big) + O\big(n^{-2}\big) + O\big(n^{-(1+\alpha)}\big).$$

We designate now the entries of the matrix D_2 in (3.38):

$$D_2 = \begin{pmatrix} d_{11}^{(2)} & d_{12}^{(2)} \\ d_{21}^{(2)} & d_{22}^{(2)} \end{pmatrix}. \tag{3.46}$$

Then, some easy calculations show that

$$S_1 = \begin{pmatrix} 0 & 1 \\ d_{21}^{(2)} & 0 \end{pmatrix}, \quad S_2 = \begin{pmatrix} d_{11}^{(2)} & 0 \\ \psi & d_{22}^{(2)} \end{pmatrix}, \quad Q = -\frac{\alpha}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.47}$$

where ψ is a certain real number whose exact value will not be needed for the sequel.

Next, we will construct the asymptotic formulae for solutions of Eq. (1.1) using the introduced notations (3.46). The exact values for the entries of the matrix D_2 will be obtained in the separate paragraph at the end of this section. We can now proceed analogously to what was done in [7]. Suppose first that

$$\alpha = 1. \tag{3.48}$$

System (3.45) takes the form

$$w_1(n+1) = \Big[I + \big(S_1 + Q\big)n^{-1} + O\big(n^{-2}\big)\Big]w_1(n), \tag{3.49}$$

The eigenvalues of the matrix $S_1 + Q$ are

$$\hat{\nu}_{1,2} = \pm\sqrt{d_{21}^{(2)} + \frac{1}{4}}, \tag{3.50}$$

where $\sqrt{\cdot}$ means (3.11). If $d_{21}^{(2)} = -\frac{1}{4}$, we have the multiple zero eigenvalue. By the appropriate transformation with constant coefficients $w_1 = Pw_2$, we reduce system (3.49) to the form

$$w_2(n+1) = \Big[I + Jn^{-1} + O\big(n^{-2}\big)\Big]w_2(n), \tag{3.51}$$

where

$$J = \begin{cases} \operatorname{diag}(\hat{\nu}_1, \hat{\nu}_2), & d_{21}^{(2)} \neq -\frac{1}{4}, \\ J_0, & d_{21}^{(2)} = -\frac{1}{4}. \end{cases} \tag{3.52}$$

and the matrix J_0 is described in (3.9). We should consider two cases.

a) $d_{21}^{(2)} \neq -\frac{1}{4}$.

System (3.51) has L-diagonal form (2.30) with $\hat{\Lambda}(n) = I + Jn^{-1}$. The fundamental matrix of this system has the following asymptotic representation:

$$W_2(n) = \big[I + o(1)\big]\operatorname{diag}\big(\exp\{\hat{\nu}_1 \ln n\}, \exp\{\hat{\nu}_2 \ln n\}\big), \qquad n \to \infty.$$

Solutions of the initial Eq. (1.1) have the following asymptotics as $n \to \infty$:

$$x(n) = c_1 n^{\frac{1}{2}} \exp\{\hat{\nu}_1 \ln n\}\big(1 + o(1)\big) + c_2 n^{\frac{1}{2}} \exp\{\hat{\nu}_2 \ln n\}\big(1 + o(1)\big) + O\big(q_1^n\big). \tag{3.53}$$

Here c_1, c_2 are arbitrary real (or complex) constants, $0 < q_1 < 1$ is a certain real number and quantities $\hat{\nu}_{1,2}$ are defined by formula (3.50). Hence, if $d_{21}^{(2)} + \frac{1}{4} > 0$, then Eq. (1.1) has unbounded nonoscillating solutions with the asymptotic estimate $O\big(n^{\frac{1}{2}+\sqrt{d_{21}^{(2)}+\frac{1}{4}}}\big)$ as $n \to \infty$. If $d_{21}^{(2)} + \frac{1}{4} < 0$, Eq. (1.1) has unbounded oscillating solutions with the asymptotic estimate $O\big(n^{\frac{1}{2}}\big)$ for the oscillation amplitude as $n \to \infty$.

b) $d_{21}^{(2)} = -\frac{1}{4}$.

System (3.51) has form (3.28); therefore, we obtain the asymptotic formula (3.29) for its fundamental matrix. Hence, the behaviour of solutions of Eq. (1.1) is described by the asymptotic representation (3.30).

We proceed now to the case

$$\alpha < 1. \tag{3.54}$$

In what follows we assume that

$$d_{21}^{(2)} \neq 0. \tag{3.55}$$

This yields that the eigenvalues of the matrix S_1 in system (3.45) are distincts. By applying the results from [4, pp. 204–206], we conclude that there exists for sufficiently large n matrix $P(n)$ with the following properties:

(i) $P(n) \to P$ as $n \to \infty$, where P is the constant matrix whose columns are the eigenvectors of the matrix S_1;

(ii) $\Delta P(n) \in \ell_1$;

(iii) $P^{-1}(n)\big[S_1 n^{-\alpha} + \cdots + S_m n^{-m\alpha} + Q n^{-1}\big]P(n) = \Lambda_2(n)$, where $\Lambda_2(n)$ is a diagonal matrix whose entries are the eigenvalues of the matrix $S_1 n^{-\alpha} + \cdots + S_m n^{-m\alpha} + Q n^{-1}$.

It can be shown that

$$\Delta P(n) = O\Big(\Delta\big(S_1 + S_2 n^{-\alpha} + \cdots + S_m n^{-(m-1)\alpha} + Q n^{\alpha-1}\big)\Big) = O\big(n^{-(1+\alpha)}\big) + O\big(n^{\alpha-2}\big)$$

as $n \to \infty$. By the change of variable $w_1(n) = P(n) w_2(n)$, we reduce system (3.45) to L-diagonal form

$$w_2(n+1) = \Big[I + \Lambda_2(n) + R_3(n)\big)\Big] w_2(n). \tag{3.56}$$

Here the matrix $\Lambda_2(n)$ has the following asymptotics as $n \to \infty$:

$$\Lambda_2(n) = \sqrt{d_{21}^{(2)}}\,\mathrm{diag}(1,-1) n^{-\alpha}\big[1 + O\big(n^{-\alpha}\big)\big] + \frac{d_{11}^{(2)} + d_{22}^{(2)}}{2} I n^{-2\alpha} + O\big(n^{-3\alpha}\big) + O\big(n^{-(1+\alpha)}\big) + O\big(n^{\alpha-2}\big) \tag{3.57}$$

and the symbol $\sqrt{\cdot}$ is defined in (3.11). The matrix $R_3(n)$ from ℓ_1 has the following asymptotic estimate as $n \to \infty$:

$$R_3(n) = O\big(n^{-(m+1)\alpha}\big) + O\big(n^{-2}\big) + O\big(n^{-(1+\alpha)}\big) + O\big(n^{\alpha-2}\big).$$

We also note that the asymptotic order symbol $O\big(n^{-\alpha}\big)$ inside the square brackets in (3.57) is a certain real function. Moreover, since certain terms in (3.57) belong to ℓ_1, they may be added to the matrix $R_3(n)$. The asymptotic summation of system (3.56) may be handled in the same way as it was described at the end of Section 2. Therefore, we will write only the final asymptotic formulae for solutions of Eq. (1.1).

Thus, if

$$\frac{1}{2} < \alpha < 1,$$

we have the following asymptotics as $n \to \infty$:

$$x(n) = c_1 n^{\frac{\alpha}{2}} \exp\Big\{\frac{\sqrt{d_{21}^{(2)}}}{1-\alpha} n^{1-\alpha}\Big\}\big(1+o(1)\big) + c_2 n^{\frac{\alpha}{2}} \exp\Big\{-\frac{\sqrt{d_{21}^{(2)}}}{1-\alpha} n^{1-\alpha}\Big\}\big(1+o(1)\big) + O\big(q_1^n\big). \tag{3.58}$$

If

$$\alpha = \frac{1}{2},$$

we obtain the following asymptotic representation as $n \to \infty$:

$$x(n) = n^{\frac{1}{2}(\frac{1}{2}+d_{11}^{(2)}+d_{22}^{(2)}-d_{21}^{(2)})}\Big[c_1 \exp\Big\{2\sqrt{d_{21}^{(2)} n}\Big\}\big(1+o(1)\big) + c_2 \exp\Big\{-2\sqrt{d_{21}^{(2)} n}\Big\}\big(1+o(1)\big)\Big] + O\big(q_1^n\big). \tag{3.59}$$

Finally, if

$$\alpha < \frac{1}{2},$$

the following asymptotic formula holds as $n \to \infty$:

$$x(n) = n^{\frac{\alpha}{2}} \exp\Big\{\frac{d_{11}^{(2)}+d_{22}^{(2)}-d_{21}^{(2)}}{2(1-2\alpha)} n^{1-2\alpha} + O\Big(\int_{n_0}^{n} s^{-3\alpha} ds\Big)\Big\}\Big[c_1 \exp\Big\{\frac{\sqrt{d_{21}^{(2)}}}{1-\alpha} n^{1-\alpha}\big(1+O(n^{-\alpha})\big)\Big\}\big(1+o(1)\big)$$

$$+ c_2 \exp\Big\{-\frac{\sqrt{d_{21}^{(2)}}}{1-\alpha} n^{1-\alpha}\big(1+O(n^{-\alpha})\big)\Big\}\big(1+o(1)\big)\Big] + O\big(q_1^n\big). \tag{3.60}$$

In all these asymptotic formulae, c_1, c_2 are arbitrary real (or complex) constants and $0 < q_1 < 1$ is a certain real number.

We now analyze formulae (3.58)–(3.60), constructed for case (3.54). If $d_{21}^{(2)} > 0$, then Eq. (1.1) has unbounded nonoscillating solutions with the asymptotic estimate $O\big(\exp\{\frac{\sqrt{d_{21}^{(2)}}}{1-\alpha} n^{1-\alpha}\big(1+o(1)\big)\}\big)$ as $n \to \infty$. If $d_{21}^{(2)} < 0$, Eq. (1.1) has oscillating solutions. In this case, if $\frac{1}{2} < \alpha < 1$, then there exist unbounded solutions with the asymptotic estimate $O\big(n^{\frac{\alpha}{2}}\big)$ for the oscillation amplitude as $n \to \infty$. If $\alpha = \frac{1}{2}$, then either Eq. (1.1) has unbounded oscillating solutions or all solution of this equation tend to zero as $n \to \infty$ (or are simply bounded). In this situation, the sign of the quantity $\frac{1}{2} + d_{11}^{(2)} + d_{22}^{(2)} - d_{21}^{(2)}$ defines the dynamics of solutions and we have the asymptotic estimate $O\big(n^{\frac{1}{2}(\frac{1}{2}+d_{11}^{(2)}+d_{22}^{(2)}-d_{21}^{(2)})}\big)$ for the oscillation amplitude as $n \to \infty$. Finally, if $\alpha < \frac{1}{2}$, then again either Eq. (1.1) has unbounded oscillating solutions or all solution of this equation tend to zero as $n \to \infty$ (or are simply bounded). The dynamics of solutions is now defined by the sign of the quantity

$$d_{11}^{(2)} + d_{22}^{(2)} - d_{21}^{(2)}$$

and we obtain the asymptotic estimate $O\big(\exp\big\{\frac{d_{11}^{(2)}+d_{22}^{(2)}-d_{21}^{(2)}}{2(1-2\alpha)}n^{1-2\alpha}\big(1+o(1)\big)\big\}\big)$ for the oscillation amplitude as $n\to\infty$.

Calculation of the entries of the matrix D_2

As it follows from the constructed above asymptotic formulae, the dynamics of solutions of Eq. (1.1), if $\alpha\le 1$ and $p_0=0$, is defined by the entries $d_{11}^{(2)}$, $d_{21}^{(2)}$ and $d_{22}^{(2)}$ of matrix (3.46). In this paragraph we will calculate these entries.

We recall that the matrix D_2 is described by formula (2.28) with account of (2.29), where matrices $D_1(n)$ and $D_2(n)$ have form (2.22). We start with calculating the quantity $\mathrm{M}[D_1(n)Y_1(n)]$ in (2.28). Letting

$$Y_1(n)=\begin{pmatrix} y_{11}(n) & y_{12}(n)\\ y_{21}(n) & y_{22}(n)\end{pmatrix}, \tag{3.61}$$

with account of (2.22) we conclude that

$$\mathrm{M}\big[D_1(n)Y_1(n)\big]=\mathrm{M}\left[\begin{pmatrix}\beta_1 p(n)\big(y_{11}(n)+y_{21}(n)\big) & \beta_1 p(n)\big(y_{12}(n)+y_{22}(n)\big)\\ \beta_2 p(n)\big(y_{11}(n)+y_{21}(n)\big) & \beta_2 p(n)\big(y_{12}(n)+y_{22}(n)\big)\end{pmatrix}\right]. \tag{3.62}$$

Here β_1 and β_2 are defined in (2.24). Matrix (3.61) is the solution of Eq. (2.29), where $D_1=0$, having zero mean value. We write this matrix difference equation in the form of the system of scalar difference equations:

$$\begin{aligned}
&y_{11}(n+1)-y_{11}(n)-y_{21}(n)=\beta_1 p(n),\\
&y_{11}(n+1)+y_{12}(n+1)-y_{12}(n)-y_{22}(n)=\beta_1 p(n),\\
&y_{21}(n+1)-y_{21}(n)=\beta_2 p(n),\\
&y_{21}(n+1)+y_{22}(n+1)-y_{22}(n)=\beta_2 p(n).
\end{aligned} \tag{3.63}$$

We introduce the functions $p_1(n)$ and $p_2(n)$ as the solutions of the equations

$$\Delta p_1(n)=p(n),\qquad \Delta p_2(n)=p_1(n), \tag{3.64}$$

having zero mean values. In the sequel, we will use the symbol $\mathrel{\hat{=}}$ to define quantity up to the additive term that will not affect the final result. Particularly, we deduce from (3.64) that

$$p_1(n)\mathrel{\hat{=}}\sum_{j=1}^{n-1}p(j),\qquad p_2(n)\mathrel{\hat{=}}\sum_{j=1}^{n-1}p_1(j). \tag{3.65}$$

Solving system (3.63) does not meet any difficulties. We start with the third equation from the top in (3.63) and then find all the unknown functions. Finally, we obtain

$$\begin{gathered}
y_{11}(n)=\beta_1 p_1(n)+\beta_2 p_2(n),\quad y_{12}(n)\mathrel{\hat{=}}-(\beta_1+\beta_2)p_2(n)-2\beta_2\sum_{j=1}^{n-1}p_2(j),\\
y_{21}(n)=\beta_2 p_1(n),\qquad y_{22}(n)=-\beta_2 p_2(n).
\end{gathered} \tag{3.66}$$

We remark two equalities that may be easily obtained by using the summation by parts formula (Abel transformation) and (3.64):

$$\mathrm{M}\big[p(n)p_2(n)\big] = -\,\mathrm{M}\big[p_1(n+1)p_1(n)\big], \qquad \mathrm{M}\Big[p(n)\sum_{j=1}^{n-1} p_2(j)\Big] = -\,\mathrm{M}\big[p_1(n+1)p_2(n)\big]. \tag{3.67}$$

With account of these equalities we get the following expression for matrix (3.62):

$$\begin{gathered}\mathrm{M}\big[D_1(n)Y_1(n)\big] = \mathrm{M}\big[p(n)p_1(n)\big]\begin{pmatrix}\beta_1\beta_2+\beta_1^2 & 0\\ \beta_1\beta_2+\beta_2^2 & 0\end{pmatrix}\\ +\,\mathrm{M}\big[p_1(n)p_1(n+1)\big]\begin{pmatrix}-\beta_1\beta_2 & 2\beta_1\beta_2+\beta_1^2\\ -\beta_2^2 & \beta_1\beta_2+2\beta_2^2\end{pmatrix} + \mathrm{M}\big[p_1(n+1)p_2(n)\big]\begin{pmatrix}0 & 2\beta_1\beta_2\\ 0 & 2\beta_2^2\end{pmatrix}.\end{gathered} \tag{3.68}$$

Since the entries $d_{11}^{(2)}$ and $d_{22}^{(2)}$ of the matrix D_2 appear in asymptotic formulae in the form of a sum, it makes sense to write the expression for the trace of matrix (3.68). Taking in account (3.65), we have

$$\begin{gathered}\operatorname{tr}\mathrm{M}\big[D_1(n)Y_1(n)\big] = (\beta_1\beta_2+\beta_1^2)\,\mathrm{M}\big[p(n)p_1(n)\big] + 2\beta_2^2\,\mathrm{M}\big[p_1(n)p_1(n+1)\big]\\ +2\beta_2^2\,\mathrm{M}\big[p_1(n+1)p_2(n)\big] = (\beta_1\beta_2+\beta_1^2)\,\mathrm{M}\big[p(n)p_1(n)\big] + 2\beta_2^2\,\mathrm{M}\Big[p_1(n+1)\sum_{j=1}^{n} p_1(j)\Big].\end{gathered} \tag{3.69}$$

Recalling (2.24), we get

$$\beta_1\beta_2+\beta_1^2 = \frac{8(2k+1)(k-1)}{9(k+1)^2}, \quad \beta_1\beta_2+\beta_2^2 = \frac{8(1-k)}{3(k+1)^2}, \quad \beta_2^2 = \frac{4}{(k+1)^2}. \tag{3.70}$$

Our next goal is to calculate the mean value of the matrix $D_2(n)$ that is defined in (2.22). First, we need to compute the entries $h_{1,1}(n)$ and $h_{1,2}(n)$ of the matrix $H_1(n)$ from (2.17). We substitute (2.17) in Eq. (2.13) and collect terms with factors $n^{-\alpha}$, omitting the terms from ℓ_1. We, then, obtain the following equation for matrix $H_1(n)$:

$$H_1(n+1)D - AH_1(n) - p(n)(\Phi\Psi - I)B\Phi = 0. \tag{3.71}$$

Here the constant matrix B is decribed in (2.3) and matrices Φ and Ψ have form (2.5). The entries of the matrix $H_1(n)$ are either T-periodic functions or discrete trigonometric polynomials depending on the corresponding property of the function $p(n)$ in (1.8). It is known (see [12, Lemmas 2.2 and 2.3]) that Eq. (3.71) is uniquely solvable in the class of $(k+1)\times 2$ matrices whose columns belong to the subspace $\mathcal{Q}$. The latter condition, due to (2.9), may be written as follows:

$$\Psi H_1(n) \equiv 0. \tag{3.72}$$

If we write matrix equation (3.71) as a system of equations for the entries of the matrix $H_1(n)$, recalling notation (2.23), we obtain the following:

$$
\begin{aligned}
&h_{1,1}(n+1) - h_{2,1}(n) - p(n)\frac{2k-2}{3}\beta_2 = 0,\\
&h_{1,1}(n+1) + h_{1,2}(n+1) - h_{2,2}(n) - p(n)\frac{2k-2}{3}\beta_2 = 0,\\
&\vdots\\
&h_{k,1}(n+1) - h_{k+1,1}(n) - p(n)\frac{2k-(3k-1)}{3}\beta_2 = 0,\\
&h_{k,1}(n+1) + h_{k,2}(n+1) - h_{k+1,2}(n) - p(n)\frac{2k-(3k-1)}{3}\beta_2 = 0,\\
&h_{k+1,1}(n+1) + \frac{1}{k}h_{1,1}(n) - \frac{k+1}{k}h_{k+1,1}(n) - p(n)\frac{k-1}{6}\beta_2 = 0,\\
&h_{k+1,1}(n+1) + h_{k+1,2}(n+1) + \frac{1}{k}h_{1,2}(n) - \frac{k+1}{k}h_{k+1,2}(n) - p(n)\frac{k-1}{6}\beta_2 = 0.
\end{aligned}
\tag{3.73}
$$

We emphasize that only the formulae for the functions $h_{1,1}(n)$ and $h_{1,2}(n)$ are of interest for us in this system. We will not describe in detail the arithmetic transformations that allow us to obtain formulae for the mentioned functions because they are quite standard. We write only the final result:

$$
h_{1,1}(n) = h_{k+1,1}(n-k) + \frac{\beta_2}{3}\Big[\big(2k-(3k-1)\big)p(n-k) + \cdots + (2k-2)p(n-1)\Big],
\tag{3.74}
$$

$$
\begin{aligned}
h_{1,2}(n) = h_{k+1,2}(n-k) - kh_{k+1,1}(n-k) - \frac{\beta_2}{3}\Big[&(k-1)\big(2k-(3k-1)\big)p(n-k)\\
&+ \cdots + 2(2k-8)p(n-3) + (2k-5)p(n-2)\Big].
\end{aligned}
\tag{3.75}
$$

Here the functions $h_{k+1,1}(n)$ and $h_{k+1,2}(n)$ are the solutions of the following $(k+1)$-th order scalar difference equations:

$$
\begin{aligned}
&h_{k+1,1}(n+k+1) - \frac{k+1}{k}h_{k+1,1}(n+k) + \frac{1}{k}h_{k+1,1}(n)\\
&= \frac{\beta_2}{3}\Big[\frac{k-1}{2}p(n+k) - \frac{1}{k}\Big(\big(2k-(3k-1)\big)p(n) + \cdots + (2k-2)p(n+k-1)\Big)\Big],
\end{aligned}
\tag{3.76}
$$

$$
\begin{aligned}
&h_{k+1,2}(n+k+1) - \frac{k+1}{k}h_{k+1,2}(n+k) + \frac{1}{k}h_{k+1,2}(n) = -h_{k+1,1}(n+k+1) + h_{k+1,1}(n)\\
&+ \frac{\beta_2}{3}\Big[\frac{k-1}{2}p(n+k) + \frac{1}{k}\Big((k-1)\big(2k-(3k-1)\big)p(n) + \cdots + (2k-5)p(n+k-2)\Big)\Big].
\end{aligned}
\tag{3.77}
$$

We note that equations (3.76) and (3.77) should be solved with additional conditions

$$
\mathrm{M}\big[h_{k+1,1}(n)\big] = \mathrm{M}\big[h_{k+1,2}(n)\big] = 0
\tag{3.78}
$$

that imply the uniqueness of the solutions. Namely, since $p_0 = \mathrm{M}\big[p(n)\big] = 0$, due to (3.72) and (3.73), we get the linear homogeneous algebraic system for the mean values of the entries of the matrix $H_1(n)$ (generally speaking, overdetermined system). It follows from [12] that this system is uniquely solvable in the subspace $\mathcal{Q}$. Hence, the zero solution is the only solution.

Let $\mathcal{L}$ denote the difference operator that is defined by the left-hand sides of equations (3.76) and (3.77):

$$\mathcal{L}x(n) := x(n+k+1) - \frac{k+1}{k}x(n+k) + \frac{1}{k}x(n) \tag{3.79}$$

We also introduce the functions $\varphi(n)$ and $\varphi_1(n)$ as the solutions of equations

$$\mathcal{L}\varphi(n) = p(n), \qquad \mathcal{L}\varphi_1(n) = \varphi(n), \tag{3.80}$$

having zero mean values. These solutions can be easily found. Namely, if

$$p(n) = \sum_{j=1}^{M} p_j e^{\mathrm{i}\omega_j n}, \quad \omega_j \neq 2\pi l, \quad l \in \mathbb{Z},$$

then

$$\varphi(n) = \sum_{j=1}^{M} \frac{p_j}{L(e^{\mathrm{i}\omega_j})} e^{\mathrm{i}\omega_j n}, \quad \varphi_1(n) = \sum_{j=1}^{M} \frac{p_j}{\big(L(e^{\mathrm{i}\omega_j})\big)^2} e^{\mathrm{i}\omega_j n}, \tag{3.81}$$

where the characteristic polynomial $L(\lambda)$ is defined by (1.9). Returning to equations (3.76) and (3.77), we obtain the following representations for their solutions:

$$h_{k+1,1}(n) = \frac{\beta_2}{3}\Big[\frac{k-1}{2}\varphi(n+k) - \frac{1}{k}\Big(\big(2k-(3k-1)\big)\varphi(n) + \cdots + (2k-2)\varphi(n+k-1)\Big)\Big] \tag{3.82}$$

and

$$\begin{aligned} h_{k+1,2}(n) = \frac{\beta_2}{3}\Big[&\frac{1-k}{2}\varphi_1(n+2k+1) + \frac{1}{k}\Big(\big(2k-(3k-1)\big)\varphi_1(n+k+1) + \cdots + (2k-2)\varphi_1(n+2k)\Big) \\ &+ \frac{k-1}{2}\varphi_1(n+k) - \frac{1}{k}\Big(\big(2k-(3k-1)\big)\varphi_1(n) + \cdots + (2k-2)\varphi_1(n+k-1)\Big) + \frac{k-1}{2}\varphi(n+k) \\ &+ \frac{1}{k}\Big((k-1)\big(2k-(3k-1)\big)\varphi(n) + (k-2)\big(2k-(3k-4)\big)\varphi(n+1) + \cdots + (2k-5)\varphi(n+k-2)\Big)\Big]. \end{aligned} \tag{3.83}$$

We then substitute (3.82) and (3.83) in (3.74) and (3.75) respectively to obtain the following formulae for the entries $h_{1,1}(n)$ and $h_{1,2}(n)$ of the matrix $H_1(n)$:

$$\begin{aligned} h_{1,1}(n) = \frac{\beta_2}{3}\Big[&\frac{k-1}{2}\varphi(n) - \frac{1}{k}\Big(\big(2k-(3k-1)\big)\varphi(n-k) + \cdots + (2k-2)\varphi(n-1)\Big) \\ &+\big(2k-(3k-1)\big)p(n-k) + \cdots + (2k-2)p(n-1)\Big] \end{aligned} \tag{3.84}$$

and

$$\begin{aligned} h_{1,2}(n) = \frac{\beta_2}{3}\Big[&\frac{1-k}{2}\varphi_1(n+k+1) + \frac{1}{k}\Big(\big(2k-(3k-1)\big)\varphi_1(n+1) + \cdots + (2k-2)\varphi_1(n+k)\Big) + \frac{k-1}{2}\varphi_1(n) \\ &- \frac{1}{k}\Big(\big(2k-(3k-1)\big)\varphi_1(n-k) + \cdots + (2k-2)\varphi_1(n-1)\Big) + \frac{1}{k}\Big((2k-1)\big(2k-(3k-1)\big)\varphi(n-k) \\ &+ \cdots + (k+1)(2k-5)\varphi(n-2)\Big) + (2k-2)\varphi(n-1) - \frac{(k-1)^2}{2}\varphi(n) - (k-1)\big(2k-(3k-1)\big)p(n-k) \\ &- \cdots - 2(2k-8)p(n-3) - (2k-5)p(n-2)\Big]. \end{aligned} \tag{3.85}$$

Finally, by using (2.22), (2.28), (3.46), and (3.68), we can get formulae for the quantities $d_{11}^{(2)}$, $d_{21}^{(2)}$, and $d_{22}^{(2)}$. We remark that, due to the form of the asymptotic formulae (3.58)–(3.60), we need the explicit expressions only for $d_{21}^{(2)}$ and for the quantity $d_{11}^{(2)} + d_{22}^{(2)} - d_{21}^{(2)}$. For this reason we give here only these expressions. We have

$$\begin{aligned} d_{21}^{(2)} &= (\beta_1\beta_2+\beta_2^2)\,\mathrm{M}\big[p(n)p_1(n)\big] - \beta_2^2\,\mathrm{M}\big[p_1(n)p_1(n+1)\big] + \beta_2\,\mathrm{M}\big[p(n)h_{1,1}(n)\big] = (\beta_1\beta_2+\beta_2^2)\,\mathrm{M}\big[p(n)p_1(n)\big] \\ &- \beta_2^2\,\mathrm{M}\big[p_1(n)p_1(n+1)\big] + \frac{\beta_2^2}{3}\Big\{\frac{k-1}{2}\,\mathrm{M}\big[p(n)\varphi(n)\big] - \frac{1}{k}\Big(\big(2k-(3k-1)\big)\mathrm{M}\big[p(n)\varphi(n-k)\big] \\ &+\cdots+(2k-2)\,\mathrm{M}\big[p(n)\varphi(n-1)\big]\Big) + \big(2k-(3k-1)\big)\mathrm{M}\big[p(n)p(n-k)\big]+\cdots+(2k-2)\,\mathrm{M}\big[p(n)p(n-1)\big]\Big\}. \end{aligned} \tag{3.86}$$

Next, with account of (3.69), we get

$$\begin{aligned} d_{11}^{(2)} + d_{22}^{(2)} - d_{21}^{(2)} &= \operatorname{tr}\mathrm{M}\big[D_1(n)Y_1(n)\big] - (\beta_1\beta_2+\beta_2^2)\,\mathrm{M}\big[p(n)p_1(n)\big] + \beta_2^2\,\mathrm{M}\big[p_1(n)p_1(n+1)\big] \\ &+ (\beta_1-\beta_2)\,\mathrm{M}\big[p(n)h_{1,1}(n)\big] + \beta_2\,\mathrm{M}\big[p(n)h_{1,2}(n)\big] = (\beta_1^2-\beta_2^2)\,\mathrm{M}\big[p(n)p_1(n)\big] + 2\beta_2^2\,\mathrm{M}\Big[p_1(n+1)\sum_{j=1}^{n}p_1(j)\Big] \\ &+ \beta_2^2\,\mathrm{M}\big[p_1(n)p_1(n+1)\big] + \frac{\beta_2^2}{3}\Big\{\frac{1-k}{2}\,\mathrm{M}\big[p(n)\varphi_1(n+k+1)\big] + \frac{1}{k}\Big(\big(2k-(3k-1)\big)\mathrm{M}\big[p(n)\varphi_1(n+1)\big] \\ &+\cdots+(2k-2)\,\mathrm{M}\big[p(n)\varphi_1(n+k)\big]\Big) + \frac{k-1}{2}\,\mathrm{M}\big[p(n)\varphi_1(n)\big] - \frac{1}{k}\Big(\big(2k-(3k-1)\big)\mathrm{M}\big[p(n)\varphi_1(n-k)\big] \\ &+\cdots+(2k-2)\,\mathrm{M}\big[p(n)\varphi_1(n-1)\big]\Big)\Big\} + \frac{\beta_2}{3}\Big\{\frac{1-k}{2}(k\beta_2-\beta_1)\,\mathrm{M}\big[p(n)\varphi(n)\big] \\ &+\frac{1}{k}\Big((2k\beta_2-\beta_1)\big(2k-(3k-1)\big)\mathrm{M}\big[p(n)\varphi(n-k)\big]+\cdots+\big((k+1)\beta_2-\beta_1\big)(2k-2)\,\mathrm{M}\big[p(n)\varphi(n-1)\big]\Big) \\ &+ (\beta_1-k\beta_2)\big(2k-(3k-1)\big)\mathrm{M}\big[p(n)p(n-k)\big]+\cdots+(\beta_1-2\beta_2)(2k-5)\,\mathrm{M}\big[p(n)p(n-2)\big] \\ &+ (\beta_1-\beta_2)(2k-2)\,\mathrm{M}\big[p(n)p(n-1)\big]\Big\}. \end{aligned} \tag{3.87}$$

Despite the cumbersome appearance of expressions (3.86) and (3.87), they may be effectively calculated by using the mathematical packages (for example, Wolfram Mathematica). In the next section we give some examples of equation (1.1), where we need to calculate the mentioned expressions.

4. Examples

Example 1.

The first example deals with equation (1.1), where the function $q(n)$ has form (1.8) and

$$p(n) = (-1)^n, \quad n \in \mathbb{N}. \tag{4.1}$$

Since $p_0 = \mathrm{M}\big[p(n)\big] = 0$, the asymptotic formulae for the solutions are constructed according to Section 3.2. Thus, if $\alpha > 1$ then asymptotics for the solutions of Eq. (1.1) as $n \to \infty$ has form (3.4). To obtain the asymptotic formulae for the case $\alpha \leq 1$, first, we need to calculate quantity (3.86). The function $p_1(n)$, being the solution of equation (3.64) with zero mean value, is defined by the formula

$$p_1(n) = \frac{(-1)^{n+1}}{2}.$$

Hence,

$$\mathrm{M}\big[p(n)p_1(n)\big] = -\frac{1}{2}, \quad \mathrm{M}\big[p_1(n)p_1(n+1)\big] = -\frac{1}{4}, \quad \mathrm{M}\big[p(n)p(n-j)\big] = (-1)^j, \tag{4.2}$$

where $j = 1, \dots, k$.

Then we need to calculate the function $\varphi(n)$ that is the solution of equation (3.79), (3.80) having zero mean value. It is easily seen that

$$\varphi(n) = \frac{(-1)^n}{L(-1)},$$

where, according to (1.9),

$$L(-1) = \frac{1}{k}\big[(2k+1)(-1)^{k+1} + 1\big] = \begin{cases} -2, & k \text{ is even}, \\ \dfrac{2(k+1)}{k}, & k \text{ is odd}. \end{cases} \tag{4.3}$$

Therefore,

$$\mathrm{M}\big[p(n)\varphi(n-j)\big] = \frac{(-1)^j}{L(-1)}, \quad j = 0, \dots, k. \tag{4.4}$$

By substituting (4.2) and (4.4) in (3.86), with account of (3.70) and (4.3), we conclude that

$$\begin{aligned} d_{21}^{(2)} &= \frac{k^2\big(2 - 2(-1)^k L(-1)\big) + k\big((-1)^{k+1}L(-1) + 2(-1)^k + 2\big) + (-1)^k - 1}{3k(k+1)^2 L(-1)} \\ &= \begin{cases} -\dfrac{1}{k+1}, & k \text{ is even}, \\ \dfrac{k}{(k+1)^2}, & k \text{ is odd}. \end{cases} \end{aligned} \tag{4.5}$$

Speaking about the asymptotics in the case $\alpha = 1$, we remark that, due to (4.5), the inequality $d_{21}^{(2)} \neq -\frac{1}{4}$ holds. Thus, if $\alpha = 1$, the dynamics of the solutions of Eq. (1.1) is described by the asymptotic formula (3.53). We also note that inequality $d_{21}^{(2)} + \frac{1}{4} < 0$ holds only when $k = 2$.

If $\frac{1}{2} < \alpha < 1$, the asymptotics of the solutions has form (3.58). In this case, we should also keep in mind that, due to (4.5), the quantity $d_{21}^{(2)}$ has different sign depending on the parity of the number k.

Finally, if $\alpha \leq \frac{1}{2}$, we need to compute quantity (3.87). We note that only the case of even k is of interest in this situation. Indeed, if k is odd then $d_{21}^{(2)} > 0$. Since the asymptotics is described by formulae (3.59), (3.60), Eq. (1.1) has unbounded nonoscillating solutions with the asymptotic estimate $O\big(\exp\{\frac{\sqrt{d_{21}^{(2)}}}{1-\alpha} n^{1-\alpha}\big(1 + o(1)\big)\}\big)$ as $n \to \infty$. Thus, we assume further that $\alpha \leq \frac{1}{2}$ and k is even. The solution $\varphi_1(n)$ of equation (3.79), (3.80) with zero mean value is

$$\varphi_1(n) = \frac{(-1)^n}{\big(L(-1)\big)^2} = \frac{(-1)^n}{4}.$$

Then

$$\mathrm{M}\big[p(n)\varphi_1(n-j)\big] = \frac{(-1)^j}{4}, \quad j = 0, \dots, k+1. \tag{4.6}$$

Moreover, since k is even, we have

$$\varphi_1(n+k+1-j) = -\varphi_1(n-j), \quad j = 0, \dots, k. \tag{4.7}$$

Finally,

$$\mathrm{M}\Big[p_1(n+1)\sum_{j=1}^{n} p_1(j)\Big] = -\frac{1}{8}. \tag{4.8}$$

Substituting (4.2), (4.4), (4.6), (4.7), and (4.8) in (3.87), we deduce that

$$d_{11}^{(2)} + d_{22}^{(2)} - d_{21}^{(2)} = \frac{5k+4}{3(k+1)}. \tag{4.9}$$

To compute quantities (4.5) and (4.9) we used Wolfram Mathematica 10 package for symbolic calculations. Since quantity (4.9) is positive for all k, it follows from formulae (3.59) and (3.60) that Eq. (1.1) has unbounded oscillating solutions if $\alpha \leq \frac{1}{2}$ and k is even.

Example 2.

Consider Eq. (1.1) with function $q(n)$ having form (1.8), where

$$p(n) = \sin \omega n, \quad 0 < \omega < 2\pi, \quad \omega \neq \pi, \quad n \in \mathbb{N}. \tag{4.10}$$

As in the previous example, we have $p_0 = \mathrm{M}\big[p(n)\big] = 0$. Therefore, the asymptotic formulae are constructed according to Section 3.2. If $\alpha > 1$, the asymptotics for the solutions of Eq. (1.1) as $n \to \infty$ has form (3.4). If $\alpha \leq 1$, we should calculate the quantity (3.86). Function $p_1(n)$, being the solution of Eq. (3.64) with zero mean value, is determined by the formula

$$p_1(n) = -\frac{\cos\big(n-\frac{1}{2}\big)\omega}{2\sin\frac{\omega}{2}}.$$

Hence,

$$\mathrm{M}\big[p(n)p_1(n)\big] = -\frac{1}{4}, \quad \mathrm{M}\big[p_1(n)p_1(n+1)\big] = \frac{\cos\omega}{8\sin^2\frac{\omega}{2}}, \quad \mathrm{M}\big[p(n)p(n-j)\big] = \frac{\cos\omega j}{2}, \tag{4.11}$$

where $j = 1, \dots, k$. We have the following expression for the function $\varphi(n)$ that is the solution of equation (3.79), (3.80) with zero mean value:

$$\varphi(n) = \mathrm{Im}\Big\{\frac{e^{\mathrm{i}\omega n}}{L(e^{\mathrm{i}\omega})}\Big\}.$$

Here the quantity $L(e^{\mathrm{i}\omega})$ is calculated according to (1.9):

$$L(e^{\mathrm{i}\omega}) = e^{\mathrm{i}(k+1)\omega} - \frac{k+1}{k}e^{\mathrm{i}k\omega} + \frac{1}{k}. \tag{4.12}$$

Hence,

$$\mathrm{M}\big[p(n)\varphi(n-j)\big] = \frac{1}{2}\,\mathrm{Im}\Big\{\frac{\mathrm{i}e^{-\mathrm{i}\omega j}}{L(e^{\mathrm{i}\omega})}\Big\}, \quad j=0,\dots,k. \tag{4.13}$$

Substituting (4.11) and (4.13) in (3.86), with account of (3.70), we get

$$d_{21}^{(2)} = -\frac{k\big((k+1)\cos(k\omega) - k\cos((k+1)\omega) - 1\big)}{2(k+1)\big(k^2-(k+1)k\cos\omega + k\cos((k+1)\omega) - (k+1)\cos(k\omega) + k + 1\big)}. \tag{4.14}$$

Again we used Wolfram Mathematica 10 to obtain expression (4.14). We see at once that quantity $d_{21}^{(2)}$ as a function of ω is 2π-periodic. Moreover, the following asymptotic relations hold:

$$d_{21}^{(2)}(\omega) = -\frac{2}{(1+k)^2\omega^2} + \frac{7k^2+7k+1}{18(1+k)^2} + O\big(\omega^2\big), \qquad \omega\to 0, \tag{4.15}$$

$$d_{21}^{(2)}(k) = -\frac{\sin\big((k+\frac{1}{2})\omega\big)}{2k\sin(\frac{\omega}{2})} + O\big(k^{-2}\big), \qquad k\to\infty. \tag{4.16}$$

Here we assume that the second parameter is fixed.

Since the quantity $d_{21}^{(2)}$ may be of different sign, in the case $\alpha=1$, the asymptotics for solutions of Eq. (1.1) is defined either by formula (3.53) or by formula (3.30). If $\frac{1}{2}<\alpha<1$, the behavior of solutions is described by the asymptotic formula (3.58). If $\alpha\le\frac{1}{2}$, the asymptotics has either form (3.59) or (3.60). In this situation the qualitative behavior of the solutions is not clear provided that $d_{21}^{(2)}<0$. Hence, we need to calculate the quantity (3.87).

By computing the function $\varphi_1(n)$ that is the solution of equations (3.79) and (3.80) with the zero mean value, we get

$$\varphi_1(n) = \mathrm{Im}\Big\{\frac{e^{\mathrm{i}\omega n}}{\big(L(e^{\mathrm{i}\omega})\big)^2}\Big\},$$

where $L(e^{\mathrm{i}\omega})$ is defined according to (4.12). Consequently,

$$\mathrm{M}\big[p(n)\varphi_1(n+j)\big] = \frac{1}{2}\,\mathrm{Im}\Big\{\frac{\mathrm{i}e^{\mathrm{i}\omega j}}{\big(L(e^{\mathrm{i}\omega})\big)^2}\Big\}, \quad j=-k,\dots,k+1. \tag{4.17}$$

Furthermore,

$$\mathrm{M}\Big[p_1(n+1)\sum_{j=1}^{n}p_1(j)\Big] = -\frac{1}{16\sin^2\frac{\omega}{2}}. \tag{4.18}$$

By substituting (4.11), (4.13), (4.17), and (4.18) in (3.87), we obtain

$$d_{11}^{(2)}+d_{22}^{(2)}-d_{21}^{(2)}=-\frac{4(k-1)(k+2)}{9(k+1)^2}-\frac{1}{2(k+1)^2\sin^2\frac{\omega}{2}}+\frac{\cos\omega}{2(k+1)^2\sin^2\frac{\omega}{2}}+\frac{4}{3(k+1)^2}\Big\{\frac{1-k}{4}\operatorname{Im}\Big\{\frac{\mathrm{i}e^{\mathrm{i}\omega(k+1)}}{\big(L(e^{\mathrm{i}\omega})\big)^2}\Big\}$$
$$+\operatorname{Im}\Big\{\frac{\mathrm{i}}{2k\big(L(e^{\mathrm{i}\omega})\big)^2}\sum_{j=1}^{k}\big(2k-(3j-1)\big)e^{\mathrm{i}\omega(k-j+1)}\Big\}+\frac{k-1}{4}\operatorname{Im}\Big\{\frac{\mathrm{i}}{\big(L(e^{\mathrm{i}\omega})\big)^2}\Big\}$$
$$-\operatorname{Im}\Big\{\frac{\mathrm{i}}{2k\big(L(e^{\mathrm{i}\omega})\big)^2}\sum_{j=1}^{k}\big(2k-(3j-1)\big)e^{-\mathrm{i}\omega j}\Big\}\Big\}-\frac{2}{3(k+1)}\Big\{\frac{k-1}{2}\Big(\frac{k}{k+1}+\frac{2k+1}{3(k+1)}\Big)\operatorname{Im}\Big\{\frac{\mathrm{i}}{L(e^{\mathrm{i}\omega})}\Big\}$$
$$-\operatorname{Im}\Big\{\frac{\mathrm{i}}{kL(e^{\mathrm{i}\omega})}\sum_{j=1}^{k}\Big(\frac{k+j}{k+1}+\frac{2k+1}{3(k+1)}\Big)\big(2k-(3j-1)\big)e^{-\mathrm{i}\omega j}\Big\}+\sum_{j=1}^{k}\Big(\frac{2k+1}{3(k+1)}+\frac{j}{k+1}\Big)\big(2k-(3j-1)\big)\cos\omega j\Big\}. \tag{4.19}$$

Expression (4.19) looks rather cumbersome and unfortunately cannot be simplified. Despite this fact, we can use Wolfram Mathematica to deal with it. Evidently, quantity (4.19) as the function of ω is 2π-periodic. Moreover, provided the parameter k is fixed, the following limit relation holds:

$$d_{11}^{(2)}(\omega)+d_{22}^{(2)}(\omega)-d_{21}^{(2)}(\omega)=-\frac{86k^3+129k^2+51k+4}{270(k+1)^2}+O\big(\omega^2\big),\qquad \omega\to 0. \tag{4.20}$$

It follows from formulae (4.15) and (4.20) that quantities (4.14) and (4.19) are negative provided that k is fixed and ω is sufficiently small. Hence, due to (3.60), equation (1.1) has oscillating solutions if $\alpha<\frac{1}{2}$. Moreover, all solutions of this equation tend to zero as $n\to\infty$.

In Table, we give explicit formulae of quantities (4.14) and (4.19) for the first few values of the parameter k.

Table . Formulae of quantities (4.14) and (4.19) for $k=1,2,3,4$.

k	$d_{21}^{(2)}$	$d_{11}^{(2)}+d_{22}^{(2)}-d_{21}^{(2)}$
1	$\frac{\cos(2\omega)-2\cos\omega+1}{4\big(\cos(2\omega)-4\cos\omega+3\big)}$	$-\frac{1}{4}$
2	$\frac{2\cos(3\omega)-3\cos(2\omega)+1}{3\big(2\cos(3\omega)-3\cos(2\omega)-6\cos\omega+7\big)}$	$-\frac{40\cos(2\omega)+200\cos\omega+153}{9\big(4\cos\omega+5\big)^2}$
3	$\frac{3\big(3\cos(4\omega)-4\cos(3\omega)+1\big)}{8\big(3\cos(4\omega)-4\cos(3\omega)-12\cos\omega+13\big)}$	$-\frac{63\cos(4\omega)+336\cos(3\omega)+1036\cos(2\omega)}{16\big(3\cos(2\omega)+8\cos\omega+7\big)^2}-\frac{1832\cos\omega+1101}{16\big(3\cos(2\omega)+8\cos\omega+7\big)^2}$
4	$\frac{2\big(4\cos(5\omega)-5\cos(4\omega)+1\big)}{5\big(4\cos(5\omega)-5\cos(4\omega)-20\cos\omega+21\big)}$	$-\frac{3\big(16\cos(6\omega)+88\cos(5\omega)+281\cos(4\omega)\big)}{5\big(4\cos(3\omega)+11\cos(2\omega)+20\cos\omega+15\big)^2}-\frac{2040\cos(3\omega)+3580\cos(2\omega)+4904\cos\omega+2721}{5\big(4\cos(3\omega)+11\cos(2\omega)+20\cos\omega+15\big)^2}$

In Figure 1 we give the graphs of quantities (4.14) and (4.19) as functions $f(\omega)$ of the parameter ω for $k = 1, 2, 3, 4$ (plots were prepared in Wolfram Mathematica 10). The zeroes of these quantities that seem to be coinciding in this figure are actually distinct but closely located. In Figure 2 we give the enlarged fragment of the graphs for $k = 3$ that contains zeroes located in the interval $\omega \in (2.6, 2.65)$.

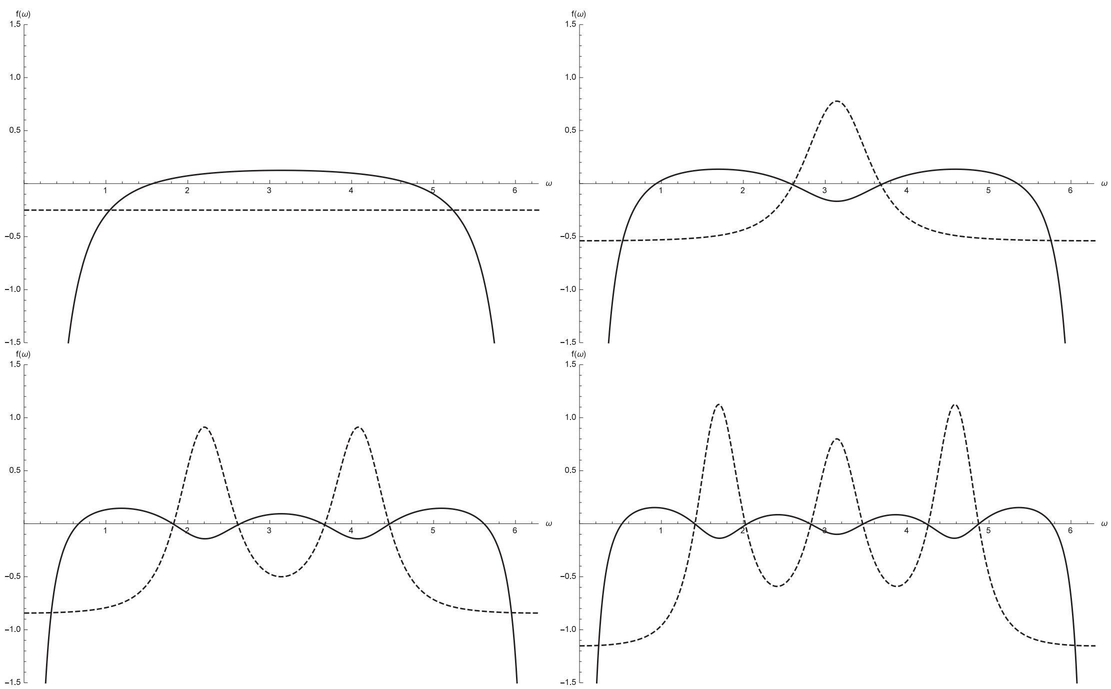

Figure 1. Graphs of quantities (4.14) and (4.19) as functions of ω for $k = 1$ (left, top) and $k = 2$ (right, top), and also for $k = 3$ (left, bottom) and $k = 4$ (right, bottom): solid line corresponds to quantity (4.14) and dashed line corresponds to quantity (4.19).

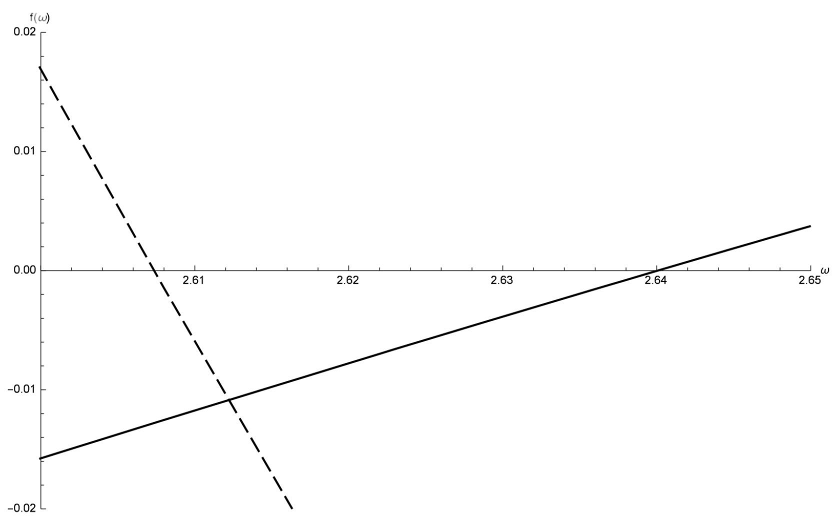

Figure 2. The enlarged fragment of the graphs of quantities (4.14) and (4.19) as functions of ω for $k = 3$ that contains closely located zeroes of these functions: solid line corresponds to quantity (4.14) and dashed line corresponds to quantity (4.19).

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