

Various results for series expansions of the error functions with the complex variable and some of their implications

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Abstract: This scientific investigation deals with introducing certain basic information relating to the error functions in z -plane, establishing extensive relations between various series expansions of the complex error functions and presenting a number of their implications.

Key words: Complex plane, open unit disk, error functions with complex variable, analytic function, special functions with complex variable, series expansions, inequalities in complex plane

1. Introduction, definitions, notations and motivation

As is well known, equations and inequalities basically have important roles both in nearly all sciences and engineering. Specially, if there are some requirements between equations (*or* inequalities), naturally, certain relations between them can also arise. Of course, similar relations also have a different importance for mathematical science and these appear in the literature as various propositions. The basic content of this scientific research relates to certain applications of various propositions which will be determined by various relationships between certain series, which are stated by the error functions in the complex plane and generally used in many fields of science and technology. In other words, to determine some novel-nonlinear relations between some series specified by the error functions and certain inequalities in the complex plane and then to emphasize some possible implications of their possible propositions. Now, before presenting the relevant propositions, let us begin to introduce some basic information and definitions about the well-known error functions in the complex plane.

In the literature, as we know, the well-known error functions are basically encountered in two forms. The first is the error function with real (*or* complex) variable (*or* parameter) and the second is the complementary error function with real (*or* complex) variable (*or* parameter). These functions, covered by special functions in mathematics, were (are) frequently used in science and technology. We note that they are one of the basic subjects in theoretical physics and especially in the theory of statistics and probability. For example, see the studies in [5, 12–14, 18, 28, 29] for primary resources, and see also the earlier results relating to both those and other sciences in [1–3, 6, 8, 10, 11, 16, 17, 19, 22, 23, 25, 26, 30, 32, 35, 36, 38] in the references.

For our main results, there is a need to introduce (*or* remind) a number of notations and notions which will be related to our investigation. For those, firstly, the usual notations \mathbb{C} , \mathbb{R} , \mathbb{N} , and \mathbb{U} refer to the well-

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known set of the set of all complex numbers, the set of all real numbers, the set of all integers and open unit disk (in the complex plane \mathbb{C}), respectively.

Next, there is a need to introduce certain complex functions and complex series expansions, which are analytic and uniform convergence in \mathbb{C} , respectively. For more information, see the results given in [4, 7, 10, 11, 15, 21, 23, 24, 26, 34, 37]. Some of those can be also given by the following notations and notions, which are below.

The *error* function with complex variable z (or the complex error function) is denoted by $erf(z)$ and defined by

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\eta^2) d\eta \quad (1.1)$$

for an arbitrary integration path (from the point $\xi = 0$ to any point $\xi = z$) in any domain of the complex plane \mathbb{C} .

By taking into account the Taylor–Maclaurin series expansion of the following form:

$$\exp(-\xi^2) = 1 - \frac{\eta^2}{2!} + \frac{\eta^4}{4!} - \cdots + \frac{\eta^{2n}}{(2n)!} + \cdots$$

and also by using term-by-term integration of any function series which is uniformly convergent on any interval of the set \mathbb{R} , the following series expansion of the complex error function $erf(z)$ in the form:

$$\begin{aligned} erf(z) &= \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots + \frac{(-1)^n z^{2n+1}}{n!(2n+1)} + \cdots \right) \\ &= \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)} \end{aligned} \quad (1.2)$$

can be easily derived, where $z \in \mathbb{C}$. This definition is also known as the second definition of the complex error function, i.e. $erf(z)$, in the literature.

In view of the error function with complex variable z , defined by (1.1), the complementary error function with complex variable z (or the complementary complex error function) is denoted by $erfc(z)$ and is defined by

$$erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \exp(-\eta^2) d\eta \quad (z \in \mathbb{C}). \quad (1.3)$$

Through the instrumentality of the familiar result

$$\int_0^{\infty} \exp(-\eta^2) d\eta = \frac{\sqrt{\pi}}{2}$$

and the property

$$\int_0^{\infty} \exp(-\eta^2) d\eta = \int_0^z \exp(-\eta^2) d\eta + \int_z^{\infty} \exp(-\eta^2) d\eta,$$

the basic relation between these complex functions ($erf(z)$ and $erfc(z)$), which is given by

$$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z \exp(-\eta^2) d\eta$$

$$\begin{aligned}
&= \frac{2}{\sqrt{\pi}} \left(\int_0^\infty \exp(-\eta^2) d\eta - \int_z^\infty \exp(-\eta^2) d\xi \right) \\
&= 1 - \operatorname{erfc}(z) ,
\end{aligned} \tag{1.4}$$

can be easily seen, where $z \in \mathbb{C}$.

Moreover, by means of the series expansion of the complex error function $\operatorname{erf}(z)$, given by (1.1), the second definition of the complementary complex error function, i.e. $\operatorname{erfc}(z)$, given by the following-complex-series expansion:

$$\begin{aligned}
\operatorname{erfc}(z) &= 1 - \frac{2}{\sqrt{\pi}} \left(z - \frac{z^3}{3} + \frac{z^5}{10} - \frac{z^7}{42} + \cdots + \frac{(-1)^n z^{2n+1}}{n!(2n+1)} + \cdots \right) \\
&= 1 - \frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{n!(2n+1)}
\end{aligned} \tag{1.5}$$

can be also determined, where $z \in \mathbb{C}$.

With a simple observation, it is easily seen that both the complex series, given in (1.2) and (1.5), are uniformly convergent on any region of the set \mathbb{C} .

As we have indicated before, this investigation includes various novel results consisting of certain series and inequalities in related with the error functions in the complex plane. We note here that certain special results derived by the complex error functions and also their implications may be an interesting investigation for some researchers whose scientific fields are relevant. Specially, by means of an unusual proof's technique and also by making use of the well-known assertion given by [27], which is the main lemma (Lemma 1.1 below), various unusual results (consisting of certain special inequalities) specified by some relationships between the series expansions of the complex error functions are given by (1.2) and (1.5). In addition, see the earlier results, which were used for proving, given in [19–21] as examples.

Lemma 1.1 [27] *Let a function $p(z)$ be in the form:*

$$p(z) = 1 + a_n z^n + a_{n+1} z^{n+1} + \cdots \quad (a_n \in \mathbb{C}; n \in \mathbb{N}). \tag{1.6}$$

Then, if there exists a point z_0 in \mathbb{U} such that

$$\Re(p(z)) > 0 \quad (|z| < |z_0|) \quad , \quad \Re(p(z_0)) = 0 \quad \text{and} \quad p(z_0) \neq 0, \tag{1.7}$$

then

$$p(z_0) = ir \quad \text{and} \quad zp'(z) \Big|_{z=z_0} = i \frac{s}{2} \left(r + \frac{1}{r} \right) p(z) \Big|_{z=z_0} , \tag{1.8}$$

where $s \geq 1$ and $r \in \mathbb{R}^ := \mathbb{R} - \{0\}$.*

2. The main results and related implications

We now begin by setting and then by proving our main results associating with the complex error functions defined by (1.1) and (1.3) (or, given by (1.2) and (1.5)). The first is Theorem 1 (below), which can be easily proven by considering the complex series of the error function $\operatorname{erf}(z)$ given by (1.2).

Theorem 2.1 *Let z be in the disk \mathbb{U} . Then, the following implication:*

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \Re \{z^{2n}\} \right) > \Theta(\tau) \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!(2n+1)} \Re \{z^{2n}\} \right) > \hat{\Xi}(\tau) \quad (2.1)$$

holds true, where

$$0 \leq \tau < \frac{2}{\sqrt{\pi}} \quad , \quad \hat{\Xi}(\tau) := \frac{\sqrt{\pi}}{2} \tau \quad \text{and} \quad \Theta(\tau) := \frac{1}{2} \left(\frac{3\sqrt{\pi}}{2} \tau - 1 \right). \quad (2.2)$$

Proof For the proof of Theorem 2.1, it is sufficient to use Lemma 1 and consider the complex error function, i.e. the function $erf(z)$ defined by (1.2). For this, with the help of the series expansion of the related complex function, define an implicit function $p(z)$ as in the form:

$$\frac{erf(z)}{z} = \tau + \left(\frac{2}{\sqrt{\pi}} - \tau \right) p(z) \quad \left(z \in \mathbb{U}; 0 \leq \tau < \frac{2}{\sqrt{\pi}} \right). \quad (2.3)$$

With focusing on the function $p(z)$ defined by (2.3), it is easy to see that $p(z)$ has the series form given by (1.6) of Lemma 1.1 and it is an analytic function in the open disk \mathbb{U} . In consideration of Lemma 1.1, it follows from (2.3) that

$$\begin{aligned} \frac{d}{dz} \left(\frac{erf(z)}{z} \right) &= \frac{z \frac{d}{dz} [erf(z)] - erf(z)}{z^2} \\ &= \left(\frac{2}{\sqrt{\pi}} - \tau \right) \frac{d}{dz} [p(z)] \\ &= \left(\frac{2}{\sqrt{\pi}} - \tau \right) p'(z), \end{aligned}$$

and, by a simple computation, the statement (just above) along with (2.3) also gives us

$$\begin{aligned} erf'(z) &= \frac{erf(z)}{z} + \left(\frac{2}{\sqrt{\pi}} - \tau \right) zp'(z) \\ &= \tau + \left(\frac{2}{\sqrt{\pi}} - \tau \right) p(z) + \left(\frac{2}{\sqrt{\pi}} - \tau \right) zp'(z), \end{aligned} \quad (2.4)$$

where $z \in \mathbb{U}$ and $0 \leq \tau < 2/\sqrt{\pi}$.

We now assume that there exists a point $z_0 \in \mathbb{U}$ satisfying the conditions given by (1.7) of Lemma 1.1. By taking cognizance of the assertions given by (1.8) of Lemma 1.1, viz, the following assertions:

$$p(z_0) = ir \quad \text{and} \quad zp'(z) \Big|_{z=z_0} = i \frac{s}{2} \left(r + \frac{1}{r} \right) p(z) \Big|_{z=z_0},$$

for the equation obtained by (2.4), the following result:

$$erf'(z) \Big|_{z:=z_0} = \tau + \left(\frac{2}{\sqrt{\pi}} - \tau \right) p(z) + \left(\frac{2}{\sqrt{\pi}} - \tau \right) zp'(z) \Big|_{z:=z_0}$$

$$= \tau - \frac{s}{2}(1+r^2)\left(\frac{2}{\sqrt{\pi}} - \tau\right) + i\left(\frac{2}{\sqrt{\pi}} - \tau\right) \quad (2.5)$$

is then determined, where $s \geq 1$, $r \in \mathbb{R}^*$ and $z_0 \in \mathbb{U}$.

By considering the values of the parameters s and r in Lemma 1.1, the real part of (??) gives us the following inequality:

$$\Re\left(\operatorname{erf}'(z_0)\right) \leq \frac{3}{2}\tau - \frac{1}{\sqrt{\pi}},$$

or, equivalently,

$$\frac{2}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \Re\{z^{2n}\} \right) \leq \frac{3}{2}\tau - \frac{1}{\sqrt{\pi}}, \quad (2.6)$$

where τ is given by (10). However, the inequality given by (2.6) is a contradiction with the hypothesis of Theorem 2.1 given by (2.1). Thus, the statement, given by (2.3), immediately requires the inequality:

$$\Re\left(\frac{\operatorname{erf}(z)}{z}\right) > \tau,$$

or, equivalently,

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!(2n+1)} \Re\{z^{2n}\} \right) > \hat{\Xi}(\tau),$$

which is the provision of Theorem 2.1 given by (2.1), where $z \in \mathbb{U}$, τ is given by (2.2) and also $\hat{\Xi}(\tau)$ is defined by (2.2). Thereby, this completes the proof of Theorem 2.1. \square

The second is Theorem 2.2, which is below and in relation with (the series of) the complex error function $\operatorname{erf}(z)$ given by (1.2).

Theorem 2.2 *Let $z \in \mathbb{U}$. Then, the following implication:*

$$\sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \Im\{z^{2n}\} \right) = 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!(2n+1)} \Re\{z^{2n}\} \right) > \hat{\Xi}(\tau)$$

holds true, where τ is given by (2.2) and $\hat{\Xi}(\tau)$ is defined by (2.2).

Proof Through the instrument of the statement determined by (2.5), the desired proof of Theorem 2.2 can be easily achieved by the the proof of Theorem 2.1. Its detail is omitted here. \square

The third is the following theorem (Theorem 2.1 below), which also relates to (the series of) the complex error function $\operatorname{erf}(z)$ given by (1.2).

Theorem 2.3 *Let $z \in \mathbb{U}$. Then, the following implication:*

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(n-1)!} \Re\{z^{2n-1}\} \right) > \Delta(\tau) \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \Re\{z^{2n}\} \right) > \hat{\Xi}(\tau)$$

holds true, where

$$0 \leq \tau < \frac{2}{\sqrt{\pi}} \quad , \quad \hat{\Xi}(\tau) := \frac{\sqrt{\pi}}{2}\tau \quad \text{and} \quad \Delta(\tau) := \frac{1}{4}\left(\frac{\sqrt{\pi}}{2}\tau - 1\right). \quad (2.7)$$

Proof For the proof of Theorem 2.3, it is sufficient to use Lemma 1.1 and consider the complex error function $erf(z)$ defined by (1.2). For this, by the help of the series expansion of the complex function $erf(z)$, if one defines a function $p(z)$ in the form:

$$p(z) = \frac{\frac{d}{dz}(erf(z)) - \tau}{\frac{2}{\sqrt{\pi}} - \tau} \quad \left(z \in \mathbb{U}; 0 \leq \tau < \frac{2}{\sqrt{\pi}}\right), \quad (2.8)$$

it is easily seen that the function $p(z)$ above both has the series form given by (1.6) of Lemma 1.1 and is an analytic function in \mathbb{U} . By differentiating both sides of (2.8) with respect to the complex variable, the following statement:

$$z \frac{d^2}{dz^2}[erf(z)] = \left(\frac{2}{\sqrt{\pi}} - \tau\right)zp'(z)$$

is then obtained. By using the similar steps used in the proof of Theorem 2.1 for the result just above, the desired proof of Theorem 2.2 can be easily constituted. For this reason, its detail is omitted here. \square

The fourth is Theorem 2.4 (below). Its proof can be also presented by considering (the series of) the complex error function $erf(z)$ together with the Proof of Theorem 2.3. It is omitted here again.

Theorem 2.4 *Let z be in \mathbb{U} . Then, the following implication:*

$$\sum_{n=1}^{\infty} \left(\frac{(-1)^n}{(n-1)!} \Im \{z^{2n-1}\} \right) \neq 0 \quad \Rightarrow \quad \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \Re \{z^{2n}\} \right) > \hat{\Xi}(\tau)$$

holds true, where $\hat{\Xi}(\tau)$ is defined by (2.7).

As certain implications and recommendations, in the first two sections, firstly, certain fundamental information appertaining to the series expansions of the complex error functions has been presented and then several statements in relation with those complex series have been proven. In addition, we also need extra information directly associated with the complex error functions. For those, it is time to recall certain relations between them. Clearly, these complex functions are differentiable functions in the whole complex plane \mathbb{C} and also have various connections relating to certain special functions with complex variable. First of all, for convenience, let

$$\Xi(z) := erf(z) \quad \text{and} \quad \Xi_c(z) := erfc(z) \quad (z \in \mathbb{C}).$$

Then, for the scope of this investigation, there is a need to remind only some of them, which can be presented by the following forms:

$$\Xi(-z) = -\Xi(z), \quad (2.9)$$

$$\Xi_c(-z) = -\Xi_c(z), \quad (2.10)$$

$$\Xi(\bar{z}) = \overline{\Xi(z)}, \quad (2.11)$$

$$\Xi_c(\bar{z}) = \overline{\Xi_c(z)}, \quad (2.12)$$

$$\Xi(z) = 1 - \Xi_c(z), \quad (2.13)$$

$$\Xi_c(z) = 1 - \Xi(z), \quad (2.14)$$

$$\frac{d}{dz}(\Xi(z)) = \frac{2}{\sqrt{\pi}} z \exp(-z^2), \quad (2.15)$$

$$\frac{d}{dz}(\Xi_c(z)) = -\frac{2}{\sqrt{\pi}} z \exp(-z^2), \quad (2.16)$$

$$\int \Xi(z) dz = z \Xi(z) + \frac{1}{\sqrt{\pi}} \exp(-z^2), \quad (2.17)$$

$$\Xi(z) = \frac{2z}{\sqrt{\pi}} {}_1F_1(1/2; 3/2; -z^2), \quad (2.18)$$

$$\Xi_c(z) = 1 - \frac{2z}{\sqrt{\pi}} {}_1F_1(1/2; 3/2; -z^2), \quad (2.19)$$

$$\Xi(z) = \frac{1}{\sqrt{\pi}} \Gamma(1/2, z^2) \quad (\Re(z) > 0), \quad (2.20)$$

$$\Xi_c(z) = 1 - \frac{1}{\sqrt{\pi}} \Gamma(1/2, z^2) \quad (\Re(z) > 0), \quad (2.21)$$

$$\Xi(z) = \frac{2}{\sqrt{\pi}} z^2 e^{-z^2} {}_1F_1(1; 3/2; z^2) \quad (2.22)$$

and

$$\Xi_c(z) = 1 - \frac{2}{\sqrt{\pi}} z^2 e^{-z^2} {}_1F_1(1; 3/2; z^2), \quad (2.23)$$

and so on, where the special functions:

$${}_1F_1(a; b; z) \quad \text{and} \quad \Gamma(a, z)$$

are known as the confluent hypergeometric function of the first kind and incomplete gamma function, respectively. For more properties and also some of their applications, one may refer to the studies given by [3, 7–9, 13, 14, 17, 18, 21, 25, 28, 30, 37] in the references.

In the light of the information in the first section, the main results in the second section and also the properties given by (2.9)-(2.23), various extensive results (together with suitable examples) can be exposed. It is impossible to present all of them. Nevertheless, we want to specify only one of them and bring other special results to the attentions of the interested researchers.

As a simple example, by reevaluating the properties given by (2.13), (2.15), and (2.16) in the results given by Theorem 2.1, the following proposition, which is directly related to the complex error functions $erf(z)$ and $erfc(z)$, can be easily constituted by the basic form:

Proposition 2.5 *Let $z \in \mathbb{U}$. Then, the following implication:*

$$\Re \left(\frac{d}{dz} (\operatorname{erfc}(z)) \right) < \frac{2}{\sqrt{\pi}} \Theta(\tau) \quad \Rightarrow \quad \Re \left(\frac{1 - \operatorname{erfc}(z)}{z} \right) > \tau,$$

or, equivalently,

$$\begin{aligned} \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!} \Re \{ z^{2n} \} \right) &< -\Theta(\tau) \\ \Rightarrow \sum_{n=0}^{\infty} \left(\frac{(-1)^n}{n!(2n+1)} \Re \{ z^{2n} \} \right) &> \hat{\Xi}(\tau) \end{aligned}$$

is true, where τ is given by (2.2), and $\hat{\Xi}(\tau)$ and $\Theta(\tau)$ are defined by (2.2).

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